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Some problems of analytic number theory

by

K. Ramachandra (Bombay)

Dedicated with deepest regards to the memory of Prof. Paul Turán

1. Introduction and statements of results. A few years ago Professor A. E. Ingham explained to me the proof of theorems like

$$\pi(x+x^{\varphi})-\pi(x) \sim x^{\varphi}(\log x)^{-1} \quad \text{with} \quad \varphi=\tfrac{5}{8} \tag{1}$$

and

$$\frac{1}{X}\int\limits_{X}^{2X}\Big(\pi(x+h)-\pi(x)-\frac{h}{\log x}\Big)^2 dx = O_A\big(h^2(\log X)^{-A}+X^{2\varphi'}\big) \tag{2}$$

uniformly for $1 \leqslant h \leqslant X(\log X)^{-A}$. (Here $x \geqslant 2$, $X \geqslant 2$ and $\pi(x)$ denotes the number of prime numbers not exceeding x; $A > 0$ are arbitrary constants and $\varphi' = \frac{1}{4}$.) The pioneering work in this direction was the proof, due to G. Hoheisel (for references see [4]), of (1) with some positive constant φ less than 1. The two results mentioned above are due to A. E. Ingham and A. Selberg respectively. In fact the special case $F(s) = -\big(\zeta'(s)\big)\big(\zeta(s)\big)^{-1}$ of the theorem to be stated below, is due to them.

In a recent paper [6] Motohashi raises the following question. Let $N_0(x)$ denote the number of numbers which are either squares of integers or sums of two squares of two integers, not exceeding x. Then is it true that there exists a positive constant a less than 1 such that if $h = x^a$ then

$$N_0(x+h)-N_0(x) \sim Ch(\log x)^{-1/2}, \tag{3}$$

where $C = 2^{-1/2}\prod\limits_{p\equiv 1(4)}(1-p^2)^{-1/2}$. More recently this problem was solved independently by myself (who proved this with $a = 33/53+\varepsilon$) and by Huxley and Hooley (who proved this with $a = 7/12+\varepsilon$, note $7/12 < 33/53$). I also considered the analogue of Selberg's result for $N_0(x)$ and I proved it with $\varphi' = 9/29+\varepsilon$.

The result of Huxley and Hooley is stated without proof in Hooley's paper [1]. Their proof is not likely to be published. They have educated me on their method and allowed me make some comments about their proof. The main difference between my proof and theirs is that they use instead of my contour (used by me in an earlier draft of this paper) an ingeneous contour which I will call the Huxley–Hooley contour (or briefly the H–H contour) which I will explain in Section 4.

The main purpose of writing this note is to modify the H–H contour so as to prove for instance results like

$$\sum_{x\leqslant n\leqslant x+h}\mu(n) = O_{\varepsilon,A}\big(h(\log x)^{-A}+x^{7/12+\varepsilon}\big) \tag{4}$$

and

$$\frac{1}{X}\int_X^{2X}\Big|\sum_{x\leqslant n\leqslant x+h}\mu(n)\Big|^2 dx = O_{\varepsilon,A}\big(h(\log X)^{-A}+X^{1/6+\varepsilon}\big). \tag{5}$$

The results (4) and (5) seem to be new. The new contour enables also to prove the results of Ingham and Selberg without the aid of explicit formulae for $\pi(x)$ and related functions. Thus it not only solves all the questions raised above, but also provides an alternative approach which is perhaps simpler.

We will state our result in a somewhat general form. Let $L(s,\chi)$ be the Dirichlet L-series (for the principal character we take $\zeta(s)$ when the modulus of definition is 1) where χ is a character mod q ($q\geqslant 1$), and let s be a complex variable. Let $0\leqslant a\leqslant 1$, $T\geqslant 3$ and let $N_\chi(a,T)$ denote the number of zeros of $L(s,\chi)$ with real part $\geqslant a$ and imaginary part not exceeding T in absolute value. Let

$$N_\chi(a,T) = O\big(T^{B(1-a)}(\log T)^D\big), \tag{6}$$

where B and D are absolute numerical constants and the O-constant is independent of T and a but may depend on χ and q. The constant B is extremely important in analytic number theory. The smallest value of B known today is 12/5. This is the famous work of Montgomery and Huxley see [4] and [2]. Huxley proves his results for the zeta function. But it is not difficult to extend it for the L-series. It is clear that B has to be greater than or equal to 2. The assertion of (6) with $B=2$ (or $2+\varepsilon$ and $D=D_\varepsilon$) is a famous hypothesis called density hypothesis.

