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Characterizations of reflexivity *

by

R. C. JAMES (Claremont)

A Banach space B is isometric with a subspace of its second conjugate space under the "natural mapping" for which the element of B^{**} that corresponds to the element x of B is the linear functional F_x defined by $F_x(f) = f(x)$ for each f of B^*. If each F of B^{**} is of this form, then B is said to be *reflexive* and B is isometric with B^{**} under this natural mapping. It is known that for a Banach space B each of the following conditions is equivalent to reflexivity:

I. *The unit sphere of B is weakly compact* [1].

II. *Each decreasing sequence of bounded closed convex sets has a non-empty intersection* [5].

If the unit sphere is weakly compact, then each continuous linear functional attains it sup on the unit sphere. Klee used II to show that if each continuous linear functional attains its sup on the unit sphere of any isomorph of the space, then the space is reflexive [4]; a result that was first known for spaces with bases [2]. Later it was shown by the author that if each continuous linear functional attains its sup on the unit sphere of a separable Banach space, then the space is reflexive [3]. The proof used a characterization of reflexivity that will also be used in the proof of Theorem 1 of this paper. Roughly, this characterization is that a Banach space is reflexive if and only if its unit sphere does not contain a "large flat region" ([3], Lemma 1). Theorem 1 gives a somewhat stronger but similar characterization of reflexivity that will be used in proving that an arbitrary Banach space is reflexive if each continuous linear functional attains its sup on the unit sphere (Theorem 5).

No specific results for complex Banach spaces are included in this paper. However, all results can be translated easily if (where appropriate) one uses linear functionals that are real-valued. Except where specifically restricted to being real, the spaces (c_0) and $l^{(1)}$ may be either real or complex. It should be recalled that all complex linear functionals on

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a complex Banach space are of type $F(x)=f(x)-if(ix)$, where f is a real-valued linear functional. Moreover, $|F|$ attains its sup on the unit sphere if and only if there is a point on the unit sphere at which both $|F|$ and f attain their sups; if f attains its sup, then $|F|$ attains its sup at the same point.

THEOREM 1. *A Banach space B is non-reflexive if and only if, for each number $r<1$, there exists a sequence $\{z_i\}$ of elements with unit norms and a sequence $\{f_i\}$ of continuous linear functionals with unit norms such that*

(1) $$f_n(z_i)>r \quad \text{if} \quad n\leqslant i, \quad f_n(z_i)=0 \quad \text{if} \quad n>i.$$

Proof. Suppose first that such sequences $\{z_i\}$ and $\{f_i\}$ exist. Let S_n denote $\mathrm{cl}[\mathrm{conv}\{z_n, z_{n+1}, \ldots\}]$. Then $\{S_n\}$ is a decreasing sequence of bounded closed convex sets that has an empty intersection, since, if w belongs to the intersection, $f_n(w)\geqslant r$ for all n and also $\lim_{n\to\infty} f_n(w)=0$.

Suppose now that B is non-reflexive. Then it follows from Lemma 1 of [3] that, for some $\sigma>0$ and for all positive numbers Δ and δ with $\Delta>\delta$, there exists a sequence $\{x_i\}$ of members of B for which:

(i) $\delta\leqslant\|\xi\|\leqslant\Delta$ for all $\xi\in\mathrm{conv}\{x_i\}$;

(ii) for each n, there is an $N(n)$ such that

$$\varrho(\mathrm{lin}\{x_1, \ldots, x_n\}, \xi)\geqslant\sigma\|\xi\|,$$

for all $\xi\in\mathrm{conv}\{x_N, x_{N+1}, \ldots\}$.

It follows from (i) and (ii) that

(2) $$\varrho(\mathrm{lin}\{x_1, \ldots, x_n\}, \mathrm{conv}\{x_N, x_{N+1}, \ldots\})\geqslant\sigma\delta$$

for all n. Let

(3) $$\varepsilon=\inf_{\omega\in B}\lim_{i\to\infty}\varrho(\omega, \mathrm{conv}\{x_i, x_{i+1}, \ldots\}).$$

Then $\varepsilon\geqslant\frac{1}{2}\sigma\delta$. To show this, let us suppose that $\varepsilon<\frac{1}{2}\sigma\delta$, which together with (2) implies that there is an $\omega\in B$ and members η and ζ of $\mathrm{conv}\{x_1, x_2, \ldots\}$ such that $\|\omega-\eta\|<\frac{1}{2}\sigma\delta$, $\|\omega-\zeta\|<\frac{1}{2}\sigma\delta$, and $\|\eta-\zeta\|\geqslant\sigma\delta$. The first two of these inequalities imply $\|\eta-\zeta\|<\sigma\delta$.

