Accordingly, $g^+ \geqslant f_0 \geqslant g \geqslant g^+$, that is, $g = g^+ = f_0$. Dually for infima, so $CI(X)$ is a regular sublattice of $C(X)$. ■

We recall from [7] that the minimal Boolean extension $B(L)$ of any $L \in D_{01}$ may be obtained from PL by "forgetting the order": more precisely, $B(L)$ is isomorphic to the algebra of *all* clopen subsets of PL. Minimal Boolean extensions are connected with the second question raised above in the following way:

THEOREM 13. *If the embedding of L into its minimal Boolean extension $B(L)$ is regular, then $CI(PL)$ is a regular sublattice of $C(PL)$.*

Proof. The assumption is equivalent to PL being an I-space, see Proposition 17 of [7]. ■

Theorems 12 and 13 are not fully satisfactory since they provide only sufficient conditions. The problem of giving exact characterizations of the spaces and lattices in question remains open.

We conclude by remarking that it is possible to generalize our key theorems (8 and 9) to the case where L lacks universal bounds. One would then consider the lattices $CI(P(L_{01}))$, where L_{01} is obtained from L by adjoining a zero and a unit regardless of the fact that L already may — but need not — have such elements. Much of the theory developed would remain valid, but we feel that the generality gained by such a procedure does not compensate the required technical clumsiness.

References

[1] R. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, Miss. 1974.

[2] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, N. J. 1960.

[3] J. L. Kelley, *General Topology*, Van Nostrand, Princeton, N. J. 1955.

[4] L. Nachbin, *Topology and Order*, Van Nostrand, Princeton, N. J. 1965.

[5] H. A. Priestley, *Separation theorems for semicontinuous functions on normally ordered topological spaces*, J. London Math. Soc. 3 (1971), pp. 371–377.

[6] — *Representation of distributive lattices by means of ordered Stone spaces*, Bull. London Math. Soc. 2 (1970), pp. 186–190.

[7] — *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. 24 (1972), pp. 507–530.

[8] G. L. Seever, *Measures on F-spaces*, Trans. Amer. Math. Soc. 133 (1968), pp. 267–280.

[9] M. H. Stone, *Boundedness properties in function lattices*, Canadian J. Math. 1 (1949), pp. 176–186.

[10] R. C. Walker, *The Stone-Čech Compactification*, Springer, 1974.

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On CE-images of the Hilbert cube and characterization of Q-manifolds

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Abstract. It is shown that a locally compact ANR, X, admitting arbitrarily small (i.e. close to id_X) maps $f, g\colon X \to X$ with $f(X) \cap g(X) = \emptyset$ is a Q-manifold. This is applied to show that if $\mathcal{A}$ is a semicontinuous decomposition of Q such that each $A \in \mathcal{A}$ has trivial shape then $Q/\mathcal{A} \cong Q$ (resp. $Q/\mathcal{A} \times [0, 1] \cong Q$) provided $Q/\mathcal{A} \in \mathrm{AR}$ and the union of non-degenerate elements of $\mathcal{A}$ is contained in a countable union of Z-sets in Q (resp. in a countable union of finite-dimensional compacta). A short proof of the Curtis-Schori hyperspace theorem is included in the Appendix.

In 1975, R. D. Edwards established the following profound result (see [11] and [8], § 43):

EDWARDS' THEOREM. *If M is a manifold modelled on the Hilbert cube Q and $\pi\colon M \to X$ is a proper* CE*-map of M onto a locally compact* ANR, *then*

$$\pi \times \mathrm{id}_Q\colon M \times Q \to X \times Q$$

is a limit of homeomorphisms and, in particular, $X \times Q$ is a Q-manifold.

However, it is of interest to know under what additional conditions on π the space X is itself a Q-manifold. Specifically, for the case $M = Q$, the following problems were posed in [1]:

(a) Suppose that the union $S(\pi)$ of non-degenerate point inverses of π is contained in a countable union of Z-sets. Is then $X \cong Q$ (1)?

(b) Under what conditions on π is $X \times [0, 1] \cong Q$?

In connection with (a) it follows from a theorem of J. E. West that $X \cong Q$ if $S(\pi)$ is contained in a single Z-set of Q; see [19] and [8], § 42. In connection with (b) it was shown by J. L. Bryant and by T. A. Chapman that $X \times [0, 1] \cong Q$ and $X \times X \cong Q$ if π has only one non-degenerate point inverse A which is an arc. This was subsequently generalized by Z. Čerin [9] to the case $A \cong [0, 1]^n$, $n < \infty$. In [1] it is mentioned that R. D. Edwards has proved that $Q/A \times [0, 1] \cong Q$ for any finite-dimensional compactum A in Q of trivial shape (unpublished).

In this note we solve (a) in affirmative and we also show that if $S(\pi)$ is a countable union of finite-dimensional compacta, then $X \times [0, 1] \cong Q$ (see § 4). In fact we

(1) We write $X \cong Y$ to denote that X and Y are homeomorphic.

show that, under the same assumption, $[0, 1]$ may be replaced by any compact AR-space Y containing more than one point. This implies the existence of spaces X which are topologically different from Q and become homeomorphic to Q after taking product with any non-trivial AR. (Consider $X = Q/A$, where A is any arc containing Wong's [22] wild Cantor set). These results are obtained by considering the projection p_X: $X\times Q \to X$, rather than π itself, and by using Edwards' ANR-theorem and Bing's shrinking criterion, along with some information carried by π.

