and

$$\tau(n) \geqslant 2^{\log\log x - u\sqrt{\log\log x}}$$

then n belongs to the first class; otherwise it belongs to the second class. If $u \to \infty$ with x, all but $o(x)$ integers $n \leqslant x$ satisfy both the conditions above, and so the second class contains only $o(x)$ numbers. Now let Σ' run over elements of the first class. Then

$$\sum_{n\leqslant x}{}' \tau^{2-2\lambda}(n) 2^{-\omega(n)} \ll x(\log\log x)^4$$

and so

$$\sum_{n\leqslant x}{}' 1 \ll x(\log\log x)^4 2^{(2\lambda-1)\log\log x + (3-2\lambda)u\sqrt{\log\log x}} = o(x)$$

if $\lambda < \frac{1}{2}$, and u increases more slowly than $\sqrt{\log\log x}$, say

$$u = u(x) = (\log\log x)^{1/4}.$$

This completes the proof of Theorem 1.

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On the average length of finite continued fractions

by

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Let a and n be positive integers, $1 \leqslant a < n$, $(a, n) = 1$, and let

$$\frac{a}{n} = \cfrac{1}{c_1 + \cfrac{1}{c_2 + \ddots + \cfrac{1}{c_{l(a,n)}}}},$$

where $c_1, c_2, \ldots, c_{l(a,n)}$ are positive integers, $c_{l(a,n)} > 1$. Put

$$L(n) = \sum_{\substack{1\leqslant a<n\\(a,n)=1}} l(a, n).$$

Denote by $r(n)$ the number of solutions of the equation

$$n = xx' + yy'$$

in positive integers x, x', y, y', for which $x > y$, $x' > y'$, $(x, y) = 1$, $(x', y') = 1$.

Recently H. Heilbronn [3] proved that if $n > 2$, then

$$L(n) = \tfrac{3}{2}\varphi(n) + 2r(n) \tag{1}$$

and

$$r(n) = \frac{6\ln 2}{\pi^2}\varphi(n)\ln n + O\left(n\left(\sum_{d|n}\frac{1}{d}\right)^3\right),$$

where φ is Euler's function.

For the numbers a and n we compute positive integers q_i and r_i such that

$$r_0 = n,\ r_1 = a;\quad r_{i-1} = q_i r_i + r_{i+1}\ (i = 1, 2, \ldots, m)$$

and

$$r_0 > r_1 > \ldots > r_m > r_{m+1} = 0.$$

This is the known Euclidean algorithm. Here m is the number of steps in the algorithm. Evidently, $m = l(a, n)$.

J. D. Dixon [1], [2] proved theorems about $l(a, n)$, namely (see [2]): For all positive ε there exists $c_0 > 0$ such that

$$|l(a, n) - 12\pi^{-2}\ln 2\ln n| < (\ln n)^{\frac{1}{2}+\varepsilon}$$

for all except at most

$$x^2 \exp\{-c_0(\ln x)^{\varepsilon/2}\}$$

pairs of integers a, n with $1 \leqslant a \leqslant n \leqslant x$.

Here we shall prove the following

THEOREM.

$$L(n) = \frac{12\ln 2}{\pi^2}\varphi(n)\ln n + O(\sigma(n)),$$

where $\sigma(n) = n\sum_{d|n} 1/d$.

Preliminary results. We formulate two known lemmas.

LEMMA 1 (of I. M. Vinogradov [5]). *Let r be positive integer, α and β real numbers and let $0 < \Delta < 0.25$, $\Delta \leqslant \beta - \alpha \leqslant 1 - \Delta$. Then there exists a periodic function $\psi(x)$ with period 1 and with following properties:*

1) $\psi(x) = 1$ *in interval* $\alpha + 0.5\Delta \leqslant x \leqslant \beta - 0.5\Delta$;

2) $0 < \psi(x) < 1$ *in intervals* $\alpha - 0.5\Delta < x < \alpha + 0.5\Delta$ *and* $\beta - 0.5\Delta < x < \beta + 0.5\Delta$;

3) $\psi(x) = 0$ *in the interval* $\beta + 0.5\Delta \leqslant x \leqslant 1 + \alpha - 0.5\Delta$;