Consider the set S_1 of all L-series. Now we can define $\log L(s,\chi)$ in the half-plane $\operatorname{Re} s>1$ by the series

$$\sum_m\sum_p\frac{\chi(p^m)}{mp^{ms}}, \tag{7}$$

where the sum over m is over all positive integers and p runs over all

primes. More generally we can (by analytic continuation) define $\log L(s,\chi)$ in any simply connected domain containing $\operatorname{Re} s>1$ and not containing any zero or pole of $L(s,\chi)$. For any complex constant z we can define $\big(L(s,\chi)\big)^z$ as $\exp\big(z\log L(s,\chi)\big)$. Let S_2 consist of the set of all derivatives of $L(s,\chi)$ for all L-series and let S_3 denote the set of logarithms as defined above for all L-series.

Let $P_1(s)$ be any finite power product (with complex exponents) of functions of S_1. Let $P_2(s)$ be any finite power product (with non-negative integral exponents) of functions of S_2. Also let $P_3(s)$ denote any finite power product (with non-negative integral exponents) of functions of S_3. Let b_n ($n=1,2,\ldots$) be complex numbers which are $O_\varepsilon(n^\varepsilon)$ for every positive constant ε and suppose that $F_0(s)=\sum_{n=1}^\infty b_n n^{-s}$ is absolutely convergent in $\operatorname{Re} s>\frac12$. Finally put

$$F(s) = P_1(s)P_2(s)P_3(s)F_0(s) = \sum_{n=1}^\infty a_n n^{-s}, \tag{8}$$

and

$$N(x) = \sum_{n\leqslant x} a_n \quad\text{for}\quad x\geqslant 0.$$

Then we have the following

MAIN THEOREM. *Let x and X be sufficiently large and $1\leqslant h\leqslant x$. Consider a circle of positive radius (a constant depending only on $F(s)$) with 1 as centre which has no singularities of $F(s)$ (except possibly $s=1$) in its interior and on the boundary. From this circle remove the point of intersection with the real axis which lies to the left of 1. Let C_0 denote the contour got by traversing the remaining portion of the circle in the anti-clockwise direction. Let*

$$I(x,h) = \frac{1}{2\pi i}\int_0^h\Big(\int_{C_0}F(s)(v+x)^{s-1}ds\Big)dv. \tag{9}$$

Then we have with $\varphi = 1-\dfrac{1}{B}+\varepsilon$ and $\varphi' = 1-\dfrac{2}{B}+\varepsilon$

$$N(x+h)-N(x) = I(x,h)+O_\varepsilon\big(he^{-(\log x)^{1/6}}+x^{\varphi}\big) \tag{10}$$

and

$$\frac{1}{X}\int_X^{2X}|N(x+h)-N(x)-I(x,h)|^2dx = O_\varepsilon\big(h^2e^{-(\log X)^{1/6}}+X^{2\varphi'}\big). \tag{11}$$

Here B is the constant occurring in (6) and ε is an arbitrary small positive constant such that φ and φ' are less than $\frac12$.

Remark 1. The result (10), is due to Huxley and Hooley, in the special case $F(s) = \left(\zeta(s)L(s,\chi)\right)^{1/2}F_0(s)$. Their result is unpublished. For a suitable L-series and a suitable $F_0(s)$ this answers the question raised by Motohashi. Motohashi's question concerns $N_0(x)$, the number of integers, represented by the simplest binary quadratic form $X_1^2+X_2^2$, not exceeding x. This function $N_0(x)$ was first studied by E. Landau and later by S. Ramanujan. The function $F(s)$ connected with this problem has been generalized by Luthar [3] so as to include the integers represented by norm forms in any imaginary quadratic field. These results combined with the above theorem enable us to answer Motohashi's question for these more general binary quadratic forms.

