Nun ergibt sich auch endlich das in der Einleitung als Satz I angekündigte Hauptergebnis dieser Arbeit:

Satz 5. *Es sei* $|M| = \mathfrak{k}_\alpha$, *und es sei die Potenzmenge* $\mathfrak{P}(M)$ *von* M *auf* $\aleph_\alpha$ *viele Klassen verteilt:* $\mathfrak{P}(M) = \bigcup T_\nu |\ \nu < \omega_\alpha$. *Dann gibt es ein* T_ν, *das zu jedem* $\tau < \omega_{\alpha+1}$ *eine Teilmenge vom Typ* $\tau + \tau^*$ *umfaßt.*

Insbesondere hat sich also ergeben: Es gibt ein T_ν, *das "Universalmenge" ist für alle wohlgeordneten und für alle inverswohlgeordneten Mengen der Mächtigkeit* $\aleph_\alpha$.

Beweis. Wegen des Satzes II der Einleitung hat $\mathfrak{P}(M)$ eine Teilmenge, die zu C_α ähnlich ist, und aus Satz 4 folgt dann sofort Satz 5.

Literaturverzeichnis

[1] P. Erdös and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. 62 (1956), S. 427-489.

[2] E. Harzheim, *Dualzerlegungen in totalgeordneten Mengen*, Fund. Math. 53 (1963), S. 81-91.

[3] — *Beiträge zur Theorie der Ordnungstypen, insbesondere der* η_α*-Mengen*, Math. Ann. 154 (1964), S. 116-134.

[4] — *Einbettungssätze für totalgeordnete Mengen*, Math. Ann. 158 (1965), S. 90-108.

[5] — *Kombinatorische Betrachtungen über die Struktur der Potenzmenge*, Math. Nachr. 34 (1967), S. 123-141.

[6] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914.

[7] C. Kuratowski, *Sur la notion de l'ordre dans la théorie des ensembles*, Fund. Math. 2 (1921), S. 161-171.

[8] G. Kurepa, *Partitive sets and ordered chains*, Rad Jugoslav. Akad. Znan. Umjet. Odjel. Mat. Fiz. Techn. Nauke 6 (302), (1957), S. 197-235.

[9] W. Sierpiński, *Sur un problème concernant les sous-ensembles croissants du continu*, Fund. Math. 3 (1922), S. 109-112.

[10] — *Sur une propriété des ensembles ordonnés*, Fund. Math. 36 (1949), S. 56-67.

[11] R. P. Dilworth and A. M. Gleason, *A generalized Cantor theorem*, Proc. Amer. Math. Soc. 13 (1962), S. 704-705.

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Wallman spaces and compactifications

by

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It is possible to obtain a compactification for any T_1-space by employing a method introduced by Wallman [9]. Frink [3] has generalized this method to provide Hausdorff compactifications for Tychonoff spaces. His procedure uses a *normal base* of closed sets instead of the family of all closed sets as employed by Wallman. He shows that the Alexandroff and Stone–Čech compactifications can be obtained in this way.

In a recent paper, Njåstad [7] gives a condition for a Hausdorff compactification to be of the Wallman type as defined by Frink. This conditions is on the corresponding proximity. He shows that many compactifications satisfy this condition; among them those of Alexandroff, Stone–Čech, Freudenthal [2], Fan–Gottesman [1], and Gould [5].

In this paper we present a generalization of the Frink procedure which starts with a simple notion of a Wallman space. Necessary and sufficient conditions for a Wallman space to be a compactification are given. We will call these *Wallman compactifications*. Our compactifications are obtained by using *separating families* of closed sets as defined in [8]. This condition is considerably less restrictive than that of a normal base. In this way, compactifications for spaces other than Tychonoff spaces are obtained. Necessary and sufficient conditions for a given compactification to be a Wallman compactification are also given Hausdorff Wallman compactifications are then shown to be those of Frink.

In [7], Njåstad expresses the doubt that many common compactifications are Wallman compactifications, and in particular, the disk. We will show that the closed disk is a Wallman compactification of each of its dense subspaces. In fact, we show that any product of compact subsets of real numbers is a Wallman compactification of any of its dense subspaces. This is done by using the idea of a regular Wallman compactification.

By using different families of closed sets in a given topological space, many Wallman compactifications are obtained. We find necessary and sufficient conditions for two families to give the same compactification.

Thus, Wallman compactifications may be compared by looking at their defining families rather than at the compactifications themselves. This allows us to show that the Stone-Čech compactification of the real numbers is regular Wallman. We are also able to show that not all Wallman compactifications are regular.

1. Wallman spaces. Let X be a topological space and $\mathcal{F}$ be a distinguished family of non-empty closed subsets. Define $w(X, \mathcal{F})$ to be the set of all non-empty subfamilies of $\mathcal{F}$ which are maximal with respect to the finite intersection property (finite intersections are non-empty). Now if $F \in \mathcal{F}$, let $F^* = \{\mathcal{A} \in w(X, \mathcal{F}): F \in \mathcal{A}\}$. The sets F^* will constitute a subbase for a topology on $w(X, \mathcal{F})$. We will call $w(X, \mathcal{F})$ with this topology a *Wallman space*. By a *ring of sets* we will mean a family closed under finite unions and finite intersections.