4) $\psi(x)$ *has an expansion into Fourier series of the form*

$$\psi(x) = \beta - \alpha + \sum_{m=1}^{\infty}(g_m e^{2\pi imx} + h_m e^{-2\pi imx}),$$

where

$$|g_m| \leqslant \frac{1}{\pi m}, \quad |g_m| \leqslant \beta - \alpha, \quad |g_m| < \frac{1}{\pi m}\left(\frac{r}{\pi mr}\right)^r;$$

$$|h_m| \leqslant \frac{1}{\pi m}, \quad |h_m| \leqslant \beta - \alpha, \quad |h_m| < \frac{1}{\pi m}\left(\frac{r}{\pi mr}\right)^r.$$

LEMMA 2 (see [4], p. 43). *Let q, b, x be integers, $q > 1$, $(x, q) = 1$. If $0 < q_1 < q$, then*

$$\sum_{x \leqslant q_1} e^{\frac{2\pi ibx^{-1}}{q}} = O(\sqrt{q}\,\tau(q)\ln q\sqrt{(b, q)}),$$

where $xx^{-1} \equiv 1 \pmod q$.

We shall use also

LEMMA 3.

$$\sum_{d|n}\frac{\mu(d)}{d}\ln d = -\frac{\varphi(n)}{n}\sum_{p|n}\frac{\ln p}{p-1},$$

where p is prime.

This lemma follows from the identity

$$\frac{d}{ds}\left(\sum_{d|n}\frac{\mu(d)}{d^s}\right)_{s=1} = \frac{d}{ds}\left(\prod_{p|n}\left(1 - \frac{1}{p^s}\right)\right)_{s=1}.$$

Proof of the theorem. We have

$$r(n) = \sum_{(A)} 1,$$

where (A) designates the following triplet of conditions:

$$n = xx' + yy'; \; x > y, \; x' > y'; \; (x, y) = (x', y') = 1.$$

Then

$$r(n) = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \tag{2}$$

where

$$\Sigma_1 = \sum_{\substack{(A)\\ x+y<\sqrt{n}\\ x'+y'<\sqrt{n}}} 1, \tag{3}$$

$$\Sigma_2 = \sum_{\substack{(A)\\ x+y<\sqrt{n}\\ x'+y'\geqslant\sqrt{n}}} 1, \tag{4}$$

$$\Sigma_3 = \sum_{\substack{(A)\\ x+y\geqslant\sqrt{n}\\ x'+y'<\sqrt{n}}} 1, \tag{5}$$

$$\Sigma_4 = \sum_{\substack{(A)\\ x+y\geqslant\sqrt{n}\\ x'+y'\geqslant\sqrt{n}}} 1. \tag{6}$$

Consider the sum Σ_1 from (3). We have

$$x < \sqrt{n}, \; x' < \sqrt{n}, \; y < \tfrac{1}{2}\sqrt{n}, \; y' < \tfrac{1}{2}\sqrt{n}.$$

If $x < \frac{1}{2}\sqrt{n}$, then $xx' + yy' < \frac{1}{2}n + \frac{1}{4}n < n$. Consequently $x \geqslant \frac{1}{2}\sqrt{n}$. For fixed x, y, $(x, y) = 1$, the number of the pairs x', y', for which $y' < \frac{1}{2}\sqrt{n}$, is less than $\sqrt{n}/2x+1$ and we get

$$0 \leqslant \Sigma_1 < \sum_{\frac{1}{2}\sqrt{n}\leqslant x<\sqrt{n}} \; \sum_{y<\frac{1}{2}\sqrt{n}} \left(\frac{\sqrt{n}}{2x}+1\right) = O(n). \tag{7}$$

Now consider the sum Σ_4 from (6). We have

$$x > \tfrac{1}{2}\sqrt{n},\ x' > \tfrac{1}{2}\sqrt{n},\ x < n/x' < 2\sqrt{n},\ y < 2\sqrt{n},\ x' < 2\sqrt{n},\ y' < 2\sqrt{n}$$

and we get

$$0 \leqslant \Sigma_4 < \sum_{\frac{1}{2}\sqrt{n}<x<2\sqrt{n}} \; \sum_{y<2\sqrt{n}} \left(\frac{2\sqrt{n}}{x}+1\right) = O(n). \tag{8}$$