Remark 2. It is not hard to get an asymptotic series expansion as $u\to\infty$ for the integral

$$\int_{C_0} F(s)u^{s-1}ds. \tag{12}$$

Because this reduces to the asymptotic evaluation with a good error term in

$$\int_{C_0} u^{s-1}(s-1)^z\left(\log(s-1)\right)^n ds \tag{13}$$

where z is a complex constant and n is a non-negative integer. This can be done by first treating the case $n=0$ and then we can differentiate the resulting formula (with respect to z) n times.

Remark 3. The result (10), say, with $F(s)$ a negative power of $\zeta(s)$ seems to be new, although the methods of the present paper are not very different from the method of Huxley and Hooley. The result (11) needs some further modifications in proof.

Remark 4. The quantities $e^{-(\log x)^{1/6}}$ and $e^{-(\log X)^{1/6}}$ appearing in (10) and (11) can be improved to

$$\exp\left(-g\left(\frac{\log x}{\log\log x}\right)^{1/3}\right) \quad \text{and} \quad \exp\left(-g\left(\frac{\log X}{\log\log X}\right)^{1/3}\right)$$

respectively, using the latest zero free regions for L-series. Here g denotes a positive constant.

Remark 5. The result $B = 12/5$ due to Montgomery and Huxley gives $\varphi = 7/12+\varepsilon$ and $\varphi' = 1/6+\varepsilon$. These seem to be the best known results in this direction.

We next make some remarks on the deep results used in the proof. We owe very much to the deep results and methods of I. M. Vinogradov (see the relevant chapters of [7], [8] and also the book of A. Walfisz, for reference see [5]). Due to his methods it was possible for H. E. Richert to obtain the estimate

$$\zeta(s) = O\left(T^{100(1-\sigma)^{3/2}}(\log T)^{2/3}\right) \tag{14}$$

uniformly in σ and T $(0\leqslant\sigma\leqslant 1,\ 2\leqslant|t|\leqslant T,\ T\geqslant 3)$. Similar results are also true of L-series. A new method of proving zero-free regions (for L-series) of the type (starting from (14)),

$$\sigma\geqslant 1-g_0(\log T)^{-2/3}(\log\log T)^{-1/3} \tag{15}$$

in $|t|\leqslant T$, $T\geqslant 3$ (g_0 is a positive constant), was discovered by Montgomery (see p. 87, cor. 11.4 of [5]). We have also to use the results of Hálasz, Turán and Montgomery (see p. 102, cor. 12.5 of [5]) in the form

$$N_\chi(\sigma, T) = O\left(T^{167(1-\sigma)^{3/2}}(\log T)^{17}\right) \tag{16}$$

uniformly with respect to σ and T in $0\leqslant\sigma\leqslant 1$ and $T\geqslant 3$.

Some of the more elementary but very important results used in our proof are the Borel-Carathéodory theorem (see Titchmarsh's book: *Theory of functions*, Oxford 1939, p. 174), and maximum modulus principle, Cauchy's theorem and so on.

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2. Notation.

If a_1, a_2, a_3, a_4 satisfy $a_1 < a_2$, $a_3 < a_4$ we denote by $R(a_1, a_2, a_3, a_4)$ the rectangle $a_1\leqslant \operatorname{Re}s\leqslant a_2$, $a_3\leqslant \operatorname{Im}s\leqslant a_4$. The positive constants a, b, θ and ε will be chosen in that order in the end. T will ultimately be a fixed positive power of x (resp. X) and x (resp. X) will be arbitrarily large.