LEMMA 1. *Every Wallman space $w(X, \mathcal{F})$ is compact. If $\mathcal{F}'$ is the ring generated by $\mathcal{F}$, then $w(X, \mathcal{F}')$ is homeomorphic to $w(X, \mathcal{F})$.*

Proof. Let $\mathcal{G} = \{F_i^*: i \in I\}$ have the finite intersection property. If $\mathcal{A} \in F_1^* \cap F_2^* \cap \ldots \cap F_n^*$, then $F_1, \ldots, F_n \in \mathcal{A}$. Thus $\{F_i: i \in I\}$ has the finite intersection property. There exists a maximal family $\mathcal{B} \in w(X, \mathcal{F})$ such that $\{F_i: i \in I\} \subset \mathcal{B}$. Thus $\mathcal{B} \in \bigcap \{F_i^*: i \in I\}$. By Alexander's theorem (see [6], p. 139), $w(X, \mathcal{F})$ is compact.

For the second part of the theorem, let $\mathcal{A} \in w(X, \mathcal{F})$ and define $f(\mathcal{A})$ to be the collection of all supersets in $\mathcal{F}'$ of elements in the ring generated by $\mathcal{A}$. It follows that $f(\mathcal{A}) \in w(X, \mathcal{F}')$. If $\mathcal{B} \in w(X, \mathcal{F}')$, then $\mathcal{B} \cap \mathcal{F} \in w(X, \mathcal{F})$ and $f(\mathcal{B} \cap \mathcal{F}) = \mathcal{B}$. Thus f is onto. It is easy to see that f is one-to-one. If $F \in \mathcal{F}$ and $G \in \mathcal{F}'$, define $F^* = \{\mathcal{A} \in w(X, \mathcal{F}): F \in \mathcal{A}\}$ and $G^{**} = \{\mathcal{B} \in w(X, \mathcal{F}'): G \in \mathcal{B}\}$. We now see that $f(F^*) = F^{**}$ and if $G^{**} = \bigcup \bigcap \{F_{ij}^{**}\}$, for a finite number of $F_{ij} \in \mathcal{F}$, then $f^{-1}(G^{**}) = \bigcup \bigcap \{F_{ij}^*\}$. Hence f is a homeomorphism.

We remark that if $\mathcal{F}$ is a ring, the elements of $w(X, \mathcal{F})$ are $\mathcal{F}$-ultrafilters and the sets F^* form a base for the topology.

In order to get a closer connection between $w(X, \mathcal{F})$ and X, for each $x \in X$ define $\varphi(x) = \{F \in \mathcal{F}: x \in F\}$.

A family $\mathcal{F}$ of closed subsets will be called *disjunctive in* $\mathcal{F}$ if $x \notin F \in \mathcal{F}$ implies that there exists a set $G \in \mathcal{F}$ such that $x \in G \subset X - F$.

LEMMA 2. *If $\mathcal{F}$ is a ring of closed subsets of X, then φ is a mapping of X into $w(X, \mathcal{F})$ if and only if $\mathcal{F}$ is disjunctive in $\mathcal{F}$. When φ is a mapping into $w(X, \mathcal{F})$, it is continuous and $\varphi[X]$ is dense in $w(X, \mathcal{F})$.*

Proof. In order that φ be a mapping into $w(X, \mathcal{F})$, it is necessary and sufficient that $\varphi(x)$ be maximal with respect to the finite intersection property for each $x \in X$. This can be seen to be equivalent to $\mathcal{F}$ being disjunctive in $\mathcal{F}$.

If φ is a mapping into $w(X, \mathcal{F})$, then $\varphi^{-1}[F^*] = F$, since $\varphi(x) \in F^*$ is and only if $F \in \varphi(x)$ if and only if $x \in F$. Thus whenever φ is a mapping, it is continuous. To show that $\varphi[X]$ is dense in $w(X, \mathcal{F})$, it suffices to show that when $\varphi[X] \subset \bigcup \{F_i^*: i \in I\}$, I finite, then $w(X, \mathcal{F}) = \bigcup \{F_i^*: i \in I\}$. If $\varphi[X] \subset \bigcup \{F_i^*\}$, then $x \in X$ implies that $\varphi(x) \in F_i^*$, or $F_i \in \varphi(x)$, for some $i \in I$. Thus $\varphi[X] \subset \bigcup \{F_i^*\}$ implies that $X = \bigcup \{F_i\}$. Suppose $\mathcal{A} \in w(X, \mathcal{F})$ and $\mathcal{A} \notin \bigcup \{F_i^*: i \in I\}$. Then there is an $A_i \in \mathcal{A}$ such that $F_i \cap A_i = \emptyset$ for each $i \in I$. This implies that $\bigcap \{A_i\} \subset \bigcap \{(X - F_i)\} = \emptyset$ which contradicts the finite intersection property of $\mathcal{A}$. Consequently, $\bigcup \{F_i^*: i \in I\} = w(X, \mathcal{F})$ and $\varphi[X]$ is dense in $w(X, \mathcal{F})$.

The mapping φ will be referred to as the *Wallman mapping*.