From (4) and (5) it follows the identity

$$\Sigma_2 = \Sigma_3. \tag{9}$$

Consider in detail the sum Σ_2 from (4). The inequality $x' > y'$ is equivalent to the inequality $y' < n/x+y$. The condition $(x', y') = 1$ may be expressed by the Möbius' function μ and by (4), (3) and (7), we obtain

$$\Sigma_2 = \sum_{d|n} \mu(d) S_d + O(n), \tag{10}$$

where

$$S_d = \sum_{\substack{\frac{n}{d}=xx'+yy',\, x>y,\, (x,y)=1 \\ x+y<\sqrt{n},\, y'<\frac{n}{d(x+y)}}} 1. \tag{11}$$

Decompose the sum (11) into two sums:

$$S_d = S'_d + S''_d, \tag{12}$$

where

$$S'_d = \sum_{\substack{\frac{n}{d}=xx'+yy',\, x>y,\, (x,y)=1 \\ x+y<\sqrt{\frac{n}{d}},\, y'<\frac{n}{d(x+y)}}} 1, \tag{13}$$

$$S''_d = \sum_{\substack{\frac{n}{d}=xx'+yy',\, x>y,\, (x,y)=1 \\ \sqrt{\frac{n}{d}}\leqslant x+y<\sqrt{n},\, y'<\frac{n}{d(x+y)}}} 1. \tag{14}$$

We have for arbitrary sufficiently small positive number ε

$$\sum_{\substack{\frac{n}{d}=xx'+yy',\, (x,y)=1 \\ \left(\frac{n}{d},x\right)\geqslant n^{2\varepsilon}}} 1 = \sum_{\substack{\Delta \mid \frac{n}{d} \\ \Delta\geqslant n^{2\varepsilon}}} \; \sum_{\frac{n}{d\Delta}=\frac{x}{\Delta}x'+y\frac{y'}{\Delta}} 1$$

$$= \sum_{\Delta \mid \frac{n}{d},\, \Delta\geqslant n^{2\varepsilon}} \; \sum_{\frac{x}{\Delta}\cdot x'=X} \; \sum_{y\cdot\frac{y'}{\Delta}=Y} \; \sum_{\frac{n}{d\Delta}=X+Y} 1$$

$$\leqslant \sum_{\Delta \mid \frac{n}{d},\, \Delta\geqslant n^{2\varepsilon}} \left(\max_{m\leqslant n/d\Delta} \tau(m)\right)^2 \frac{n}{d\Delta} \ll \frac{n}{d}\, n^{\varepsilon/2} \sum_{\Delta \mid \frac{n}{d},\, \Delta\geqslant n^{2\varepsilon}} \frac{1}{\Delta}$$

$$\leqslant \frac{n}{d}\, n^{\varepsilon/2} \frac{1}{n^{2\varepsilon}} \sum_{\Delta \mid \frac{n}{d}} 1 \ll \frac{n}{d}\, n^{\varepsilon/2} \frac{1}{n^{2\varepsilon}}\, n^{\varepsilon/2} \lessapprox n^{1-\varepsilon};$$

$$\sum_{\substack{\frac{n}{d}=xx'+yy',\, \sqrt{\frac{n}{d}}\leqslant x+y \\ y<\left(\frac{n}{d}\right)^{(1-\varepsilon)/2},\, y'<\frac{n}{d(x+y)}}} 1 < \sum_{\substack{\frac{n}{d}=xx'+yy',\, y<\left(\frac{n}{d}\right)^{(1-\varepsilon)/2} \\ y'<\sqrt{\frac{n}{d}}}} 1 < \left(\max_{m<n} \tau(m)\right)^2 \left(\frac{n}{d}\right)^{1-\varepsilon/2} \ll \frac{n}{d}$$

and by (14) we obtain

$$S''_d = S'''_d + O\left(\frac{n}{d}\right), \tag{15}$$

where

$$S'''_d = \sum_{\substack{\frac{n}{d}=xx'+yy',\, x>y,\, (x,y)=1 \\ \sqrt{\frac{n}{d}}\leqslant x+y<\sqrt{n},\, y'<\frac{n}{d(x+y)} \\ \left(\frac{n}{d},x\right)<n^{2\varepsilon},\, y\geqslant\left(\frac{n}{d}\right)^{(1-\varepsilon)/2}}} 1. \tag{16}$$