Now let λ denote a positive number for which $(\varepsilon-\lambda)/(\varepsilon+\lambda)>r$. Because of the definition of ε, it is possible to choose w and $\{\eta_i\}$, with $\eta_i\in\mathrm{conv}\{x_i, x_{i+1}, \ldots\}$ for all i, such that

(4) $$\|\eta_i-w\|<\varepsilon+\lambda \quad \text{for all} \quad i.$$

It also follows from the definition of ε that, for each choice of $\{\zeta_1, \ldots, \zeta_{n-1}\}$ as a subset of $\{\eta_i\}$, and each choice of u from $\mathrm{lin}\{\zeta_1-w, \ldots, \zeta_{n-1}-w\}$, there exists an integer N such that

(5) $$\varrho\big(u, \mathrm{conv}\{\eta_N-w, \eta_{N+1}-w, \ldots\}\big)\geqslant\varepsilon-\tfrac{1}{2}\lambda,$$

since otherwise the right member of (3) would not be greater than $\varepsilon-\frac{1}{2}\lambda$ when ω is replaced by $u+w$. Moreover, suppose we let U be a finite subset of $\mathrm{lin}\{\zeta_1-w, \ldots, \zeta_{n-1}-w\}$ with enough members that, if $x\in\mathrm{lin}\{\zeta_1-w, \ldots, \zeta_{n-1}-w\}$ and $\|x\|<2\varepsilon$, then there is a $u\in U$ with $\|x-u\|<\frac{1}{2}\lambda$. Then if we let N^* be the largest N needed to satisfy (5) for any u of U, then (5) will be valid with N replaced by N^*, with $\varepsilon-\frac{1}{2}\lambda$ replaced by $\varepsilon-\lambda$, and for all $u\in\mathrm{lin}\{\zeta_1-w, \ldots, \zeta_{n-1}-w\}$. Then we let $\zeta_n=\eta_{N^*}$. By induction, this defines a subsequence $\{\zeta_i\}$ of $\{\eta_i\}$ such that

(6) $$\varrho(u, \mathrm{conv}\{\zeta_n-w, \zeta_{n+1}-w, \ldots\})\geqslant\varepsilon-\lambda$$

for all n and all $u\in\mathrm{lin}\{\zeta_1-w, \ldots, \zeta_{n-1}-w\}$. Therefore if we let $z_n=(\zeta_n-w)/\|\zeta_n-w\|$ and use (6) and (4), we have

$$\|u+\sum_n a_i z_i\|\geqslant(\varepsilon-\lambda)\sum_n\frac{a_i}{\|\zeta_i-w\|}\geqslant\frac{\varepsilon-\lambda}{\varepsilon+\lambda}\sum_n a_i$$

for all positive integers n, all $u\in\mathrm{lin}\{z_1, \ldots, z_{n-1}\}$, and all finite sequences $\{a_i\}$ of non-negative numbers. Therefore, for all n,

(7) $$\varrho(\mathrm{lin}\{z_1, \ldots, z_{n-1}\}, \mathrm{conv}\{z_n, z_{n+1}, \ldots\})\geqslant\frac{\varepsilon-\lambda}{\varepsilon+\lambda}>r.$$

As a first step toward defining the desired continuous linear functional f_n for a particular n, let us define a new norm on $\mathrm{lin}\{z_i\}$ as follows. Let S_{n-1} denote $\mathrm{lin}\{z_1, \ldots, z_{n-1}\}$ and, when $z=\sum a_i z_i$, let

$$|||z|||=\max\Big[\varrho(z, S_{n-1}), \Big\|\sum_1^{n-1} a_i z_i\Big\|\Big].$$

If $z\in S_{n-1}$, then $|||z|||=\|z\|$. If $|||z|||=0$, then $\varrho(z, S_{n-1})=0$ and $z\in S_{n-1}$, so that $|||z|||=\|z\|$. Therefore $|||z|||\neq0$ if $z\neq0$. To establish the triangle inequality, let us write $x=\sum a_i z_i$ and $y=\sum b_i z_i$. Then