3. Some preparations

LEMMA 1. *Let* $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. *Then* $a_n = O(n^\varepsilon)$ *for every positive* ε *(our O-constants hereafter depend on* ε *though we do not state this explicitly). Then for* $T\geqslant 2$, $x\geqslant 1$ *and* $c>1$ *we have*

$$N(x) = \sum_{n\leqslant x} a_n = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)\frac{x^s}{s}ds + O_{c,\varepsilon}\left(\frac{x^{c+\varepsilon}}{T}+x^\varepsilon\right). \tag{17}$$

Proof. The assertion $a_n = O(n^\varepsilon)$ is trivial. For the proof of the next assertion see Lemma 3.12, page 53 of Titchmarsh's book [8]. Here a slight change is necessary. We have to use (for positive y)

$$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{y^s}{s}ds - E(y)\right| \leqslant \min\left(\frac{y^c}{T|\log y|}, 2y^c\right)$$

where $E(y) = 1$, $1/2$ or 0 according as y exceeds 1, equal to 1 or is less than 1. Actually the result $O_c(y^c \log T)$ in place of $2y^c$ is sufficient for our purpose.

LEMMA 2. *We have with $c = 1+\varepsilon$ and $1 \leqslant h \leqslant x$*

$$N(x+h)-N(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)\,\frac{(x+h)^s - x^s}{s}\,ds + O\left(\frac{x^{1+2\varepsilon}}{T} + x^{2\varepsilon}\right). \tag{18}$$

Proof follows from Lemma 1.

4. The Huxley-Hooley contour and its modification. The Huxley-Hooley contour that will work for the problem of Motohashi is as follows. We take the rectangle $1/2 \leqslant \sigma \leqslant 1$, $|t| \leqslant T+2000(\log T)^2$ and divide it into equal rectangles of height $40(\log T)^2$ (the smaller rectangles at the ends we ignore) seeing that the real line cuts into two equal portions one of these rectangles R_0. Let R^n $(n = -n_1, \ldots, n_1)$ be these rectangles. In a typical rectangle R^n (with $|n| < n_1$) we fix a new right side and obtain a new rectangle $R^{n,0}$ as follows. Take R^{n-1}, R^n, R^{n+1} whenever all are defined and in the union of these rectangles, pick out a zero ϱ_n with the greatest real part β_n, of all the L-series involved in $F(s)$ (1). Consider only the right edges of these rectangles and join the ends of these edges by horizontal lines. These form the contour with the change that the contour shall not cross the real line but shall traverse from β_0 below the real axis and then makes a circular detour round the point 1 and comes back to β_0 from above the real axis.

We now make the following changes. Let a, b and θ be positive constants to be chosen later, satisfying for the present $\theta < 1$, a shall be small and b shall be close to 1 but less than 1. If $\beta_n < \theta$ then in place of β_n we shall take $\beta_n' = \beta_n + 3a(1-\beta_n)$. If $\beta_n > \theta$ then in place of β_n we shall take $\beta_n' = \beta_n + b(1-\beta_n)$. This contour will work from the proof of the main theorem. It is clear that if C_0' denotes the contour from β_0' below the real line making a circular detour round the point 1 and coming back to β_0' from above the real axis then

$$\int_{C_0'} F(s)\,\frac{(x+h)^s - x^s}{s}\,ds = \int_0^h du\left(\int_{C_0'} F(s)(x+u)^{s-1}\,ds\right)$$

is the same as C_0' replaced by C_0 with an error $O\big(h\exp(-(\log x)^{1/2})\big)$, provided T is chosen to be a positive constant power of x. This follows from Cauchy's theorem and the fact that $1-\beta_0 = O\big((\log\log T)^{-10}\big)$. Thus we get the main term $I(x, h)$.

(1) On R^n we shall fix the new right side $\sigma = \beta_n$ instead of $\sigma = 1$.

5. The main part of the proof. We will now join the points $c \pm iT$ to the modified contour by horizontal lines H_1, H_2 of shortest distance. We assume that $T \leqslant x$ and that T is bounded below by a positive constant power of x. We also assume that x is sufficiently large. We will denote the modified contour by M. It consists of C_0' and the portion M_1 which lies strictly above the real axis and the portion M_2 which lies strictly below the real axis. On H_1, H_2 we prove that

$$F(s) = O(T^\varepsilon).$$

We divide the contour M_1 into 3 parts $M_{1,1}, M_{1,2}$ and $M_{1,3}$ according as $\sigma \leqslant \theta$, $\theta < \sigma \leqslant \theta + b(1-\theta)$ or $\sigma > \theta + b(1-\theta)$. Similarly we divide M_2 into 3 parts $M_{2,1}$, $M_{2,2}$ and $M_{2,3}$. We prove that on $M_{1,1}, M_{1,2}, M_{2,1}$ and $M_{2,2}$,

$$F(s) = O(T^\varepsilon).$$

We also prove on $M_{1,3}$ and $M_{2,3}$,

$$F(s) = O\left(\exp\left((\log T)^{2(1-b)}\right)\right).$$

All these will be shown to follows from elementary considerations in Section 6.