Consider the sum S'''_d from (16). Let x be fixed, $x < \sqrt{n}$. Decompose the interval $[1, x)$ of variation of y into subintervals Y of the form $[y_0, y_0+y_0^{1-4\varepsilon}\ln^2 n)$. The last subinterval may be incomplete, but we may eliminate it, omitting $O(n^{1-\varepsilon})$ solutions. In fact, one subinterval Y contains no more than $y_0^{1-4\varepsilon}\ln^2 n < n^{(1-4\varepsilon)/2}\ln^2 n$ numbers y. But $y' < \sqrt{n/d}$, consequently, the number of the pairs y, y' is $\leqslant n^{1-2\varepsilon}\ln^2 n$. To every y, y' it corresponds no more than $\tau\left(\frac{n}{d}-yy'\right) = O(n^{\varepsilon/2})$ pairs x, x'. Consequently, the number of the corresponding quadruples x, x', y, y' is $\ll n^{1-\varepsilon}$.

Count the number of the subintervals Y. The interval of variation of y contains $\ll \ln n$ subintervals of the form $[y_0, 2y_0)$ and one interval $[y_0, 2y_0)$ contains no more than $\dfrac{y_0}{y_0^{1-4\varepsilon}\ln^2 n} = \dfrac{y_0^{4\varepsilon}}{\ln^2 n} < \dfrac{n^{2\varepsilon}}{\ln^2 n}$ subintervals Y. Consequently

$$\sum_Y 1 = O\left(\frac{n^{2\varepsilon}}{\ln n}\right). \tag{17}$$

Let $y \in Y$. If

$$\frac{n}{d(x+y)} < y' < \frac{y}{d(x+y_0)}$$

then the number of the numbers y' is less than

$$\frac{n}{d(x+y_0)} - \frac{n}{d(x+y)} = \frac{n}{d}\,\frac{y-y_0}{(x+y)(x+y_0)} < \frac{n}{d}\,\frac{y^{1-4\varepsilon}\ln^2 n}{y_0^2} = \frac{n}{d}\,\frac{\ln^2 n}{y_0^{1+4\varepsilon}}$$

and the number of the numbers y is less than $y_0^{1-4\varepsilon}\ln^2 n$; the number of the corresponding solutions x, x', y, y' will be

$$\ll \frac{n\ln^2 n}{dy_0^{1+4\varepsilon}}\, y_0^{1-4\varepsilon}\ln^2 n\cdot n^{\varepsilon^2/2} = \frac{n}{d}\,\frac{\ln^4 n}{y_0^{8\varepsilon}}\, n^{\varepsilon^2/2} < \left(\frac{n}{d}\right)^{1-4\varepsilon+4\varepsilon^2} \ln^4 n\cdot n^{\varepsilon^2/2} < n^{1-3\varepsilon}$$

because $y_0 > \left(\dfrac{n}{d}\right)^{(1-\varepsilon)/2}$, ε is sufficiently small, and by (16) and (17) we get

$$S_d''' = \sum_{x<\sqrt{n}}\sum_Y\sum_{y\in Y}\sum_{\substack{\frac{n}{d}=xx'+yy'\\ x>y,\,(x,y)=1\\ \sqrt{\frac{n}{d}}\leqslant x+y<\sqrt{n}\\ \left(\frac{n}{d},x\right)<n^{2\varepsilon}\\ y'<\frac{n}{d(x+y_0)}}} 1 + O(n^{1-\varepsilon}). \tag{18}$$

We cut off the segment x as many times as possible (denoting it by q_{x,y_0}) from the beginning of the interval $\left(0, \dfrac{n}{d(x+y_0)}\right)$ of variation of y' and designate by E the received interval (E may be empty) and by F the remainder.