When considering the projection p_X: $X\times Q\to X$ we arrive in § 2 at a general characterization of Q-manifolds stating that a locally compact $X\in$ ANR is a Q-manifold if it admits arbitrarily "small" (i.e. close to the identity) maps into its Z-sets or, equivalently, if any two maps f, g: $Q\to X$ may be approximated by maps with disjoint images. This characterization readily implies the before-mentioned results; it also yields the fact that an ANR-compactification of an l_2-manifold with a Z_σ-remainder is a Q-manifold (see § 3).

Let us note that the same characterization can be used to give short proofs of many, if not all, of the earlier results concerning identifying Q-manifolds (see [17]). As an example we include in the Appendix an argument for the result of D. W. Curtis and R. M. Schori on hyperspaces of Peano continua.

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§ 1. **Preliminaries.** In this section we shall fix the notation and state some known facts which will be needed later.

By I we denote the segment $[0, 1]$, by I^k the k-cube, and by ∂I^k the boundary of I^k, $k<\infty$. We let $Q = I^\infty$, the Hilbert cube. Any product $\prod_{i<n} X_i$, $n\leqslant\infty$, of metric spaces (X_i, ϱ_i) will be considered in the product metric

$$\varrho((x_i), (y_i)) = \max\{\min(2^{-i}, \varrho_i(x_i, y_i)) \colon i<n\}.$$

p_X: $X\times Y\to X$ denotes the natural projection.

We write $\operatorname{cov}(X)$ for the family of all covers of X by open sets and $C(M, X)$ for the set of all maps (i.e. continuous functions) from M to X. We topologize $C(M, X)$ by the "limitation topology" in which each $f\in C(M, X)$ has

$$\{V(f, \mathscr{U}) \colon \mathscr{U}\in\operatorname{cov}(X)\}$$

as a basis of neighbourhoods, where

$$V(f, \mathscr{U}) = \{g\in C(M, X) \colon \forall x\in M \exists U\in\mathscr{U} \text{ with } f(x), g(x)\in U\}.$$

The limitation topology can, equivalently, be described as the one induced by the family $\{\hat{\varrho}\colon \varrho\in \operatorname{Metr}(X)\}$ of metrics on $C(M, X)$, where $\operatorname{Metr}(X)$ is the set of all bounded metrics compatible with the topology of X and

$$\hat{\varrho}(f, g) = \sup\{\varrho(f(x), g(x)) \colon x\in M\}$$

for $f, g\in C(M, X)$ (see [17], § I.1 or [2], p. 121).

All function spaces will be considered in the limitation topology. If M is compact, then the limitation topology of $C(M, X)$ coincides with the compact-open one and may be metrized by any of the metrics $\hat{\varrho}$, $\varrho\in\operatorname{Metr}(X)$.

(A) *If X is complete-metrizable, then $C(M, X)$ has the Baire property (i.e. the intersection of a countable family $\{U_i\colon i\in N\}$ of dense open sets in $C(M, X)$ is dense).*

Outline of the proof (Details in [17]). Fix $f\in C(M, X)$ and a complete $\varrho\in\operatorname{Metr}(X)$ and modify the proof of the classical Baire theorem to get a $\hat{\varrho}$-Cauchy sequence in $V = \{g\in C(M, X)\colon \hat{\varrho}(g, f)\leqslant 1\}$ that converges to an $h\in V\cap\bigcap U_i$. Since V may be required as small as we wish, f is in the closure of $\bigcap U_i$.

A subset A of X will be called a Z_∞-*set* in X if $\{f\in C(Q, X)\colon f(Q)\cap A = \emptyset\}$ is dense in $C(Q, X)$. Closed Z_∞-sets will be called Z-*sets* and the family of all Z-sets in X (resp. the family of all countable unions of Z-sets in X) will be denoted by $\mathscr{Z}(X)$ (resp. $\mathscr{Z}_\sigma(X)$). By (A), if X is complete and K is a closed set contained in an $L\in\mathscr{Z}_\sigma(X)$, then $K\in\mathscr{Z}(X)$.

An $f\in C(M, X)$ will be called a Z-*map* if $f(M)\in\mathscr{Z}(X)$. Embeddings will always be assumed to be closed and homeomorphisms to be surjective; id_X denotes the identity map of X. By an ANR we mean any absolute neighbourhood retract for metric spaces, c.f. [2], p. 66.

(B) *Let X be a locally compact space and let M be an* ANR. *Then, given $A\in\mathscr{Z}_\sigma(X)$, the set $\{f\in C(M, X)\colon f(M)\cap A = \emptyset\}$ is dense in $C(M, X)$.*

Proof. The lemma is a special case of known facts (see e.g. [18]).