Proof of the first part. We have from these remarks and Lemma 2, the following

LEMMA 3. *If T is bounded both above and below by positive constant powers of x, then*

$$N(x+h)-N(x) = I(x,h) + O\left(h\exp\left(-(\log x)^{1/2}\right)\right) + \frac{1}{2\pi i}\int_{M-C_0'} F(s)\,\frac{(x+h)^s - x^s}{s}\,ds + O\left(\frac{x^{1+2\varepsilon}}{T} + x^{2\varepsilon}\right). \tag{19}$$

Proof. On the contours H_1 and H_2, $F(s) = O(T^\varepsilon)$ and this completes the proof.

LEMMA 4. *We have if $1 \leqslant h \leqslant x$*

$$\int_{M-C_0'} F(s)\,\frac{(x+h)^s - x^s}{s}\,ds = \int_0^h G(x+u)\,du \tag{20}$$

where

$$G(v) = \int_{M-C_0'} F(s)\,v^{s-1}\,ds. \tag{21}$$

Proof. Trivial.

It now suffices to prove that $G(u)$ for u lying between x and $x+h$ is

$$O\left(h\exp\left(-(\log x)^{1/6}\right)\right)$$

provided T is a suitable power of x. For this purpose it suffices to treat the portion M_1, and M_2 requires only a similar treatment. We divide the smallest vertical strip containing M_1 into vertical strips of width $1/\log T$. Consider the bits of M_1, say $M_1(\sigma')$, in the vertical strip about the abscissa σ'. Then if $N_0(\sigma, T)$ denotes the sum $\sum N_\chi(\sigma, T)$ over all L-series we have

$$\int\limits_{M_1(\sigma')} |ds| = O\big(N_0(\sigma, T)(\log T)^{10}\big) \tag{22}$$

where σ' is $\sigma+3a(1-\sigma)$ or $\sigma+b(1-\sigma)$ according as $\sigma' \leqslant \theta$ or $\sigma' > \theta$. Using the estimates stated for $F(s)$ already we have

$$G(u) = O\Bigg(T^{\varepsilon}\bigg(\frac{T^{B(1-3a)^{-1}}}{x}\bigg)^{1-\theta} + T^{\varepsilon}\bigg(\frac{T^{167(1-\theta)^{1/2}(1-b)^{-3/2}}}{x}\bigg)^{(1-b)(1-\theta)} + \\ + \exp\big((\log T)^{3(1-b)}\big)\bigg(\frac{T^{167(1-\theta)^{1/2}(1-b)^{-3/2}}}{x}\bigg)^{a_0(\log T)^{-4/5}}\Bigg) \tag{23}$$

by using the fact that the zeta and L-functions have no zero in $\sigma \geqslant 1-2a_0(\log T)^{-4/5}$ where a_0 is a positive constant (this is a deep Theorem of I. M. Vinogodov, see p. 114 of Titchmarsh's book [8] and p. 295 of Prachar's book [7]) provided for some positive constant δ

$$T = x^{1/B-\delta} \quad \text{and} \quad T^{400(1-\theta)^{1/2}(1-b)^{-3/2}} \leqslant X. \tag{24}$$

We first choose a such that $B(1/B-\delta)(1-3a)^{-1} < 1$, $0 < a < 1/100$, b such that $3(1-b) < 1/100$ and then a θ such that the second inequality in (24) holds. We choose then a small ε and this proves the first part of the theorem.