$$\begin{aligned}|||x+y|||&=\max\Big[\varrho(x+y, S_{n-1}), \Big\|\sum_1^{n-1}(a_i+b_i)z_i\Big\|\Big]\\&\leqslant\max\Big[\varrho(x, S_{n-1})+\varrho(y, S_{n-1}), \Big\|\sum_1^{n-1}a_i z_i\Big\|+\Big\|\sum_1^{n-1}b_i z_i\Big\|\Big]\\&\leqslant\max\Big[\varrho(x, S_{n-1}), \Big\|\sum_1^{n-1}a_i z_i\Big\|\Big]+\max\Big[\varrho(y, S_{n-1}), \Big\|\sum_1^{n-1}b_i z_i\Big\|\Big]\\&=|||x|||+|||y|||.\end{aligned}$$

Now let $1/\theta=\inf\{|||z|||: z\in\mathrm{conv}(z_n, z_{n+1}, \ldots)\}$. It follows from the definition of $|||\ |||$ and (7) that

(8) $$\frac{1}{\theta}=\varrho(\mathrm{conv}\{z_n, z_{n+1}, \ldots\}, S_{n-1})>r.$$

The definition of θ implies that $\mathrm{conv}\{\theta z_n, \theta z_{n+1}, \ldots\}$ contains no points x for which $|||x||| < 1$. Therefore [6] there is a linear functional f_n^* whose domain is $\mathrm{cl}[\mathrm{lin}\{z_n, z_{n+1}, \ldots\}]$ and for which

$$(9) \qquad f_n^*(z) \geqslant 1 \quad \text{for all} \quad z \in \mathrm{conv}\{\theta z_n, \theta z_{n+1}, \ldots\},$$
$$f_n^*(u) \leqslant 1 \quad \text{if} \quad |||u||| \leqslant 1.$$

Then $|f_n^*(u)| \leqslant |||u|||$ for all u. Let f_n^{**} be defined by $f_n^{**}(x) = f_n^*(x)$ when x is in the domain of f_n^*, and $f_n^{**}(z_i) = 0$ if $i < n$. Then

$$\Big|f_n^{**}\Big(\sum a_i z_i\Big)\Big| = \Big|f_n^*\Big(\sum_n a_i z_i\Big)\Big| \leqslant \Big|\Big|\Big|\sum_n a_i z_i\Big|\Big|\Big|.$$

Since $|||\sum_n a_i z_i||| = \varrho(\sum_n a_i z_i, S_{n-1}) \leqslant \|\sum a_i z_i\|$, we have

$$\Big|f_n^{**}\Big(\sum a_i z_i\Big)\Big| \leqslant \Big\|\sum a_i z_i\Big\|.$$

Therefore $\|f_n^{**}\| \leqslant 1$. Also, if $i \geqslant n$, it follows from (9) that $f_n^*(\theta z_i) \geqslant 1$, and from this and (8) that

$$f_n^{**}(z_i) = f_n^*(z_i) \geqslant \frac{1}{\theta} > r.$$

Now we can define f_n as a norm-preserving extension of $f_n^{**}/\|f_n^{**}\|$ to the entire space. This completes the proof of Theorem 1.

The following corollary gives a geometric description of Theorem 1:

COROLLARY 1. *A Banach space B is non-reflexive if and only if, for each number $r < 1$, it is possible to embed B in a space of bounded functions defined on a set A in such a way that, for each positive integer n, there is a member z_n of B with*

$$(10) \qquad z_n = (r_1^n, r_2^n, \ldots, r_{n-1}^n, 0, 0, \ldots; \{t_a^n\}),$$

where $1 \geqslant r_i^n > r$ for all i and $|t_a^n| \leqslant 1$ for all a.