(C) *Let X be an* ANR, *let T be a metric space and let T_0 be a compact set in T. Then, the restriction $f\to f|T_0$ is an open map from $C(T, X)$ to $C(T_0, X)$.*

Proof. Assume first that X is an open subset of a normed linear space E. Let $\mathscr{U}$ be a cover of X by open convex sets. Given $f\in C(T, X)$ and $h\in V(f|T_0, \mathscr{U})$, extend h to an $\bar{h}$: $T_1\to X$, where T_1 is an open neighbourhood of T_0 in T such that $\bar{h}\in V(f|T_1, \mathscr{U})$. It $\lambda\in C(T, I)$ satisfies $\lambda^{-1}(0)\supset T_0$ and $X\setminus T_1\subset\operatorname{int}\lambda^{-1}(1)$, then

$$x\mapsto \lambda(x) f(x) + (1-\lambda(x))\bar{h}(x)$$

defines a map $g\in V(f, \mathscr{U})$ with $g|T_0 = h$.

The general case now follows in a standard way by considering a retraction r: $U\to X$, where U is open in a normed linear space (see [2], p. 68), and by using the fact that $C(T, U)\in f\to r\circ f\in C(T, X)$ is continuous.

We will need the following version of Bing's shrinking criterion:

(D) *Let π: $M\twoheadrightarrow X$ be a proper map between locally compact metric spaces, and let d be a metric on M. If, given $\varepsilon>0$ and $\mathscr{U}\in\operatorname{cov}(X)$, there is a homeomorphism $h = h_{\mathscr{U}\varepsilon}$ of M such that $\pi h\in V(\pi, \mathscr{U})$ and the d-diameter of each set $h\pi^{-1}(x)$ is less than ε, then $M\cong X$.*

Proof. Let $\tilde{X}$ and $\tilde{M}$ denote the one-point compactifications of X and M, respectively. Extending π and the $h_{\mathscr{U}\varepsilon}$'s to ∞-preserving maps $\tilde{\pi}$: $\tilde{M}\to\tilde{X}$ and $\tilde{h}_{\mathscr{U}\varepsilon}$: $\tilde{M}\to\tilde{M}$, we infer from Bing's shrinking criterion as formulated in [8] or in [15] that $\tilde{\pi}$ is a limit of homeomorphisms $\tilde{M} \to \tilde{X}$ which, by an inspection of the proof, may be required to send $\{\infty\} \subset \tilde{M}$ to $\{\infty\} \subset \tilde{X}$. Thus $M \cong X$.

We shall also use the following theorem of Chapman (see [8], § 22):

(E) *Any contractible Q-manifold is homeomorphic to Q.*

Finally, we need Edwards' ANR theorem (see [11] and [8], § 44):

(F) *If X is a locally compact* ANR *then $X \times Q$ is a Q-manifold.*

§ 2. A characterization of Q-manifolds.

THEOREM 1. *Let X be a locally compact* ANR *such that, for each $k \in N$,*

(*) *the set of all Z-maps $I^k \to X$ is dense in $C(I^k, X)$.*

Then X is a Q-manifold.

The proof of the theorem is divided into the following lemmas (we assume that X is a fixed space satisfying the hypothesis of the theorem).

LEMMA 1. *Condition (*) holds with I^k replaced by any compact space K.*

Prof. Given $u\colon Q \to X$, the maps

$$x \mapsto u(x_1, \ldots, x_n, 0, 0, \ldots), \quad n = 1, 2, \ldots,$$

converge to u and may all be approximated by Z-maps. Thus (*) holds with I^k replaced by Q and there is a dense subset $\{u_1, u_2, \ldots\}$ of $C(Q, X)$ consisting of Z-maps. By (B), if K is any compactum, then each $f\colon K \to X$ is in the closure of

$$S = \{g \in C(K, X)\colon g(K) \cap u_i(Q) = \emptyset \text{ for } i \in N\};$$

clearly $g(K) \in \mathscr{Z}(X)$ for each $g \in S$.

LEMMA 2. *Given compact disjoint sets K, L in X, the set*

$$T = \{f \in C(X, X)\colon f(K) \cap f(L) = \emptyset\}$$

is dense in $C(X, X)$.

Proof. Given $h \in C(X, X)$, we may use Lemma 1 to approximate $h|K$ by a Z-map $u\colon K \to X$ and, then, we may apply (B) to approximate h by a $v\colon X \to X$ with $v(X) \cap u(K) = \emptyset$. Assuming that u and v are sufficiently close to $h|K$ and h, respectively, there is by (C) a map $f\colon X \to X$ closely approximating h and satisfying $f(x) = u(x)$ if $x \in K$ and $f(x) = v(x)$ if $x \in L$. Then $f \in T$ and hence h is in the closure of T.

LEMMA 3. *The set of all 1-to-1 Z_∞-maps $X \to X$ is dense in $C(X, X)$.*

Proof. Let $\{u_1, u_2, \ldots\}$ be a dense subset of $C(Q, X)$; by Lemma 1 we may assume that all the u_i's are Z-maps. Further let $\mathscr{V}$ be a countable basis of open sets in X such that the closure $\bar{V}$ of each $V \in \mathscr{V}$ is compact. By Lemmas 2 and (A),

$$G = \{f \in (X, X)\colon f(\bar{V}_1) \cap f(\bar{V}_2) = \emptyset \text{ for all } V_1, V_2 \in \mathscr{V} \text{ with } \bar{V}_1 \cap \bar{V}_2 = \emptyset\}$$

is a dense G_δ-set in $C(X, X)$.