As in [8], we will call a family $\mathcal{F}$ of closed subsets of X *separating* if whenever S is a closed subset of X and $x \notin S$ there exist sets $F, G \in \mathcal{F}$ such that $x \in F$, $S \subset G$ and $F \cap G = \emptyset$. We can now prove

LEMMA 3. *If $\mathcal{F}$ is a ring of closed subsets of X, then the Wallman mapping φ is a homeomorphism of X onto $\varphi[X]$ if and only if X is a T_1-space and $\mathcal{F}$ is a separating family.*

Proof. The mapping φ is one-to-one if and only if X is T_1. This follows since $\varphi(x) \neq \varphi(y)$ if and only if there exist sets $F, G \in \mathcal{F}$ such that $x \in F$, $y \in G$ and $F \cap G = \emptyset$. Now suppose that φ is one-to-one. In order that φ^{-1} be continuous it is necessary and sufficient that $\varphi[S]$ be closed for all closed subsets S of X. Since $\{F^*: F \in \mathcal{F}\}$ is a base for closed sets in $w(X, \mathcal{F})$, $\varphi[S]$ is closed if and only if $\varphi[S] = \bigcap \{F_i^*: i \in I\}$, or if and only if $S = \bigcap \{\varphi^{-1}[F_i^*]: i \in I\} = \bigcap \{F_i \in \mathcal{F}: i \in I\}$. Thus φ^{-1} is continuous if and only if $\mathcal{F}$ is a base for closed sets in X. Thus it follows from Lemma 2 that φ is a homeomorphism if and only if $\mathcal{F}$ is a base for closed sets in X and is disjunctive in $\mathcal{F}$. These last two properties can be seen to be equivalent to separating.

From the previous lemmas, we obtain

THEOREM 1. *The pair $(w(X, \mathcal{F}), \varphi)$ is a compactification of X if X is a T_1-space and $\mathcal{F}$ is a separating family. If $(w(X, \mathcal{F}), \varphi)$ is a compactification of X, then X is T_1 and the ring generated from $\mathcal{F}$ is separating.*

The compactifications of Frink [3] are constructed for Tychonoff spaces, in the same manner, with a base for closed sets $\mathcal{F}$ having the properties:

F1. $\mathcal{F}$ is a *ring*;

F2. $\mathcal{F}$ is *disjunctive* (if A is any closed set not containing x, then there is an $F \in \mathcal{F}$ such that $x \in F \subset X - A$);

F3. $\mathcal{F}$ is a *normal family* (any two disjoint members A and B of $\mathcal{F}$ are subsets respectively of disjoint complements C' and D' of members C and D of $\mathcal{F}$, that is, $A \subset C'$, $B \subset D'$, and $C' \cap D' = \emptyset$).

2. Wallman compactifications. Let us say that $\hat{X}$ is a *Wallman compactification of* X if X is T_1, dense in $\hat{X}$, and has a separating family $\mathcal{F}$ such that the mapping φ can be extended to a homeomorphism of $\hat{X}$ onto $w(X,\mathcal{F})$. We will write $\hat{X} \simeq w(X,\mathcal{F})$.

If the Wallman mapping is extendable to $\hat{X}$, the extension is unique and has the following properties.

LEMMA 4. *If φ is an extension of the Wallman mapping which maps $\hat{X}$ homeomorphically onto $w(X,\mathcal{F})$, then*

(i) $\{x\} = \bigcap\{F^-\colon F \in \varphi(x)\}$,

(ii) $\varphi(x) = \{F \in \mathcal{F}\colon x \in F^-\}$,

(iii) $\varphi^{-1}[F^*] = F^-$ *all* $F \in \mathcal{F}$.

Proof. (i). In general, if $\mathcal{A} \in w(X,\mathcal{F})$, then $\mathcal{A} = \bigcap\{F^*\colon F \in \mathcal{A}\}$. Thus $\varphi(x) = \bigcap\{F^*\colon F \in \varphi(x)\}$ and so $\{x\} = \bigcap\{\varphi^{-1}[F^*]\colon F \in \varphi(x)\}$. But, $\varphi^{-1}[F^*]$ is closed and contains F, hence it contains F^-. And so $\{x\} \supset \bigcap\{F^-\colon F \in \varphi(x)\} \neq \emptyset$ since $\varphi(x)$ has the finite intersection property and $\hat{X}$ is compact. Thus $\{x\} = \{F^-\colon F \in \varphi(x)\}$.

(ii). From (i), $\varphi(x) \subset \{F \in \mathcal{F}\colon x \in F^-\}$. Now suppose $F \in \mathcal{F}$, $x \in F^-$ and $F \notin \varphi(x)$. Since F^* is closed and $\varphi(x) \notin F^*$, there is an open set U such that $\varphi(x) \in U$ and $U \cap F^* = \emptyset$. Thus $\varphi^{-1}[U \cap F^*] = \varphi^{-1}[U] \cap \varphi^{-1}[F^*] = \emptyset$. But $x \in \varphi^{-1}[U]$ which is open and $F \subset \varphi^{-1}[F^*]$. This implies $x \notin F^-$ which is a contradiction. Hence, $\varphi(x) \supset \{F \in \mathcal{F}\colon x \in F^-\}$.