Proof of the second part. We have

$$\frac{1}{X}\int\limits_X^{2X}\Big|\int\limits_0^h G(x+u)\,du\Big|^2 dx \leqslant \frac{1}{X}\int\limits_X^{2X} h\int\limits_0^h |G(x+u)|^2\,du\,dx \\ \leqslant \frac{h}{X}\int\limits_0^h du\int\limits_X^{3X}|G(x)|^2 dx = \frac{h^2}{X}\int\limits_X^{3X}|G(x)|^2 dx \tag{25}$$

and it suffices to prove that the expression in (25) is small. It suffices to consider the portion M_1 of the contour in $G(u)$ and procedding as before we break up M_1 and observe that it suffices to prove that

$$\frac{1}{X}\int\limits_X^{3X}\Big|\int\limits_{M_1(\sigma')} F(s)x^{s-1}ds\Big|^2 dx \tag{26}$$

is small. But (26) is less than

$$\int\limits_{s_1, s_2 \text{ on } M_1(\sigma')} F(s_1)\overline{F(s_2)}X^{s_1+\bar{s}_2-2}\,\frac{3^{s_1+\bar{s}_2-1}-1}{s_1+\bar{s}_2-1}\,ds_1 d\bar{s}_2. \tag{27}$$

This is less than a constant multiple of

$$X^{2(\sigma'-1)}\int\big(|F(s_1)|^2+|F(s_2)|^2\big)\bigg|\frac{3^{s_1+\bar{s}_2-1}-1}{s_1+\bar{s}_2-1}\bigg|\,|ds_1 d\bar{s}_2|, \tag{28}$$

the line (or rather bits of lines) of integration is the same as before. As before this is

$$O\Bigg(T^{\varepsilon}\bigg(\frac{T^{B(1-3a)^{-1}}}{X^2}\bigg)^{1-\theta} + T^{\varepsilon}\bigg(\frac{T^{167(1-\theta)^{1/2}(1-b)^{-3/2}}}{X^2}\bigg)^{(1-b)(1-\theta)} + \\ + \exp\big((\log T)^{6(1-b)}\big)\bigg(\frac{T^{167(1-\theta)^{1/2}(1-b)^{-3/2}}}{X^2}\bigg)^{a_0(\log T)^{-4/5}}\Bigg) \tag{29}$$

provided for some positive constant δ'

$$T = x^{2/B-\delta'} \quad \text{and} \quad T^{400(1-\theta)^{1/2}(1-b)^{-3/2}} \leqslant X^2. \tag{30}$$

We choose a such that $B(2/B-\delta')(1-3a)^{-1} < 2$, $0 < a < 1/100$, b such that $6(1-b) < 1/100$ and then θ such that the second inequality in (30) holds. We then choose a small ε and this proves the second part.

Note that we have used in our proof something like this: For every constant δ_1, $0 < \delta_1 < \frac{1}{2}$, $R\big(\frac{1}{2}-\delta_1, \infty, U, U+(\log U)^{1/2}\big)$ contains a zero of L-series for all large U. This follows from the method of my paper *On the frequency of Titchmarsh's phenomenon for* $\zeta(s)$, Journ. London Math. Soc. (2) 8 (1974), pp. 683–690.

6. Some elementary lemmas. The main lemma which we wish to prove in this section is

LEMMA 5. *Let a and b be constants and d a variable satisfying $0 < 2a \leqslant d \leqslant b < 1$. Denote by $L(s)$ an L-series which could be the zeta function. Consider a fixed U with $U = 0$, $\pm 40(\log T)^2$, $\pm 80(\log T)^2$, $\pm 120(\log T)^2, \ldots$, $|U| \leqslant T+2000(\log T)^2$. Suppose σ is such that the rectangle $R\big(\sigma, \infty, U-60(\log T)^2, U+60(\log T)^2\big)$ contains a zero on the left edge but is otherwise zero free. Then in the rectangle $R\big(\sigma+d(1-\sigma), \infty, U-20(\log T)^2, u+20(\log T)^2\big)$ with the disc $|s-1| \leqslant (\log\log T)^{-2}$ excluded in the case of $\zeta(s)$ and L-series with principal character, we have*

$$\log L(s) = O\big((\log T)^{(1-d)(1-2a)^{-1}}\log\log T\big)$$

uniformly in d.

Remark. Putting $d = 3a$ and $d = b$ we get what was required for our proof.