Following the application of the trigonometric method of I. M. Vinogradov [5] by the dispersion method of Yu. V. Linnik [4], put

$$\mu = \frac{\operatorname{mes} F}{x}.$$

The number of the numbers y', $y' \in F$, is not more than $\operatorname{mes} F$ and if

$$\mu < \left(\frac{n}{d}\right)^{1-2\varepsilon}\frac{1}{x(x+y_0)}$$

then the number of the numbers y' shall be less than

$$\left(\frac{n}{d}\right)^{1-2\varepsilon}\frac{1}{x+y_0} < \left(\frac{n}{d}\right)^{1-2\varepsilon}\frac{1}{y_0};$$

the number of the pairs y, y' shall be less than

$$\left(\frac{n}{d}\right)^{1-2\varepsilon}\frac{1}{y_0}\cdot y_0^{1-4\varepsilon}\ln^2 n = \left(\frac{n}{d}\right)^{1-2\varepsilon}\frac{\ln^2 n}{y_0^{4\varepsilon}} < \left(\frac{n}{d}\right)^{1-4\varepsilon+2\varepsilon^2}\ln^2 n;$$

the number of the corresponding quadruples x, x', y, y' shall be

$$\ll \left(\frac{n}{d}\right)^{1-4\varepsilon+2\varepsilon^2}\ln^2 n\cdot n^{\varepsilon^2/2} < n^{1-3\varepsilon}$$

and by (15) and (14) we shall get

$$S_d''' = \sum_{x<\sqrt{n}}\sum_Y\left(q_{x,y_0}\sum_y 1 + Q_{x,y_0}\right) + O(n^{1-\varepsilon}), \tag{19}$$

where

$$Q_{x,y_0} = \sum_{\substack{\frac{n}{d}=xx'+yy',\,x>y,\,(x,y)=1\\ \sqrt{\frac{n}{d}}\leqslant x+y<\sqrt{n}\\ y\in Y,\,y'\in F,\,\frac{y'}{x}<\mu}} 1, \tag{20}$$

$$\mu \geqslant \left(\frac{n}{d}\right)^{1-2\varepsilon}\frac{1}{x(x+y_0)} \tag{21}$$

and $\sum\limits_y$ designates summation over y with the following conditions

$$y\in Y,\ y<x,\ (y,x)=1,\ \sqrt{\frac{n}{d}}-x\leqslant y<\sqrt{n}-x,\ y\geqslant\left(\frac{n}{d}\right)^{(1-\varepsilon)/2}.$$

From the equation

$$\frac{n}{d} = xx'+yy'$$

with $(x, y) = 1$, we get

$$y' \equiv \frac{n}{d}\,y^{-1} \pmod{x},$$

where

$$yy^{-1} \equiv 1 \pmod{x}.$$

The uniqueness of y' in the congruences is guaranteed by the condition $y' \in F$ and by (20) we get

$$Q_{x,y_0} = \sum_{y} \sum_{0 \leqslant \left\{\frac{n}{d}\cdot\frac{y^{-1}}{x}\right\} < \mu} 1. \tag{22}$$

Evaluate the sum in (22). Construct upper $\bar{\psi}$ and lower $\underline{\psi}$ function of Vinogradov, putting in Lemma 1

$$\alpha = -\frac{\mu n^{-\varepsilon}}{2},\ \beta = \mu + \frac{\mu n^{-\varepsilon}}{2},\ \Delta = \mu n^{-\varepsilon},\ r = 1 \quad \text{for } \bar{\psi}$$

and

$$\alpha = 0,\ \beta = \mu,\ \Delta = \mu n^{-\varepsilon},\ r = 1 \quad \text{for } \underline{\psi}.$$

We obtain

$$\sum_{y} \underline{\psi}\left(\frac{ny^{-1}}{dx}\right) \leqslant Q_{x,y_0} \leqslant \sum_{y} \bar{\psi}\left(\frac{ny^{-1}}{dx}\right), \tag{23}$$

where, designating $\bar{\psi}$ or $\underline{\psi}$ by ψ,

$$\psi(\xi) = \beta - \alpha + \sum_{m=1}^{\infty} (g_m e^{2\pi i m\xi} + h_m e^{-2\pi i m\xi}).$$