Proof. First, we use Theorem 1 to choose a sequence $\{z_i\}$ of elements with unit norms and a sequence $\{f_i\}$ of continuous linear functionals with unit norms for which $f_n(z_i) > r$ if $n \leqslant i$ and $f_n(z_i) = 0$ if $n > i$. Then we choose a countable set $\{a_1, a_2, \ldots\}$ of points to be a subset of A and define $x(a_n)$ as $f_n(x)$ for each n and each x in B. Finally, we choose enough additional continuous linear functionals $\{g_a\}$ of unit norms so that, when we introduce a point a of A for each g_a and define $x(a)$ as $g_a(x)$, it follows for all x that

$$\|x\| = \sup\{x(a) \colon a \in A\},$$

so that B is isometric with a subspace of the space of bounded functions with domain A.

If $B = l^{(1)}$, then this corollary can be strengthened and (10) replaced by

$$z_n = (1, 1, \ldots, 1, -1, -1, \ldots; \{t_a^n\}),$$

as is shown in the proof of the following theorem. The proof of this theorem uses an argument that will be generalized to prove Theorem 5.

THEOREM 2. *There is a continuous linear functional g defined on $l^{(1)}$ which has the property that, if $l^{(1)}$ is a subspace of a Banach space B, then there is a norm-preserving extension of g that does not attain its sup on the unit sphere of B.*

Proof. Let $\{e_1, e_2, \ldots\}$ be the natural basis of $l^{(1)}$ and, for each n, let f_n be the continuous linear functional defined on $l^{(1)}$ for which $f_n(e_i) = +1$ if $n \leqslant i$ and $f_n(e_i) = -1$ if $n > i$. Let f be defined by

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \Big(\text{i. e., } f(x) = -\sum_1^\infty a_n \text{ if } x = \sum_1^\infty a_n e_n\Big).$$

If $l^{(1)}$ is a subspace of B, extend each f_n to all of B without increasing the norm. Then extend f to all of B by letting $f(x)$ be $\lim_{n\to\infty} f_n(x)$ whenever this limit exists, and then completing the extension of f to all of B without increasing the norm of f. Finally, let g be defined by letting

$$g(x) = \sum_1^\infty 2^{-k} f_k(x) - f(x).$$

Then $|g(x)| \leqslant 2\|x\|$, so that $\|g\| \leqslant 2$. Since

$$g(e_n) = \sum_1^n 2^{-k} - \sum_{n+1}^\infty 2^{-k} + 1 = 2 - 2^{-(n-1)},$$

it follows that $\|g\| = 2$ and that g has the same norm on $l^{(1)}$ as on B. Now suppose that $\|u\| = 1$ and $g(u) = 2$. Since $|f(u)| \leqslant 1$ and $|f_k(u)| \leqslant 1$ for all k, we must have $f_k(u) = 1$ for all k. But then $f(u) = 1$ and $g(u) = 0$. Therefore g does not attain its sup on the unit sphere of B.

The following theorem shows that Theorem 2 is not valid if $l^{(1)}$ is replaced by (c_0). Although there are spaces B in which (c_0) can be embedded so that all norm-preserving extensions of continuous linear functionals defined on (c_0) attain their sups on the unit sphere of B, it follows from Theorems 4 and 2 that, whenever a Banach space contains a subspace isometric with (c_0), there is a continuous linear functional defined on B that does not attain its sup on the unit sphere of B. Of course, this also follows from Theorem 5.

THEOREM 3. *With (c_0) considered as a subspace of (m), each norm-preserving extension of a continuous linear functional defined on (c_0) attains its sup on the unit sphere of (m).*

Proof. Suppose f is a continuous linear functional defined on (c_0) with $\|f\|=1$. Then there are numbers $\{f_i\}$ such that $\sum|f_i|=1$ and $f(x)=\sum f_i x_i$ if x is in (c_0) and $x=(x_1,x_2,\ldots)$. Let $\operatorname{sign}(0)=0$ and $\operatorname{sign}(a)=\bar{a}/|a|$ if a is non-zero. Also let

$$u=[\operatorname{sign}(f_1),\operatorname{sign}(f_2),\operatorname{sign}(f_3),\ldots],$$
$$u_n=[\operatorname{sign}(f_1),\ldots,\operatorname{sign}(f_n),0,0,0,\ldots].$$

Then $f(\tfrac{1}{2}u)+f(u_n-\tfrac{1}{2}u)=f(u_n)$ and, since

$$\|\tfrac{1}{2}u\|=\|u_n-\tfrac{1}{2}u\|=\tfrac{1}{2}\quad\text{and}\quad f(u_n)=\sum_1^n|f_n|,$$

we have $|f(u_n-\tfrac{1}{2}u)|\leqslant\tfrac{1}{2}$ and $f(\tfrac{1}{2}u)+f(u_n-\tfrac{1}{2}u)=\sum_1^n|f_n|$. Therefore

$$f(u)\geqslant 2\sum_1^n|f_n|-1.$$

By letting $n\to\infty$, we obtain $f(u)\geqslant 1$. Since $\|u\|=1$ and $\|f\|=1$, we have $f(u)=1$.