By the σ-compactness of X, the set

$$H = \{f \in C(X, X)\colon f(X) \cap u_i(Q) = \emptyset \text{ for all } i \in N\}$$

is also of type G_δ in $C(X, X)$; moreover, H is dense in $C(X, X)$, by (B). Hence $G \cap H$ is dense in $C(X, X)$ (see (A)); clearly each $f \in G \cap H$ is a 1-to-1 Z_∞-map.

LEMMA 4. *For each $n \in N$ the following condition is satisfied:*

(c_n) *Given $\mathscr{U} \in \operatorname{cov}(X)$ and $\varepsilon > 0$, there is a homeomorphism f of $X \times Q \times I^n$ such that $p_X f \in V(p_X, \mathscr{U})$ and $\operatorname{diam} p_{I^n} f(\{x\} \times Q \times I^n) < \varepsilon$, for all $x \in X$.*

Proof. We first check (c_1). Given $\mathscr{U} \in \operatorname{cov}(X)$. Since X is an ANR, any two maps which are sufficiently close may be joined by a small homotopy; thus we may use Lemma 3 to get a 1-to-1 Z_∞-map $v\colon X \to X$ homotopic to id_X via a homotopy limited by $\mathscr{U}$. Assuming that the star of $\mathscr{U}$ consists of sets with compact closures, this homotopy is proper; in particular v is an embedding and $A = v(X) \in \mathscr{Z}(X)$. If w is any homeomorphism of $Q \times I$ onto Q, then

$$(x, q, t) \mapsto (v^{-1}(x), w(q, t), 1) \quad \text{for} \quad (x, q, t) \in A \times Q \times I$$

is a homeomorphism of $A \times Q \times I$ onto $X \times Q \times \{1\}$ which, by the Anderson–Chapman isotopy theorem, may be extended to a homeomorphism g of $X \times Q \times I$ limited by $\{U \times Q \times I\colon U \in \mathscr{U}\}$; see [8], § 19. (By (F), the product $X \times Q \times I$ is a Q-manifold, and hence the isotopy theorem is applicable). Clearly, $g(A \times Q \times I) = X \times Q \times \{1\}$. Hence $\operatorname{diam} p_I g(\{x\} \times Q \times I) = 0$ for $x \in A$; to make $\operatorname{diam} p_I g(\{x\} \times Q \times I)$ small for $x \in X \setminus A$ we shall use a trick applied by Edwards in his proof and earlier by West in [20].

Put $\alpha_0(x, q) = 0$ for $(x, q) \in X \times Q$ and $F_0 = p_X g^{-1}(X \times Q \times \{0\})$; F_0 is closed in X. Moreover, $F_0 \cap A = \emptyset$, whence $g(F_0 \times Q \times I) \cap X \times Q \times \{1\} = \emptyset$ and there is a map $\alpha_1\colon X \times Q \to (0, 1)$ such that

$$g(F_0 \times X \times I) \subset \{(x, q, t) \in X \times Q \times I\colon t \leqslant \alpha_1(x, q)\} \overset{\mathrm{df}}{=} S$$

(see [10], p. 74). Put $F_1 = p_X g^{-1}(S)$ and continue likewise to get sets $F_0 \subset F_1 \subset \ldots \subset F_{k-1} \subset X \setminus A$ and maps $0 \equiv \alpha_0 < \alpha_1 < \ldots < \alpha_k \equiv 1$ such that

$$\{(x, q, t)\colon t \leqslant \alpha_i(x, q)\} \subset g(F_i \times Q \times I) \subset \{(x, q, t)\colon t \leqslant \alpha_{i+1}(x, q)\}$$

for $i = 0, 1, \ldots, k-1$.

Let h be a homeomorphism of $X \times Q \times I$ preserving the fibres of $p_{X \times Q}$ and carrying the graph of α_i onto $X \times Q \times \{i/k\}$, for all $i = 0, 1, \ldots, k$. Given $x \in X$, if i is chosen so that $x \in F_i \setminus F_{i-1}$, then

$$p_I gh(\{x\} \times Q \times I) \subset [(i-1)/k, (i+1)/k].$$

Hence $\operatorname{diam} p_I gh(\{x\} \times Q \times I) < 2/k$ for all $x \in X$ and, assuming $2/k < \varepsilon$, $f = gh$ satisfies (c_1).