(ii). From (ii) we have that $F^* = \{\varphi(y)\colon F \in \varphi(y)\} = \{\varphi(y)\colon y \in F^-\} = \varphi[F^-]$.

Suppose that $\hat{X}$ is a compactification of X and $\hat{\mathcal{F}}$ is a family of closed sets in $\hat{X}$. Then $\hat{\mathcal{F}}$ has the *trace property with respect to* X if $F = \bigcap\{F_i \in \hat{\mathcal{F}}\colon i = 1, \ldots, n\} \neq \emptyset$ implies that $F \cap X \neq \emptyset$.

We can now give necessary and sufficient conditions for a compactification to be Wallman.

THEOREM 2. *If $\hat{X}$ is a Wallman compactification of X, then $\hat{X}$ possesses a separating ring with the trace property w.r.t. X. In fact, if $\hat{X} \simeq w(X,\mathcal{F})$, then $\hat{\mathcal{F}} = \{F^-\colon F \in \mathcal{F}\}$ is such a ring. Conversely, if a compact T_1-space $\hat{X}$ possesses a separating family $\hat{\mathcal{F}}$ with the trace property w.r.t. a dense subspace X, then $\hat{X}$ is a Wallman compactification of X. In fact, $\hat{X} \simeq w(X, \hat{\mathcal{F}} \cap X)$.*

Proof. Let $\hat{X} \simeq w(X,\mathcal{F})$. By Lemma 1, we may suppose that $\mathcal{F}$ is a ring. Consider $\hat{\mathcal{F}} = \{F^-\colon F \in \mathcal{F}\}$. We will show that $\hat{\mathcal{F}}$ is a separating ring with the trace property w.r.t. X. Let $F_1, F_2 \in \mathcal{F}$ and $y \in F_1^- \cap F_2^-$. By Lemma 4, F_1 and $F_2 \in \varphi(y)$ and so $F_1 \cap F_2 \in \varphi(y)$. This implies that $y \in (F_1 \cap F_2)^-$ and hence $F_1^- \cap F_2^- = (F_1 \cap F_2)^-$. Therefore $\hat{\mathcal{F}}$ is a ring. To show the trace property, suppose $F_1, F_2 \in \mathcal{F}$ and $F_1^- \cap F_2^- \neq \emptyset$. If $y \in F_1^- \cap F_2^-$, then as above $F_1 \cap F_2 \in \varphi(y)$. Thus $(F_1^- \cap X) \cap (F_2^- \cap X) = F_1 \cap F_2 \neq \emptyset$ and $\hat{\mathcal{F}}$ has the trace property w.r.t. X.

In order to show $\hat{\mathcal{F}}$ is separating we will first show it is a base for closed sets. Let H be a closed set in $\hat{X}$ and $y \notin H$. Then $\varphi[H]$ is closed and $\varphi(y) \notin \varphi[H]$. Since the family $\{F^*\colon F \in \mathcal{F}\}$ is a base, there is an $F \in \mathcal{F}$ such that $\varphi(y) \notin F^*$ and $\varphi[H] \subset F^*$. So $H \subset \varphi^{-1}[F^*] = F^-$ and $y \notin F^-$. Hence $\hat{\mathcal{F}}$ is a base for closed sets. Since $\hat{X}$ is T_1, $\{y\}$ is closed and $X - \{y\}$ is open and contains F^-. Thus there are basic open sets covering each point of F^- which do not contain y. Since $\hat{X}$ is compact, a finite number of these will also cover F^-. The complement of this union is a finite intersection of sets from $\hat{\mathcal{F}}$ which contains y and is disjoint from F^-. Therefore $\hat{\mathcal{F}}$ is a disjunctive base for closed sets and hence separating.

To prove the converse, let $\hat{X}$ be a compact T_1-space and $\hat{\mathcal{F}}$ be a separating family of closed sets with the trace property w.r.t. a dense subspace X. We will show that $\hat{X} \simeq w(X, \hat{\mathcal{F}} \cap X)$, where $\hat{\mathcal{F}} \cap X = \{F \cap X\colon F \in \mathcal{F}\}$. Since $\hat{\mathcal{F}} \cap X$ is separating, $w(X, \hat{\mathcal{F}} \cap X)$ is a compactification of X. Extend φ to $\hat{X} - X$ as follows: if $x \in \hat{X} - X$, let $\varphi(x) = \{F \cap X\colon x \in F \in \hat{\mathcal{F}}\}$. We must show that $\varphi(x)$ is maximal with respect to the finite intersection property. If $F_i \cap X \in \varphi(x)$, $i = 1, 2, \ldots, n$, then $x \in \bigcap\{F_i\colon i = 1, 2, \ldots, n\}$ and thus $\bigcap\{F_i\colon i = 1, \ldots, n\} \cap X \neq \emptyset$ by the trace property. Hence $\varphi(x)$ has the finite intersection property. If $H \in \hat{\mathcal{F}}$ and $x \notin H$, there exists $F \in \hat{\mathcal{F}}$ such that $x \in F$, $F \cap H = \emptyset$. Thus $(F \cap X) \cap (H \cap X) \neq \emptyset$ and so $\varphi(x)$ is maximal.