THEOREM 4. *If $(c_0)\subset B$ and each norm-preserving extension to B of each continuous linear functional defined on (c_0) attains its sup on the unit sphere of B, then B contains a subspace isometric with real $l^{(1)}$.*

Proof. Embed B in the (m)-space of all bounded functions defined on a set A. Also choose A so that there are points $\{a_1,a_2,\ldots\}$ in A with the property that, if $x\in(c_0)$, then

$$x=[x(a_1),x(a_2),x(a_3),\ldots].$$

Let $f(x)=\sum\varepsilon_n x(a_n)/2^n$, where each ε_n is $+1$ or -1. If f attains its sup on the unit sphere of B, it is at an x of unit norm for which $x(a_n)=\varepsilon_n$ for all n. Let $\{f_1,f_2,\ldots\}$ be chosen so that for each n the values at a_1, $a_2,\ldots$ of the element at which f_n attains its sup on the unit sphere form alternate blocks of $+1$'s and -1's, each block with 2^n members. The closed linear span of these elements is isometric with real $l^{(1)}$.

With the hope of simplifying notation in the statement and proof of the following lemma, specific choices of certain numbers have been made, although this is not strictly necessary. We shall use the convention that sup denotes the supremum for all x with $\|x\|=1$. Also, a sequence of *non-overlapping* members of $\operatorname{conv}\{x_i\}$ is a sequence $\{y_i\}$ of members of $\operatorname{conv}\{x_i\}$ for which there is an increasing sequence of integers $\{p_i\}$ such that y_n belongs to $\operatorname{conv}\{x_{p_n},\ldots,x_{p_{n+1}-1}\}$ for all n.

LEMMA. *If B is a non-reflexive Banach space and $\{\beta_i\}$ is a sequence of positive numbers with $\beta_1=9$, then there is a sequence $\{z_i\}$ of members of B with unit norms and a sequence $\{g_i\}$ of continuous linear functionals with norms not greater than 1, such that*

(A) *$g_n(z_i)>79/80$ if $n\leqslant i$ and, for each i, there is an $N(i)$ such that $g_n(z_i)=0$ if $n>N(i)$;*

(B) *if for some k an element ξ of unit norm has the property that $g_k(\xi)\leqslant\frac{1}{4}+2^{-k}$, then there is an element y of unit norm such that*

$$\Big[\sum_1^k\beta_i g_i(y)-\overline{\lim}\,g_i(y)\Big]>2^{-(k+2)}\beta_k+\Big[\sum_1^k\beta_i g_i(\xi)-\underline{\lim}\,g_i(\xi)\Big].$$

Proof. It follows from Theorem 1 that there is a sequence $\{z_i'\}$ of elements with unit norms and a sequence $\{f_i\}$ of continuous linear functionals with unit norms for which (1) is valid with $r=79/80$. Eventually, we shall choose $\{g_i\}$ as a sequence of non-overlapping members of $\operatorname{conv}\{f_i\}$. It is then possible to choose $\{z_i\}$ as a subsequence of $\{z_i'\}$ in such a way that (A) is valid.

We shall show now that (B) is satisfied when $k=1$ if $\{g_i\}$ is any sequence of non-overlapping members of $\operatorname{conv}\{f_i\}$. To do this, we suppose that $\{g_i\}$ has been chosen and that a ξ with unit norm has the property that $g_1(\xi)\leqslant\frac{3}{4}$. We know that $\lim_{i\to\infty}g_i(z_n')=0$ and $g_1(z_n')>79/80$ if n is large enough. Therefore there is an n large enough that we can satisfy (B) with $k=1$ and $y=z_n'$, provided it is true that

$$\frac{79}{80}\beta_1>\frac{1}{8}\beta_1+[\beta_1 g_1(\xi)-\underline{\lim}\,g_i(\xi)]. \tag{11}$$

Since the right member of (11) is not greater than $\frac{1}{8}\beta_1+(\frac{3}{4}\beta_1+1)$, or $\frac{7}{8}\beta_1+1$, it is sufficient to have $(9/80)\beta_1>1$ or $\beta_1>80/9$. For $k=1$ and $\beta_1=9$, we have shown that (B) is true for all choices of $\{g_i\}$ as a sequence of non-overlapping members of $\operatorname{conv}\{f_i\}$.