Now suppose that $n>1$ and (c_1) and (c_{n-1}) are satisfied. Given $\mathscr{U}\in\operatorname{cov}(X)$, let $\mathscr{V}\in\operatorname{cov}(X)$ be a star-refinement of $\mathscr{U}$ and let f_1 be a homeomorphism of $X\times Q\times I^{n-1}$ such that $p_X f_1\in V(p_X,\mathscr{V})$ and $\operatorname{diam} p_{I^{n-1}} f_1(\{x\}\times Q\times I^{n-1})<\varepsilon/2$ for all $x\in X$. There is a $\mathscr{W}\in\operatorname{cov}(X)$ refining $\mathscr{V}$ such that $\operatorname{diam} p_{I^{n-1}} f_1(W\times Q\times I^{n-1})<\varepsilon/2$ for all $W\in\mathscr{W}$. Let f_2 be a homeomorphism of $X\times Q\times I^{n-1}\times I$ such that $p_x f_2\in V(p_X,\mathscr{W})$ and $\operatorname{diam} p_I f_2(\{x\}\times Q\times I^{n-1}\times I)<\varepsilon$, for all $x\in X$. Then $f=(f_1\times\times\operatorname{id}_I)f_2$ is as required in (c_n); this concludes the inductive step and the proof of Lemma 4.

We now complete the proof of Theorem 1. Let ϱ be a fixed metric of X. Given $\mathscr{U}\in\operatorname{cov}(X)$ and $n\in N$, with $\mathscr{U}$ consisting of sets of ϱ-diameter less than $1/n$, we can use Lemma 4 to get an $f\in H(X\times I^\infty)$ such that $p_X f\in V(p_X,\mathscr{U})$ and $\operatorname{diam} p_{I_1\times\ldots\times I_n} f(\{x\}\times I^\infty)<1/n$ for all $x\in X$. Then $\operatorname{diam} f(\{x\}\times I^\infty)<1/n$ for all $x\in X$ (we metrize I^n, I^∞ and $X\times I^\infty$ by product metrics), whence $X\cong X\times I^\infty$ by (D) used with $\pi=p_X$. Thus X is a Q-manifold by Edwards' ANR theorem (F).

Remark 1. Condition $(*)$ is satisfied for each $k\in N$ if,

$(**)$ *given $k\in N$, $x\in X$ and a neighbourhood U of x in X, there is a neighbourhood V of x such that any Z-map $\partial I^k\to V$ extends to a Z-map $I^k\to U$.*

Proof. It follows from $(**)$ by a standard induction on $\dim(K)$ that $(*)$ holds with I^k replaced by any compact polyhedron K.

Remark 2. If X is a Q-manifold, then conditions $(*)$ and $(**)$ are satisfied.

Remark 3. It follows that a locally compact $X\in\mathrm{ANR}$ is a Q-manifold iff, for each $k\in N$,

$(***)$ *any two maps $I^k\to X$ may be approximated by maps with disjoint images.*

Proof. If $(***)$ is satisfied for each $k\in N$ then it holds also with I^k replaced by Q and, given an open subset U of $C(Q,X)$, the set

$$G(U)=\{g\in C(Q,X)\colon f(Q)\cap g(Q)=\emptyset \text{ for some } f\in U\}$$

is open and dense in $C(Q,X)$. If $\{U_1,U_2,\ldots\}$ is a basis of open subsets of $C(Q,X)$ then $G=\bigcap G(U_i)$ is dense in $C(Q,X)$, by (A), and consists of Z-maps; thus $(*)$ is satisfied.

§3. Q-manifolds as local compactifications of l_2-manifolds. From Edwards' ANR-theorem (F) combined with [18], Proposition 5.1, it follows that *if X is a locally compact* ANR *and there is a Z-set A in X such $X\setminus A$ is a Q-manifold, then X is a Q-manifold itself.* Here is another result of a similar character:

Theorem 2. *Let X be a locally compact* ANR. *If there is a Z_∞-set A in X such that $X\setminus A$ is a manifold modelled on an infinite-dimensional linear metric space E, then X is a Q-manifold.*

Proof. We need the following

Sublemma. *If M is an E-manifold and K is a compact set in M, then $K\in\mathscr{Z}(M)$.*

Proof. To be a Z-set is a local property (see [12], [18]), and hence we may assume $M=E$. Given $f\colon Q\to E$. The set $B=\{x-y\colon x\in K, y\in f(Q)\}$ is compact and hence nowhere-dense in E. Thus there is a sequence $\{a_n\}$ in $E\setminus B$ converging to 0. The maps $x\mapsto f(x)+a_n$ converge to f and take values in $E\setminus K$.

We pass to the proof of the theorem. By assumption, the set T of $X\setminus A$-valued maps $Q\to X$ is dense in $C(Q,X)$. Moreover, any compact subset K of $X\setminus A$ is a Z-set in X: given $h\in C(Q,X)$ we may approximate h by an element of T and, by the sublemma, we may approximate the latter by maps $Q\to X\setminus A\setminus K$. Thus T is dense in $C(Q,X)$ and consists of Z-maps; thus Theorem 1 applies.

By a theorem of Anderson, the Hilbert space l_2 is homeomorphic to $(0,1)^\infty$ (see [2], p. 174) and hence Q can be considered as a compactification of l_2 with a Z_σ-remainder. Theorem 2 shows that, among the ANR's, Q is the only such a compactification of l_2.