Since $\hat{X}$ is T_1, and $\hat{\mathcal{F}}$ is separating, φ is one-to-one. To show that φ is onto, let $\mathcal{A} \in w(X, \hat{\mathcal{F}} \cap X)$. Consider $\{F \in \hat{\mathcal{F}}\colon F \cap X \in \mathcal{A}\}$. This family has the finite intersection property because $\mathcal{A}$ does. Since $\hat{X}$ is compact, there exists $x \in \bigcap\{F \in \mathcal{F}\colon F \cap X \in \mathcal{A}\}$. Therefore $\mathcal{A} \subset \varphi(x)$. Since $\mathcal{A}$ is maximal, $\mathcal{A} = \varphi(x)$. That φ is a homeomorphism follows from $\varphi[F] = (F \cap X)^*$ for all $F \in \hat{\mathcal{F}}$.

It follows from Lemma 4 that all Wallman compactifications are T_1. A simple example shows that not all T_1-compactifications are Wallman. Consider the positive integers with the co-finite topology $\hat{X}$ as the compactification of the even integers X. There is only one closed set in X whose closure in $\hat{X}$ contains odd integers. Hence $\hat{X}$ cannot be a Wallman compactification of X.

3. Hausdorff Wallman compactifications. If $\hat{X}$ is a Hausdorff compactification of X, it is sufficient to find a separating family of closed sets with the trace property w.r.t. X in order to show that it is a Wallman compactification. The next theorem shows that Hausdorff Wallman compactifications and those of Frink coincide.

THEOREM 3. *$\hat{X}$ is a Hausdorff Wallman compactification of a dense subspace X if and only if $\hat{X}$ possesses a normal separating ring $\hat{\mathcal{F}}$*

of closed sets with the trace property w.r.t. X. If $\hat{\mathcal{F}}$ has these properties, so does $\hat{\mathcal{F}} \cap X$.

Proof. The sufficiency is apparent. If $\hat{X}$ is a Wallman compactification, then it possesses a separating ring $\hat{\mathcal{F}}$ of closed sets with the trace property w.r.t. X. It remains to show that $\hat{\mathcal{F}}$ is a normal family.

Recall that a Hausdorff compact space is normal. If $F_1, F_2 \in \hat{\mathcal{F}}$ and $F_1 \cap F_2 = \emptyset$, then there are disjoint open sets $O_1, O_2 \subset \hat{X}$ such that $F_1 \subset O_1$ and $F_2 \subset O_2$. Now consider the disjoint closed sets F_1 and $\hat{X} - O_1$. Let $x \in F_1$. Then there are disjoint sets $F_x, G_x \in \hat{\mathcal{F}}$ such that $x \in F_x$ and $\hat{X} - O_1 \subset G_x$ since $\hat{\mathcal{F}}$ is separating. Therefore $x \in \hat{X} - G_x \subset O_1$. Repeating this for each $x \in F_1$, $F_1 \subset \bigcup \{\hat{X} - G_x \colon x \in F_1\} \subset O_1$. Since X is compact, this covering of F_1 may be replaced by a finite subcovering. Thus $F_1 \subset K_1 \subset O_1$, where $K_1 = \bigcup \{\hat{X} - G_{x_i} \colon i = 1, 2, \ldots, n\}$ is an open set and $\hat{X} - K_1 \in \hat{\mathcal{F}}$. Similarly, there is an open set K_2 such that $F_2 \subset K_2 \subset O_2$ and $\hat{X} - K_2 \in \hat{\mathcal{F}}$. Therefore $\hat{\mathcal{F}}$ is a normal family. It is also clear that $\hat{\mathcal{F}} \cap X$ is a separating ring of closed sets in X. Since $\hat{\mathcal{F}}$ is a normal family in $\hat{X}$ and has the trace property w.r.t. X, it follows that $\hat{\mathcal{F}} \cap X$ is a normal family in X.

4. Regular Wallman compactifications. In order to find families which have the trace property we define a family of *regular closed* sets to be one in which each set is the closure of its interior.

THEOREM 4. *If $\hat{X}$ possesses a separating ring of regular closed sets $\hat{\mathcal{F}}$, then $\hat{X}$ is a Wallman compactification of each of its dense subspaces.*

Proof. In view of Theorem 2, it suffices to show that $\hat{\mathcal{F}}$ has the trace property w.r.t. any dense subspace. Let X be a dense subspace of $\hat{X}$. Suppose $F_1, F_2 \in \hat{\mathcal{F}}$ and $F_1 \cap F_2 \neq \emptyset$. Since $\hat{\mathcal{F}}$ is a ring of regular closed sets, $F_1 \cap F_2 = [\operatorname{int}(F_1 \cap F_2)]^{-}$. Since X is dense in $\hat{X}$, $(F_1 \cap X) \cap (F_2 \cap X) = (F_1 \cap F_2) \cap X = [\operatorname{int}(F_1 \cap F_2)]^{-} \cap X \neq \emptyset$. This shows that $\hat{\mathcal{F}}$ has the trace property w.r.t. X.