We have for $m > m_0 = \left[\frac{n^{4\varepsilon}}{\mu}\right]$, according to Vinogradov's lemma,

$$\begin{aligned}\psi(\xi) &= \beta - \alpha + \sum_{m=1}^{m_0} (g_m e^{2\pi i m\xi} + h_m e^{-2\pi i m\xi}) + \sum_{m=m_0+1}^{\infty} (g_m e^{2\pi i m\xi} + h_m e^{-2\pi i m\xi}) \\ &= \beta - \alpha + \sum_{m=1}^{m_0} (g_m e^{2\pi i m\xi} + h_m e^{-2\pi i m\xi}) + O\left(\sum_{m > n^{4\varepsilon}/\mu} \frac{1}{m^2 \mu n^{-\varepsilon}}\right) \\ &= \mu + \sum_{m=1}^{m_0} (g_m e^{2\pi i m\xi} + h_m e^{-2\pi i m\xi}) + O\left(\frac{1}{n^{3\varepsilon}}\right)\end{aligned}$$

and by (23) we get

$$Q_{x,y_0} = \mu \sum_{y} 1 + \sum_{m=1}^{m_0} \left(g_m \sum_{y} e^{2\pi i \frac{mn}{dx} y^{-1}} + h_m \sum_{y} e^{-2\pi i \frac{mn}{dx} y^{-1}}\right) + O\left(\frac{1}{n^{3\varepsilon}} \sum_{y} 1\right). \tag{24}$$

Here we may use the inequalities $|g_m| < \mu \leqslant 1$, $|h_m| < \mu \leqslant 1$ and, according to Lemma 2,

$$\sum_{y} e^{\pm 2\pi i m \frac{ny^{-1}}{dx}} \ll \sqrt{x}\,\tau(x) \ln x \sqrt{\left(m\frac{n}{d}, x\right)} \ll \sqrt{x}\sqrt{m}\, n^{\varepsilon} \sqrt{\left(\frac{n}{d}, x\right)} < \sqrt{x}\sqrt{m}\, n^{3\varepsilon}$$

and by (24) we get

$$Q_{x,y_0} = \mu \sum_{y} 1 + O\left(\sqrt{x}\, n^{3\varepsilon} \sum_{m=1}^{m_0} \sqrt{m}\right) + O\left(\frac{1}{n^{3\varepsilon}} \sum_{y} 1\right). \tag{25}$$

But

$$\sum_{m=1}^{m_0} \sqrt{m} \ll m_0^{3/2} < \frac{n^{6\varepsilon}}{\mu^{3/2}} \ll \frac{n^{6\varepsilon}}{\left(\frac{n}{d}\right)^{3/2}} x^3 \left(\frac{n}{d}\right)^{3\varepsilon}$$

consequently

$$\sqrt{x}\, n^{3\varepsilon} \sum_{m=1}^{m_0} \sqrt{m} \ll x^{7/2} \left(\frac{n}{d}\right)^{-3/2} n^{3\varepsilon}. \tag{26}$$

By (19), (25) and (26) it follows, that

$$S_d''' = \sum_{x<\sqrt{n}} \sum_{Y} (q_{x,y_0} + \mu) \sum_{y} 1 + O\left(\sum_{x<\sqrt{n}} \sum_{Y} x^{7/2} \left(\frac{n}{d}\right)^{-3/2} n^{3\varepsilon}\right) + O(n^{1-\varepsilon}).$$

But

$$q_{x,y_0} + \mu = \frac{\operatorname{mes}(E+F)}{x} = \frac{n}{dx(x+y_0)}$$

and

$$\sum_{x<\sqrt{n}} \sum_{Y} \frac{n}{dx(x+y_0)} \sum_{y} 1 = \sum_{x<\sqrt{n}} \sum_{Y} \sum_{y} \frac{n}{dx(x+y)} + O(n^{1-\varepsilon}),$$

$$\sum_{\substack{x \geqslant y,\, (x,y)=1 \\ \sqrt{\frac{n}{d}} \leqslant x+y < \sqrt{n} \\ \left(\frac{n}{d}, x\right) < n^{2\varepsilon}}} \frac{n}{dx(x+y)} + O(n^{1-\varepsilon}) = \sum_{\substack{x \geqslant y,\, (x,y)=1 \\ \sqrt{\frac{n}{d}} \leqslant x+y < \sqrt{n}}} \frac{n}{dx(x+y)} + O(n^{1-\varepsilon}).$$