Now suppose that $\{g_1,g_2,\ldots,g_{n-1},F_n,F_{n+1},\ldots\}$ has been chosen as a sequence of non-overlapping members of $\operatorname{conv}\{f_i\}$ and that, for each sequence $\{g_n,g_{n+1},\ldots\}$ of non-overlapping members of $\operatorname{conv}\{F_n,F_{n+1},\ldots\}$, (B) is valid for all $k\leqslant n$. Let

$$\theta_n=\inf_{\{G_i\}}\sup\Big[\sum_1^{n-1}\beta_i g_i(x)+\beta_n G_n(x)-\underline{\lim}\,G_i(x)\Big], \tag{12}$$

where $\{G_i\}$ denotes a sequence $\{G_n,G_{n+1},\ldots\}$ of non-overlapping members of $\operatorname{conv}\{F_n,F_{n+1},\ldots\}$. Choose a particular $\{\bar{G}_i^*\}$ such that

$$\sup\Big[\sum_1^{n-1}\beta_i g_i(x)+\beta_n\bar{G}_n^*(x)-\underline{\lim}\,\bar{G}_i^*(x)\Big]<\theta_n+2^{-(n+3)}\beta_{n+1}.$$

If $G_n^*=\bar{G}_n^*$ and $\{G_{n+1}^*,\ldots\}$ is a sequence of non-overlapping members of $\operatorname{conv}\{G_{n+1}^*,\ldots\}$, then for all sequences $\{h_{n+1},\ldots\}$ of non-overlapping members of $\operatorname{conv}\{G_{n+1}^*, \ldots\}$ we have

$$\underline{\lim}\, h_i(x) \geqslant \underline{\lim}\, \bar{G}_i^*(x) \quad \text{for all} \quad x, \tag{13}$$
$$\sup\Big[\sum_1^{n-1} \beta_i g_i(x) + \beta_n G_n^*(x) - \underline{\lim}\, h_i(x)\Big] < \theta_n + 2^{-(n+3)}\beta_{n+1}.$$

We can choose $\{G_{n+1}^*, \ldots\}$ so that, with $g_n = G_n^*$, we also have

$$\sup\Big[\sum_1^{n} \beta_i g_i(x) + \beta_{n+1} h_{n+1}(x) - \underline{\lim}\, h_i(x)\Big] \tag{14}$$
$$< 2^{-(n+3)}\beta_{n+1} + \sup\Big[\sum_1^{n} \beta_i g_i(x) + \beta_{n+1} h_{n+1}(x) - \overline{\lim}\, h_i(x)\Big],$$

for all sequences $\{h_{n+1}, \ldots\}$ of non-overlapping members of $\operatorname{conv}\{G_{n+1}^*, \ldots\}$. To show this, we first arrange all members of $\operatorname{conv}\{\bar{G}_{n+1}^*, \ldots\}$ that have rational coefficients in a sequence $\{\varphi_1, \varphi_2, \ldots\}$. Choose $\{H_1, H_2, \ldots\}$ and an element x_1 of unit norm so that $\{H_1, H_2, \ldots\}$ is a sequence of non-overlapping members with rational coefficients of $\operatorname{conv}\{\bar{G}_{n+1}^*, \ldots\}$ which has the property that

$$\sum_1^n \beta_i g_i(x_1) + \beta_{n+1}\varphi_1(x_1) - \underline{\lim}\, H_i(x_1)$$

is within $2^{-(n+4)}\beta_{n+1}$ of being as large as possible for all such sequences and elements x_1. Then let $\{H_1^1, H_2^1, \ldots\}$ be a subsequence of $\{H_1, H_2, \ldots\}$ for which $\lim H_i^1(x_1)$ exists and equals $\underline{\lim}\, H_i(x_1)$. Now use induction to choose for each k a sequence $\{H_1^k, H_2^k, \ldots\}$ and an element x_k of unit norm with the following properties:

(a) $\{H_1^k, H_2^k, \ldots\}$ is a sequence of non-overlapping members with rational coefficients of $\operatorname{conv}\{H_2^{k-1}, H_3^{k-1}, \ldots\}$;

(b) $\Big[\sum_1^n \beta_i g_i(x_k) + \beta_{n+1}\varphi_k(x_k) - \underline{\lim}_{i\to\infty}\, H_i^k(x_k)\Big]$ is within $2^{-(n+4)}\beta_{n+1}$ of being as large as possible for all choices of $\{H_1^k, H_2^k, \ldots\}$ as a sequence of non-overlapping members of $\operatorname{conv}\{H_2^{k-1}, H_3^{k-1}, \ldots\}$;

(c) $\lim_{i\to\infty} H_i^k(x_k)$ exists.

Now let $G_{n+k}^* = H_1^k$ for each k. For any φ_k in $\operatorname{conv}\{G_{n+1}^*, \ldots\}$, the expression

$$\sum_1^n \beta_i g_i(x_k) + \beta_{n+1}\varphi_k(x_k) - \underline{\lim}_{i\to\infty}\, G_i^*(x_k)$$
$$= \sum_1^n \beta_i g_i(x_k) + \beta_{n+1}\varphi_k(x_k) - \lim_{i\to\infty} H_i^k(x_k),$$

and this expression cannot be increased by more than $2^{-(n+4)}\beta_{n+1}$ by re-

placing $\{G_{n+1}^*, \ldots\}$ by some sequence $\{h_{n+1}, \ldots\}$ of non-overlapping members of $\operatorname{conv}\{G_{n+1}^*, \ldots\}$. Also,

$$\underline{\lim}_{i\to\infty}\, h_i(x_k) = \overline{\lim}_{i\to\infty}\, h_i(x_k)$$

for any such sequence. Now for a choice of $\{h_{n+1}, \ldots\}$, we can choose φ_k so that

$$|h_{n+1}(x) - \varphi_k(x)| < 2^{-(n+5)}\beta_{n+1} \quad \text{if} \quad \|x\| \leqslant 1$$

and use φ_k and the corresponding x_k to show that if (14) is false then we could replace h_{n+1} by φ_k and obtain the false statement:

$$\sum_1^n \beta_i g_i(x_k) + \beta_{n+1}\varphi_k(x_k) - \overline{\lim}_{i\to\infty}\, h_i(x_k)$$
$$> \sup\Big[\sum_1^n \beta_i g_i(x) + \beta_{n+1}\varphi_k(x) - \overline{\lim}\, h_i(x)\Big].$$

Now suppose there exists a G^{**} in $\operatorname{conv}\{G_{n+1}^*, \ldots\}$ and a sequence $\{h_{n+1}, \ldots\}$ of non-overlapping members of $\operatorname{conv}\{G_{n+1}^*, \ldots\}$ for which

$$\sup\Big[\sum_1^{n-1} \beta_i g_i(x) + \beta_n G_n^*(x) + \beta_{n+1} G^{**}(x) - \underline{\lim}\, h_i(x)\Big] \tag{15}$$
$$< \theta_n + \Big(\frac{1}{4} + 2^{-n}\Big)\beta_{n+1}.$$

Let $H = (\beta_n G_n^* + \beta_{n+1} G^{**})/(\beta_n + \beta_{n+1})$. Then

$$\sup\Big[\sum_1^{n-1} \beta_i g_i(x) + (\beta_n + \beta_{n+1}) H(x) - \underline{\lim}\, h_i(x)\Big] < \theta_n + \Big(\frac{1}{4} + 2^{-n}\Big)\beta_{n+1}. \tag{16}$$