If $\hat{X}$ has a separating ring of regular closed sets, we will say that $\hat{X}$ is a *regular Wallman compactification* of each of its dense subspaces.

Many common compactifications are regular Wallman.

THEOREM 5. *Every compact subset of real numbers is a regular Wallman compactification of each of its dense subspaces.*

Proof. Let $\hat{X}$ be a compact (hence closed and bounded) subset of real numbers. Let $A = \{x \in \hat{X} \colon x_n \to x,\ x_n \neq x,\ x_n \in \hat{X}$ implies $x_n < x$ for all n sufficiently large or $x_n > x$ for all n sufficiently large$\}$. Roughly speaking, the set A consists of those points of $\hat{X}$ which cannot be approached from both sides. Let $Y = [r_0, s_0]$, where r_0 is rational, s_0 is irrational, and $r_0 < x < s_0$ for all $x \in \hat{X}$. Let $R = \{$rationals in $Y - A\}$ and $S = \{$irrationals in $Y - A\}$. Then R and S are dense in Y. Indeed, assume $(p, q) \cap R = \emptyset$ for some open interval (p, q) in Y. Then $x \in (p, q)$ implies $x \in A \subset \hat{X}$. Let x_0 be a rational in (p, q). Then $\{x_0 - (x_0 - p)/(n+1)\} \to x_0$ from below and $\{x_0 + (q - x_0)/(n+1)\} \to x_0$ from above contradicting the fact that $x_0 \in A$. So, R is dense in Y. A similar proof shows S is also dense in Y.

Let $\mathcal{G} = \{$finite unions of closed intervals $[r, s]$, $r \in R$, $s \in S\}$. Then $\mathcal{G}$ is a ring of regular closed sets. Now let $\hat{\mathcal{F}} = \mathcal{G} \cap \hat{X}$. We will show that $\hat{\mathcal{F}}$ is a separating ring of regular closed sets in $\hat{X}$. Certainly it is a ring. To show that each member is a regular closed set, let $F \in \hat{\mathcal{F}}$. Then $F = G \cap \hat{X}$ for some $G \in \mathcal{G}$. Suppose $x \in F - (\operatorname{int} G \cap \hat{X})$. Then x must be the endpoint of some interval $[r, s]$ in G (G is a finite union of such intervals) and so $x \in \hat{X}$, $x \notin A$. Thus there is a sequence $x_n \neq x$ of points in $\hat{X}$ converging to x from below and a sequence $y_n \neq x$ in $\hat{X}$ converging to x from above. One of the sequences is eventually in (r, s), so $x \in [\operatorname{int} G \cap \hat{X}]^{-} \subset [\operatorname{int} F]^{-}$. Thus F is a regular closed set.

To show that $\hat{\mathcal{F}}$ is separating, let H be a closed subset of $\hat{X}$, $y \in \hat{X}$, and $y \notin H$. Since R and S are dense in Y, there exist points $r', r'' \in R$ and $s', s'' \in S$ such that $x \in H$, $x < y$ imply $x < s'' < r' < y$ and $x \in H$, $y < x$ imply $y < s' < r'' < x$. Thus, $[r_0, s''] \cup [r'', s_0]$ contains H, $y \in [r', s']$ and $[r', s'] \cap ([r_0, s''] \cup [r'', s_0]) = \emptyset$. Therefore if $F_1 = [r', s'] \cap \hat{X}$ and $F_2 = ([r_0, s''] \cup [r'', s_0]) \cap \hat{X}$, then $H \subset F_2$, $y \in F_1$ and $F_1 \cap F_2 = \emptyset$. Therefore $\hat{\mathcal{F}}$ is separating. It now follows from Theorem 4 that $\hat{X}$ is a regular Wallman compactification of each of its dense subspaces.

In the special case of a closed interval $I = [0, \pi]$, the family $\mathcal{F}$ of all finite unions of closed subintervals of the form $[r, s]$, r rational, s irrational, is a separating ring of regular closed sets.

The next theorem provides more regular Wallman compactifications.

THEOREM 6. *The topological product of spaces possessing separating rings of regular closed sets has a separating of regular closed sets.*

Proof. Let X_α, $\alpha \in A$ be a topological space with a separating ring $\mathcal{F}_\alpha$ of regular closed sets. Consider the family $\mathcal{F}$ of finite unions of sets of the form $\{x \colon x_\alpha \in F_\alpha \in \mathcal{F}_\alpha,\ \alpha \in A'\}$ where A' is some finite subset of A. Then $\mathcal{F}$ is a separating ring of regular closed sets in $\times \{X_\alpha \colon \alpha \in A\}$.

COROLLARY. *Every product of compact subsets of real numbers is a regular Wallman compactification of each of its dense subspaces.*

Thus we can see that all cubes are regular Wallman compactifications. In particular, the closed disk is a regular Wallman compactification of the open disk.