Proof of Lemma 5. The case $U = 0$ requires a separate discussion for $\zeta(s)$ and L-series with principal character. We prove it for $\zeta(s)$ and leave the L-series with principal character as an exercise. Obviously we have

$$1-\sigma = O\big((\log\log T)^{-1}\big).$$

If $|t| \leqslant (\log\log T)^2$ then in the rectangle

$$R\big(\sigma+a(1-\sigma),\ \infty,\ U-20(\log T)^2,\ U+20(\log T)^2\big)$$

we have

$$\mathrm{Re}\log\big(\zeta(s)(s-1)\big) \leqslant (\log\log T)^3$$

and so by Borel–Corathéodory theorem (ref. page 282 of Titchmarsh's book [8]) we have in $|t| \leqslant \frac{1}{2}(\log\log T)^2$ and $\mathrm{Re}\, s \geqslant \sigma+2a(1-\sigma)$ the estimation

$$\log\big(\zeta(s)(s-1)\big) = O\big((\log\log T)^4\big).$$

Now when we continue $\log(s-1)$ except in $|s-1| \leqslant (\log\log T)^{-2}$ but we do not go round the point 1 we have

$$\log(s-1) = O\big((\log\log T)^2\big)$$

and so

$$\log\zeta(s) = O\big((\log\log T)^4\big).$$

This proves the result if $|t| \leqslant \frac{1}{2}(\log\log T)^2$. If

$$\tfrac{1}{2}(\log\log T)^2 \leqslant |t| \leqslant 20(\log T)^2 \quad \text{and} \quad \mathrm{Re}\, s \geqslant \sigma+a(1-\sigma)$$

it suffices to consider $\mathrm{Re}\log\zeta(s) \leqslant (\log\log T)^2$ and so in $\mathrm{Re}\, s \geqslant \sigma+2a(1-\sigma)$ we have

$$\log\zeta(s) = O\big((\log\log T)^3\big).$$

This proves the result.

We now consider the remaining cases together. In these cases we need only the estimate $1-\sigma = O\big((\log T)^{-1}\big)$. We have in $\mathrm{Re}\, s \geqslant \sigma+ +a(1-\sigma)$ the estimate

$$\mathrm{Re}\log L(s) \leqslant E_1(1-\sigma)\log T + E_2\log\log T$$

where E_1 and E_2 are constants. This with Borel–Carathéodory theorem gives in $\mathrm{Re}\, s \geqslant \sigma+2a(1-\sigma)$, $|\mathrm{Im}\, s - U| \leqslant 30(\log T)^2$,

$$\log L(s) = O\left(\log T + \frac{\log\log T}{1-\sigma}\right) = O(\log T\log\log T).$$

In the next lemma we prove that $\log L(s) = O(\log\log T)$ on the line $\mathrm{Re}\, s = 1$. If s is a point with real part $\sigma+d(1-\sigma)$ and $|\mathrm{Im}\, s - U| \leqslant 20(\log T)^2$, then we apply maximum modulus principle to the function

$$\varphi(W) = \big(\log L(W)\big)e^{(W-s)^2}Z^{(W-s)} \quad (Z \text{ a positive real number})$$

to the rectangle $\sigma+2a(1-\sigma) \leqslant \mathrm{Re}\, W \leqslant 1$ and $|\mathrm{Im}\, s - U| \leqslant 30(\log T)^2$ and choose Z suitably. We obtain

$$\varphi(s) = \log L(s) = O\Big(\big((\log T\log\log T)^{1-d}(\log\log T)^{d-2a}\big)^{(1-2a)^{-1}}\Big)$$

and this proves the lemma.