§4. Images of the Hilbert cube under CE-maps with small singularity sets. A map $\pi\colon M\to X$ is said to be CE if, for each $x\in X$, $\pi^{-1}(x)$ is a non-void compactum of trivial shape in the sense of Borsuk [5]. The following fact is known:

Lemma G (Lacher [14], p. 722). *Let $\pi\colon M\to X$ be a* CE*-map. Then, for each $n\in N$, the set $\{\pi f\colon f\in C(I^n,M)\}$ is dense in $C(I^n,X)$; consequently, if A is closed in X and $\pi^{-1}(A)\in\mathscr{Z}(M)$, then $A\in\mathscr{Z}(X)$.*

The set

$$S(\pi)=\{x\in M\colon \pi^{-1}\pi(x)\neq\{x\}\}$$

will be called the *singularity set* of π. If π is proper, then $S(\pi)=\bigcup S_i$, where each $S_i=\{x\in M\colon \operatorname{diam}\pi^{-1}\pi(x)\geqslant 1/i\}$ is closed in M.

Theorem 3. *Let $f\colon Q\to X$ be a* CE*-map with X an* AR. *If $S(f)$ is a Z_∞-set in Q, then $X\cong Q$.*

Proof. As noted above, $S(f)$ is of type F_σ. Hence $S(f)\in\mathscr{Z}_\sigma(Q)$ and, by the results of Anderson, we may assume that $S(f)\subset Q\setminus(0,1)^\infty$; see [2] p. 156 or [17], § II.2. Put $A=f(Q\setminus(0,1)^\infty)$. Then $f^{-1}(A)=Q\setminus(0,1)^\infty$ and $f\colon(0,1)^\infty\to X\setminus A$ is a homeomorphism. By Lemma G, $A\in\mathscr{Z}_\sigma(X)$, whence $X\cong Q$, by Theorem 2 and (E).

The following result also implies Theorem 3 instantly:

Theorem 4. *Let $\pi\colon M\to X$ be a* CE*-map, where X is a locally compact* ANR *and M is a metric space. In order that X be a Q-manifold it suffices that, for any map $g\colon I^n\to M$ of a finite-dimensional cell, g should be in the closure of $\{h\in C(I^n,M)\colon \pi^{-1}\pi h(I^n)\in\mathscr{Z}(M)\}$.*

Proof. Apply Theorem 1 and Lemma G.

Below, we use Theorem 4 to consider images of Q under CE-maps with countably-dimensional singularity sets. We need the following

LEMMA 5. *Let S be a compact finite-dimensional subset of Q and let A be a closed nowhere-dense subset of an* ANR-*space Y. Then $S\times A\in\mathscr{Z}(Q\times Y)$.*

Proof. By a result of Kroonenberg [13], Q has a basis of open sets consisting of sets U with the relative homology $H_*(U, U\setminus S)$ vanishing. Therefore, by a result of Mogilski [16], for every closed set C in $S\times Y$ and for every open set V in $Q\times Y$, we have $H_*(V, V\setminus C) = 0$. By the Hurewicz theorem it thus remains to prove that $Q\times Y\setminus S\times A$ is 1-ULC.

The proof is analogous to that of Corollary 2.4 of [13]: given $f\colon \partial I^2 \to Q\times Y\setminus S\times A$, we may use the fact that S disconnects no open set in Q to approximate $p_Q f$ by a $g\colon \partial I^2\to Q\setminus S$ which is so close to $p_Q f$ that the map

$$h(x) = (g(x), p_Y f(x)) \quad \text{for} \quad x\in\partial I^2$$

is homotopic to f by a small homotopy $\{\varphi_t\colon t\in I\}$ taking values in $Q\times Y\setminus S\times A$. We may then use the local contractibility of Y and of Q to get a homotopy $\{\varphi_t\colon t\in[1,3]\}$ such that

(a) $p_Q\varphi_t = g$ for $t\in[1,2]$, $p_Y\varphi_1 = p_Y f$ and $f_Y\varphi_2(\partial I^2)$ is a singleton $\{y\}\subset Y\setminus A$, and

(b) $p_Y\varphi_t(\partial I^2) = \{y\}$ for $t\in[2,3]$, $p_Q\varphi_2 = g$ and $p_Q\varphi_3(\partial I^2) = \{\text{point}\}$.

Then $\{\varphi_t\colon t\in[0,3]\}$ homotopes f to a constant map within a subset of $Q\times Y\setminus S\times A$ of a diameter which may be assumed to be small if such was the diameter of $f(\partial I^2)$.

THEOREM 5. *Let $f\colon Q\to X$ be a* CE-*map, where X is an* AR. *If $S(f)$ is a countable union of finite-dimensional compacta, then, for any compact* AR-*space Y containing more than one point, $X\times Y$ is homeomorphic to Q.*

Proof. Put $\pi = f\times \mathrm{id}_Y\colon Q\times Y\to X\times Y$. By Theorem 4 and (E) it suffices to show that $\mathrm{id}_{Q\times Y}$ may be approximated by maps $h\colon Q\times Y\to Q\times Y$ with $\pi^{-1}\pi h(Q\times Y)\in\mathscr{Z}(Q\times Y)$.