5. Equivalent Wallman compactifications. In deciding whether a given Wallman compactification is regular it is helpful to know when two families of closed sets provide the same compactification. To do this, we make the following definition. Let $\mathcal{F}$ and $\mathcal{G}$ be two rings of closed sets in X. Then $\mathcal{F}$ *separates* $\mathcal{G}$ if $G_1, G_2 \in \mathcal{G}$, $G_1 \cap G_2 = \emptyset$ implies that there are sets $F_1, F_1 \in \mathcal{F}$ such that $G_1 \subset F_1$, $G_2 \subset F_2$ and $F_1 \cap F_2 = \emptyset$.

THEOREM 7. *If $\mathcal{F}$ and $\mathcal{G}$ are two separating rings of closed sets in X, then $w(X, \mathcal{F})$ and $w(X, \mathcal{F})$ are equivalent compactifications of X if and only if $\mathcal{G}$ separates $\mathcal{F}$ and $\mathcal{F}$ separates $\mathcal{G}$.*

Proof. Let $\hat{X} = w(X, \mathcal{F})$ and identify X with its image under the Wallman mapping. By Theorem 2, the family $\hat{\mathcal{F}} = \{F^-: F \in \mathcal{F}\}$ is a separating ring with the trace property w.r.t. X. Consider $\hat{\mathcal{G}} = \{G^-: G \in \mathcal{G}\}$, where closures are taken in $\hat{X}$. We will show that $\hat{\mathcal{G}}$ is separating family with the trace property w.r.t. X. If S is closed in $\hat{X}$ and $x \notin S$, there are sets $F_1, F_2 \in \mathcal{F}$ such that $x \in F_1^-$, $S \subset F_2^-$ and $F_1^- \cap F_2^- = \emptyset$. Thus $F_1 \cap F_2 \neq \emptyset$. The family $\mathcal{G}$ separates $\mathcal{F}$, hence there are sets $G_1, G_2 \in \mathcal{G}$ such that $F_1 \subset G_1$, $F_2 \subset G_2$ and $G_1 \cap G_2 = \emptyset$. Since $\mathcal{F}$ separates $\mathcal{G}$, there exist sets $F_3, F_4 \in \mathcal{F}$ such that $G_1 \subset F_3$, $G_2 \subset F_4$ and $F_3 \cap F_4 = \emptyset$. Thus $F_3^- \cap F_4^- = \emptyset$. This implies that $G_1^- \cap G_2^- = \emptyset$, $x \in G_1^-$, $S \subset G_2^-$. Thus $\hat{\mathcal{G}}$ is separating. To show the trace property, let $G_1 \cap G_2 = \emptyset$. Then there are sets $F_1, F_2 \in \mathcal{F}$ such that $G_1 \subset F_1$, $G_2 \subset F_2$ and $F_1 \cap F_2 = \emptyset$. Since $\hat{\mathcal{F}}$ has the trace property, $F_1^- \cap F_2^- = \emptyset$. This implies that $G_1^- \cap G_2^- = \emptyset$ and $\hat{\mathcal{G}}$ has the trace property w.r.t. X. Then by Theorem 2 $\hat{X} \simeq w(X, \hat{\mathcal{G}} \cap X) = w(X, \mathcal{G})$.

To prove the converse, again let $\hat{X} = w(X, \mathcal{F})$ and suppose $\hat{X} \simeq w(X, \mathcal{G})$. Then by Theorem 2, $\hat{\mathcal{G}} = \{G^-: G \in \mathcal{G}\}$ is a separating ring of closed sets. It follows from the compactness of $\hat{X}$ that $\hat{\mathcal{G}}$ separates the family of all closed sets; hence $\mathcal{G} = \hat{\mathcal{G}} \cap X$ separates $\mathcal{F}$. Similarly $\hat{\mathcal{F}} = \{F^-: F \in \mathcal{F}\}$ is a separating family and $\mathcal{F} = \hat{\mathcal{F}} \cap X$ separates $\mathcal{G}$.

Using this theorem, we can give a simple example of a Wallman compactification which is not regular. Let X be the positive integers with the co-finite topology and $\mathcal{F}$ be the family of all closed sets. Then $w(X, \mathcal{F})$ is a compactification since X is T_1 and $\mathcal{F}$ is separating. But, since the closure of every non-empty open set in X is X, there is no family of regular closed sets which could separate $\mathcal{F}$. Thus $w(X, \mathcal{F})$ is not a regular Wallman compactification.

As another illustration of the use of Theorem 7 we prove

THEOREM 8. *The Stone-Čech compactification $\beta(R)$ of the real numbers is a regular Wallman compactification.*