Moreover

$$\left(\frac{n}{d}\right)^{3/2} \sum_{x<\sqrt{n}} \sum_{Y} x^{7/2} \ll \left(\frac{n}{d}\right)^{3/2} n^{9/4} n^{2\varepsilon} \ln n < n^{3/4} d^{3/2} n^{3\varepsilon}.$$

Consequently, if $d < n^{2\varepsilon}$ (ε is sufficiently small), then

$$\left(\frac{n}{d}\right)^{3/2} \sum_{x<\sqrt{n}} \sum_{Y} x^{7/2} = O(n^{1-\varepsilon}).$$

If $d \geqslant n^{2\varepsilon}$, then

$$0 \leqslant S_d''' < \sum_{xx'=X} \sum_{yy'=Y} \sum_{\frac{n}{d}=X+Y} 1 < \left(\max_{m<n} \tau(m)\right)^2 \frac{n}{d} \ll n^{\varepsilon} \frac{n}{n^{2\varepsilon}} = n^{1-\varepsilon}$$

and by (15) for every $d \mid n$ we get

$$S_d'' = \sum_{\substack{x>y,\,(x,y)=1 \\ \sqrt{\frac{n}{d}} \leqslant x+y < \sqrt{n}}} \frac{n}{dx(x+y)} + O\left(\frac{n}{d}\right). \tag{27}$$

It remains to evaluate the sum S_d' from (13). We have, putting $x+y = z$,

$$\begin{aligned} S_d' + S_d'' &= \sum_{\substack{x>y,\,(x,y)=1 \\ x+y<\sqrt{n}}} \frac{n}{dx(x+y)} + O\left(\frac{n}{d}\right) \\ &= \frac{n}{d} \sum_{\sqrt{n} > \delta \geqslant 1} \frac{\mu(\delta)}{\delta^2} \sum_{\substack{x>y \\ x+y<\frac{1}{\delta}\sqrt{n}}} \frac{1}{x(x+y)} + O\left(\frac{n}{d}\right) \\ &= \frac{n}{d} \sum_{\delta<\sqrt{n}} \frac{\mu(\delta)}{\delta^2} \sum_{z<\frac{1}{\delta}\sqrt{n}} \frac{1}{z} \sum_{\frac{z}{2}<x<z} \frac{1}{x} + O\left(\frac{n}{d}\right) \\ &= \frac{n}{d} \sum_{\delta\leqslant\sqrt{n}} \frac{\mu(\delta)}{\delta^2} \ln 2 \ln\left(\frac{1}{\delta}\sqrt{n}\right) + O\left(\frac{n}{d}\right) \\ &= \frac{n}{2d} \sum_{\delta\leqslant\sqrt{n}} \frac{\mu(\delta)}{\delta^2} \ln 2 \ln n + O\left(\frac{n}{d}\right) \\ &= \frac{n \ln 2 \ln n}{2d\zeta(2)} + O\left(\frac{n}{d}\right). \end{aligned} \tag{28}$$

By (28), (27), (12) and (10) it follows, that

$$\Sigma_2 = \frac{n \ln 2 \ln n}{2\zeta(2)} \sum_{d\mid n} \frac{\mu(d)}{d} + O\left(\sum_{d\mid n} \frac{n}{d}\right).$$

But, using the identity

$$\sum_{d\mid n} \frac{\mu(d)}{d} = \frac{\varphi(n)}{n},$$

we obtain

$$\Sigma_2 = \frac{\ln 2 \ln n \varphi(n)}{2\zeta(2)} + O\left(\sum_{d\mid n} d\right). \tag{29}$$

We deduce the theorem from (1), (2), (7), (8), (9) and (29).

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