To this end let Y_0 be a countable dense subset of Y. By Lemma 5, $S(f)\times Y_0\in\mathscr{Z}_\sigma(Q\times Y)$, whence there is a Z-map $h\colon Q\times Y\to Q\times Y$ which satisfies $h(Q\times Y)\cap S(f)\times Y_0 = \emptyset$ and is as close to $\mathrm{id}_{Q\times Y}$ as we wish. (Apply (B) with $A = \big(Q\setminus(0,1)^\infty\big)\times Y\big)\cup S(f)\times Y_0$. Expressing $S(f)$ as a union of finite-dimensional compacta $S_1, S_2, \ldots$, we have

$$\pi^{-1}\pi h(Q\times Y)\setminus h(Q\times Y)\subset \bigcup S_i\times T_i\,,$$

where

$$T_i = p_Y\big(h(Q\times Y)\cap S_i\times Y\big)\subset Y\setminus Y_0 \quad \text{for} \quad i\in N\,.$$

By Lemma 5, all the $S_i\times T_i$'s are Z-sets. Thus $\pi^{-1}h(Q\times Y)$ is contained in a Z_σ-set and hence is a Z-set, as required.

With the same proof, Theorem 5 may be generalized as follows:

THEOREM 5′. *Let $f\colon M\to X$ be a* CE-*map, where M is a Q-manifold and X is a locally compact* ANR, *and assume that $S(f) = \bigcup_{i\in N} S_i$, where each S_i is a closed subset of M such that $H_*(U, U\setminus S_i) = 0$ for any open subset U of M. If Y is any locally compact* ANR *with no isolated points, then $X\times Y$ is a Q-manifold.*

Similarly, using Theorem 4, one can generalize Theorem 3 as follows:

THEOREM 3′. *Let $f\colon M\to X$ be a* CE *map, where M is a Q-manifold and X is a locally compact* ANR. *If there is an $A\in\mathscr{Z}_\sigma(M)$ such that $f|M\setminus A$ is* 1-*to*-1, *then X is a Q-manifold.*

Appendix. A proof of the Curtis-Schori hyperspace theorem. In this appendix we use Theorem 1 to give a short argument for the following result, which was originally established in [5] and [6] by delicate methods developed prior to Edwards' theorem:

THEOREM (D. W. Curtis and R. M. Schori.) *Let P be Peano continuum and assume that either $X = 2^P$, the hyperspace of all non-void closed subsets of P, or P contains no free arcs and $X = \{A\in 2^P\colon A \text{ is connected}\}$. Then $X\cong Q$.*

Proof. X is topologized by the Hausdorff metric

$$d(A_1, A_2) = \max\{\varrho(x, A_i)\colon i\in\{1,2\},\ x\in A_1\cup A_2\}\,.$$

where ϱ is a metric for P which, by [3], we may assume to be convex (i.e. any pair of points of P is contained in a subspace of (P,ϱ) isometric to a segment of the reals). By a theorem of Wojdysławski [21], X is a compact AR.

Given $\varepsilon>0$, the formula

$$f_\varepsilon(A) = \{x\in P\colon \varrho(X, A)\leqslant\varepsilon\} \quad \text{for} \quad A\in X\,,$$

defines a map from X to X with $d(f_\varepsilon(A), A)\leqslant\varepsilon$ for all $A\in X$. Moreover, if $B_1, \ldots, B_n$ are ϱ-balls in P centred at points $p_1, \ldots, p_n$ of an $\frac{1}{3}\varepsilon$-net in P and having $\frac{1}{3}\varepsilon$ as radii, then image $(f_\varepsilon)\subset X_1\cup\ldots\cup X_n$ where $X_i = \{A\in X\colon A\supset B_i\}$, $i = 1, \ldots, n$. By Lemma 5.4 of [7], each X_i is a Z-set in X. Thus f_ε is a Z-map and the result follows from Theorem 1 and (E).

References

[1] R. D. Anderson et al, *Open problems in infinite-dimensional topology*, Appendix to [8].
[2] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, Warszawa 1975.
[3] R. H. Bing, *Partitioning a set*, Bull. Amer. Math. Soc. 55 (1949), pp. 1101–1110.
[4] K. Borsuk, *Theory of Shape*, Warszawa 1975.
[5] D. W. Curtis and R. M. Schori, 2^X *and* $C(X)$ *are homeomorphic to the Hilbert cube*, Bull. Amer. Math. Soc. 80 (1974), pp. 927–931.
[6] — — *Hyperspaces of Peano continua are Hilbert cubes*, Fund. Math. 101 (1978), pp. 19–38.
[7] — — *Hyperspaces which characterize simple homotopy types*, General Top. and Appl. 6 (1976), pp. 153–165.
[8] T. A. Chapman, *Lectures on Hilbert cube manifolds*, C. B. M. S. Regional Conference Series in Math. Nº 28, Amer. Math. Soc. (1976).
[9] Z. Čerin, *Hilbert cube modulo an arc*, Fund. Math. 101 (1978), pp. 111–119.