Lemma 6. *If $2 \leqslant |t| \leqslant T$ and $T \geqslant 30$ then*

$$\log L(1+it) = O(\log\log T).$$

Proof. We have with $s_0 = 1+it$, and a positive number $V \geqslant 2$,

$$\sum_m\sum_p \frac{\chi(p^m)e^{-p^mV^{-1}}}{mp^{ms_0}} = \frac{1}{2\pi i}\int_{\mathrm{Re}\, W=2}\big(\log L(s_0+W)\big)\Gamma(W)V^W dW,$$

where the sums over m and p are clear. It is well-known that $L(s) \neq 0$ in $\mathrm{Re}\, s \geqslant 1-(\log T)^{-1}$ ($|t| \leqslant T$ and T sufficiently large) and there except in $|s| \leqslant 2$, $L'(s)/L(s) = O(\log T)$ and so $\log L(s) = O(\log T)$. We first assume that $V \leqslant (\log T)^{(\log T)^{20}}$ and break off the integral at $\mathrm{Im}\, W = \pm(\log T)^{25}$. We then shift the line of integration such that $\mathrm{Re}(s_0+W) = 1-(\log T)^{-2}$. Choosing V such that

$$V^{(\log T)^{-2}} = (\log T)^8$$

we see that

$$\log L(s_0) = O\left(\sum_{m\geqslant 1}\sum_p \frac{e^{-p^mV^{-1}}}{mp^m} + 1\right) = O(\log\log T).$$

This can be seen by breaking off the series at $p^m = V(\log V)^3$ and then using $\sum_{p\leqslant x} p^{-1} = O(\log\log x)$.

References

[1] C. Hooley, *On intervals between numbers that are sums of two squares III*, Journ. Reine Angew. Math. 267 (1974), pp. 207–218.

[2] M. N. Huxley, *On the difference between consecutive primes*, Invent. Math. 15 (1972), pp. 164–170.

[3] Indar S. Luthar, *A generalization of a theorem of Landau*, Acta. Arith. 12 (1966–67), pp. 223–228.

[4] H. L. Montgomery, *Zeros of L-functions*, Invent. Math. 8 (1969), pp. 346–354.

[5] H. L. Montgomery, *Topics in multiplicative number theory*, Lecture Notes in Mathematics, Vol. 227, Springer-Verlag, Berlin 1971.
[6] Yoichi Motohashi, *On the number of integers which are sums of two squares*, Acta. Arith. 23 (1973), pp. 401–412.
[7] K. Prachar, *Primzahlverteilung*, Springer-Verlag, Berlin 1957.
[8] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Clarenden Press, Oxford 1951.

SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
Colaba, Bombay 5, India

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Some non-linear diophantine approximations

by

R. C. Baker and J. Gajraj (London)

Introduction. Throughout the paper, k denotes a positive integer, ε an arbitrary positive number, and $C(k, \varepsilon)$ a positive number depending at most on k and ε, not necessarily the same at each occurrence, similarly for $C(k)$, $C(\varepsilon)$. $\|\alpha\|$ denotes the distance between α and the nearest integer. We write $K = 2^{k-1}$.

In 1948 Heilbronn proved the following deep and important theorem [11].

Theorem 1. *For any $N \geqslant 1$ and any real θ there is an integer x satisfying*

$$1 \leqslant x \leqslant N \quad \text{and} \quad \|\theta x^2\| < C(\varepsilon) N^{-1/2+\varepsilon}.$$

Heilbronn's result is analogous to Dirichlet's theorem (Lemma 3, below) in that the degree of approximation and the constant are independent of θ, N. We can rephrase it as

$$\min_{1\leqslant x\leqslant N} \|\theta x^2\| < C(\varepsilon) N^{-1/2+\varepsilon} \quad (N \geqslant 1,\ \theta \text{ real}).$$

The method of [11] has been applied by several authors. Thus Danicic [6] and Davenport [10] proved independently:

Theorem 2.

$$\min_{1\leqslant x\leqslant N} \|\theta x^k\| < C(k, \varepsilon)\, N^{-1/K+\varepsilon} \quad (N \geqslant 1,\ \theta \text{ real});$$

and Davenport [10] proved

Theorem 3. *For any polynomial f of degree k without constant term,*

$$\min_{1\leqslant x\leqslant N} \|f(x)\| < C(k, \varepsilon) N^{-1/(2K-1)+\varepsilon} \quad (N \geqslant 1).$$

Davenport's paper forms a very good introduction to Heilbronn's method and to this paper in particular.

Simultaneous diophantine approximations of this kind have also been studied. In [15], [16] Liu (improving a result of Danicic [6], [7]) proved the following