Proof. We know that $\beta(R) \simeq w(R, \mathcal{F})$ where $\mathcal{F}$ is the family of all closed subsets of R (see Gillman, Jerison [4]). It is clear that $\mathcal{F}$ separates any family of closed sets. Thus by Theorem 7, it suffices to construct a separating ring of regular closed sets which also separates $\mathcal{F}$. Let $\mathcal{G}$ be the family of all sets which can be expressed as the union of closed intervals of the form $[r, s]$ where r is rational and s is irrational, such that there are only a finite number in any bounded set. It is not difficult to show that $\mathcal{G}$ is a ring of closed sets. Also, if $G = \bigcup\{[r_i, s_i]: i \in I\}$, then $G =$ closure $\left(\bigcup\{(r_i, s_i): i \in I\}\right)$. Hence the members of $\mathcal{G}$ are regular closed sets. Now let F_1 and F_2 be any two disjoint closed sets. Consider $F_1 \cap [p, q]$ and $F_2 \cap [p, q]$ where $[p, q]$ is an arbitrary closed interval. These are closed, bounded and disjoint sets. Since R is normal, there are open sets O_1 and O_2 such that $F_1 \cap [p, q] \subset O_1$ and $F_2 \cap [p, q] \subset O_2$. These open sets can be expressed as countable unions of disjoint open intervals. Since $F_1 \cap [p, q]$ is compact, a finite number of the intervals in O_1 will cover it. Let one of these intervals $I = (a, b)$ cover a part P of $F_1 \cap [p, q]$. Note that a or b may lie outside of $[p, q]$. Since $F_1 \cap [p, q]$ is closed and the intervals are disjoint. P is bounded away from both a and b. Thus a closed subinterval $[r', s']$ may be chosen to cover P, where r' is rational and s' is irrational. In this way, a finite number of the closed intervals cover $F_1 \cap [p, q]$. Similarly $F_2 \cap [p, q]$ can be covered with closed intervals and they will be disjoint from those chosen to cover $F_1 \cap [p, q]$. This process can be repeated along R by using overlapping closed intervals and two disjoint members of $\mathcal{G}$ will be obtained which cover F_1 and F_2 respectively. This shows that $\mathcal{G}$ separates $\mathcal{F}$ and hence also that $\mathcal{G}$ is separating.

6. Conclusion.

The compactifications shown by Njåstad [7] to be Wallman by way of the associated proximities may also be shown to be Wallman directly by finding separating families with the trace property.

As an example consider the Fan–Gottesman compactification $\hat{X}$ of a regular T_1-space X as described by Njåstad. It is defined by a distinguished base $\mathcal{A}$ of open sets in X. It can be shown that the family of finite intersections of closures (in $\hat{X}$) of the members of $\mathcal{A}$ is a separating family with the trace property w.r.t. X.

Although we have shown that not all T_1-compactifications are Wallman and that not all Wallman compactifications are regular, we have not settled these questions in the Hausdorff case.

We have shown that all cubes are Wallman compactifications, but we have not shown this for all their closed subspaces; in fact, not even for all closed subsets of the unit disk. We proved that $\beta(R)$ is a regular Wallman compactification but have not shown this for all Stone-Čech compactifications.

References

[1] Ky Fan and N. Gottesman, *On compactifications of Freudenthal and Wallman*, Indag. Math. 14 (1952), pp. 504-510.

[2] H. Freudenthal, *Kompaktisierungen und Bikompaktisierungen*, Indag. Math. 13 (1951), pp. 184-192.

[3] O. Frink, *Compactifications and semi-normal spaces*, Amer. J. Math. 86 (1964), pp. 602-607.

[4] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand (1960), Princeton.

[5] G. G. Gould, *A Stone–Čech–Alexandroff-compactification and its application to measure theory*, Proc. Lond. Math. Soc. 14 (1964), pp. 221-244.

[6] J. L. Kelley, *General Topology*, Van Nostrand (1955), New York.

[7] O. Njåstad, *On Wallman-type compactifications*, Math. Zeitschr. 91 (1966), pp. 267-276.

[8] E. F. Steiner, *Normal families and completely regular spaces*, Duke Math. J. 33 (1966), pp. 743-745.

[9] H. Wallman, *Lattices and topological spaces*, Annals of Math. 39 (1938), pp. 112-126.

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A unique factorization theorem for countable products of circles

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To say that X is a *factor space* of Y means that Y is homeomorphic with $X \times Z$ for some space Z. One can often say something about X if something is known about Y. For example, it is not hard to show that if Y is a compact, connected, locally connected metric space, then so is X. As a further example, it can be shown that every one-dimensional factor space of the Hilbert cube is a tree (non-degenerate locally connected metric continuum containing no simple closed curve). R.D. Anderson has proved that in fact every tree is a factor space of the Hilbert cube [1]. In this note we consider the analogous problem of determining the one-dimensional factor spaces of countable products of circles. It is found that the circle is the only one. The author wishes to thank R. D. Anderson and A. Lelek for their suggestions.

The notion of an inessential space [2] will prove useful. An *inessential space* is a space X such that there is a homotopy $H\colon [0,1]\times X \to X$ with the property that $H(1,x) = x$ for each $x \in X$ and $H(0,X) \neq X$; in words, X can be continuously deformed to a proper subset of itself with a homotopy starting at the identity. A space is *essential* if no such homotopy exists. It follows from Lemma 1 of [3] that a countable product of circles is essential.

LEMMA 1. *Let X be a tree and let p be an endpoint of X. Then there is a homotopy $H\colon [0,1]\times X\to X$ such that $H(1,x) = x$ for $x \in X$, $H(0,X) = p$ and $H(t,p) = p$ for each $t \in [0,1]$. Consequently X is inessential.*

Proof. This is a corollary of Theorem A of [4], which states that X can be made into a semilattice with identity and zero p.

LEMMA 2. *Every factor space of an essential space is essential.*

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* The results in this paper are contained in the author's dissertation written under the direction of Professor R. J. Koch.