[10] J. Dieudonné, *Une generalization des espaces compacts*, J. Math. Pures et Appl. 23 (1944), pp. 65–76.
[11] R. D. Edwards, to appear.
[12] J. Eells and N. H. Kuiper, *Homotopy negligible subsets of infinite-dimensional manifolds*, Compositio Math. 21 (1969), pp. 155–164.
[13] N. Kroonenberg, *Characterization of finite-dimensional Z-sets*, Proc. Amer. Math. Soc. 43 (1974), pp. 421–427.
[14] C. Lacher, *Cell-like mappings I*, Pacific J. Math. 30 (1969), pp. 717–731.
[15] A. Marin and Y. M. Visetti, *A general proof of Bing's shrinkability criterion*, Proc. Amer. Math. Soc. 53 (1975), pp. 501–507.
[16] J. Mogilski, Ph. D. dissertation, Inst. of Math., Polish Acad. of Sci., 1978 (to appear).
[17] H. Toruńczyk, *An introduction to infinite-dimensional topology*, to appear.
[18] — *Concerning locally homotopy negligible sets and characterization of l_2-manifolds*, Fund. Math. 101 (1978), pp. 93–110.
[19] J. E. West, *Mapping Hilbert cube manifolds to* ANR's: *a solution of a conjecture of Borsuk*, Ann. of Math. 106 (1977), pp. 1–18.
[20] — *Infinite products which are Hibert cubes*, Trans. Amer. Math. Soc. 150 (1970), pp. 1–25.
[21] M. Wojdysławski, *Retractes absolus et hyperspaces des continus*, Fund. Math. 32 (1939), pp. 184–192.
[22] R. Y. T. Wong, *A wild Cantor set in the Hilbert cube*, Pacific J. Math. 24 (1968), pp. 189–193.

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Provability in arithmetic and a schema of Grzegorczyk

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Abstract. S4Grz is the system that results when the schema

$$(((B \Rightarrow DA) \Rightarrow DA) \& ((-B \Rightarrow DA) \Rightarrow DA)) \Rightarrow DA$$

is added to the modal (propositional) logic S4. Let $\varnothing$ map sentence letters of modal logic to sentences in the language of P(eano) A(rithmetic). Define ${}^{\varnothing}A$, A a sentence of modal logic, by: ${}^{\varnothing}p = \varnothing(p)$; ${}^{\varnothing}(-A) = -({}^{\varnothing}A)$; ${}^{\varnothing}(A \& B) = ({}^{\varnothing}A \& {}^{\varnothing}B)$; ${}^{\varnothing}(DA) = (\mathrm{Bew}(\ulcorner {}^{\varnothing}A \urcorner) \& {}^{\varnothing}A)$, where $\mathrm{Bew}(x)$ is the standard provability predicate for PA and $\ulcorner S \urcorner$ is the numeral for the Gödel number of the sentence S.

THEOREM. *For all sentences A of modal logic,* $\vdash_{\mathrm{S4Grz}} A$ iff *for all* $\varnothing$, $\vdash_{\mathrm{PA}} {}^{\varnothing}A$.

(This result was independently obtained by R. Goldblatt.)

We shall describe a connection between provability in PA (= Peano Arithmetic, classical first-order formal arithmetic with induction) and a system of (propositional) modal logic considered (¹) by Grzegorczyk. We are interested in "readings" of the box (D) of modal logic that concern provability in PA. Accordingly, we let $\varnothing$ be a variable ranging over functions from the sentence letters of modal logic to sentences of PA and define the *provability translation* $A^{\varnothing}$ (under $\varnothing$) of a sentence A of modal logic as follows: if A is the sentence letter p, then $A^{\varnothing} = \varnothing(p)$; if $A = -B$, then $A^{\varnothing} = -(B^{\varnothing})$; if $A = (B \& C)$, then $A^{\varnothing} = (B^{\varnothing} \& C^{\varnothing})$ (and similarly for the other non-modal connectives); and if $A = DB$, then $A^{\varnothing} = \mathrm{Bew}(\ulcorner B^{\varnothing} \urcorner)$, where $\mathrm{Bew}(x)$ is the standard provability predicate for PA, and $\ulcorner S \urcorner$ is the numeral for the Gödel number of the sentence S of PA.

It is a well-known consequence of Gödel's incompleteness theorems and their proofs that not every provability translation of every theorem of the modal system S4 is a theorem of PA. For example, if $\varnothing(p)$ is the undecidable sentence S constructed by Gödel, then since $\vdash_{\mathrm{PA}}(S \leftrightarrow -\mathrm{Bew}(\ulcorner S \urcorner))$, $(Dp \to p)^{\varnothing}$, $= (\mathrm{Bew}(\ulcorner S \urcorner) \to S)$, is not a theorem of PA; and if $\varnothing(p) = 0 = 1$, then if $\vdash_{\mathrm{PA}}(Dp \to p)^{\varnothing}$ then $\vdash_{\mathrm{PA}}(\mathrm{Bew}(\ulcorner 0 = 1 \urcorner) \to 0 = 1)$, $\vdash_{\mathrm{PA}} -\mathrm{Bew}(\ulcorner 0 = 1 \urcorner)$, and (by the second incompleteness theorem) PA is inconsistent.

(¹) In [2], p. 230.