We know that (B) is valid for $k = n$ with h_i substituted for g_i when $i > n$ and H substituted for g_n. Therefore if ξ has unit norm and is such that the expression S,

$$S = \sum_1^{n-1} \beta_i g_i(x) + \beta_n H(x) - \underline{\lim}\, h_i(x),$$

is within $2^{-(n+2)}\beta_n$ of its sup when $x = \xi$, then $H(\xi) > \frac{1}{4} + 2^{-n}$ and S is increased by more than $(\frac{1}{4} + 2^{-n})\beta_{n+1}$ when the term $\beta_{n+1}H(\xi)$ is added, so it follows from (16) that

$$\sup\Big[\sum_1^{n-1} \beta_i g_i(x) + \beta_n H(x) - \underline{\lim}\, h_i(x)\Big]$$
$$< \theta_n + \Big(\frac{1}{4} + 2^{-n}\Big)\beta_{n+1} - \Big(\frac{1}{4} + 2^{-n}\Big)\beta_{n+1} = \theta_n.$$

This contradicts (12), so we can conclude that expression (15) is false for all G^{**} and all sequences $\{h_{n+1}, \ldots\}$ of non-overlapping members of $\operatorname{conv}\{G^*_{n+1}, \ldots\}$. It follows from this and (13) that

$$(17)\qquad \sup\Big[\sum_1^{n-1}\beta_i g_i(x)+\beta_n G^*_n(x)+\beta_{n+1}G^{**}(x)-\underline{\lim}\, h_i(x)\Big]$$
$$>\Big[\frac14+2^{-n}-2^{-(n+3)}\Big]\beta_{n+1}+\sup\Big[\sum_1^{n-1}\beta_i g_i(x)+\beta_n G^*_n(x)-\lim h_i(x)\Big],$$

whatever the choice of G^{**} and $\{h_{n+1}, \ldots\}$ as non-overlapping members of $\operatorname{conv}\{G^*_{n+1}, \ldots\}$. It then follows from (14) and (17), with $G^*_n = g_n$, that

$$(18)\qquad \sup\Big[\sum_1^{n}\beta_i g_i(x)+\beta_{n+1}h_{n+1}(x)-\overline{\lim}\, h_i(x)\Big]$$
$$> -2^{-(n+3)}\beta_{n+1}+\sup\Big[\sum_1^{n}\beta_i g_i(x)+\beta_{n+1}h_{n+1}(x)-\underline{\lim}\, h_i(x)\Big]$$
$$>\Big[\frac14+2^{-n}-2^{-(n+2)}\Big]\beta_{n+1}+\sup\Big[\sum_1^{n}\beta_i g_i(x)-\underline{\lim}\, h_i(x)\Big].$$

Now we are prepared to attack (B) when $k = n+1$. Suppose that $\{h_{n+1}, \ldots\}$ is a sequence of non-overlapping members of $\operatorname{conv}\{G^*_{n+1}, \ldots\}$ and that ξ with unit norm is such that $h_{n+1}(\xi) \leqslant \frac14+2^{-(n+1)}$. Then it is easy to verify directly that

$$(19)\qquad \Big[\frac14+2^{-(n+1)}\Big]\beta_{n+1}$$
$$\geqslant\Big[\sum_1^{n}\beta_i g_i(\xi)+\beta_{n+1}h_{n+1}(\xi)-\overline{\lim}\, h_i(\xi)\Big]-\Big[\sum_1^{n}\beta_i g_i(\xi)-\underline{\lim}\, h_i(\xi)\Big].$$

From (18), it follows that there is a y of unit norm such that

$$(20)\qquad \Big[\sum_1^{n}\beta_i g_i(y)+\beta_{n+1}h_{n+1}(y)-\overline{\lim}\, h_i(y)\Big]$$
$$>\Big[\frac14+2^{-n}-2^{-(n+2)}\Big]\beta_{n+1}+\sup\Big[\sum_1^{n}\beta_i g_i(x)-\underline{\lim}\, h_i(x)\Big].$$

It follows from (19) and (20), and the equality of $\frac14+2^{-n}-2^{-(n+2)}$ and $2^{-(n+2)}+[\frac14+2^{-(n+1)}]$, that

$$\Big[\sum_1^{n}\beta_i g_i(y)+\beta_{n+1}h_{n+1}(y)-\overline{\lim}\, h_i(y)\Big]$$
$$>2^{-(n+2)}\beta_{n+1}+\Big[\sum_1^{n}\beta_i g_i(\xi)+\beta_{n+1}h_{n+1}(\xi)-\overline{\lim}\, h_i(\xi)\Big].$$

Therefore condition (B) is valid for $k = n+1$ and for all choices of $\{g_{n+1}, g_{n+2}, \ldots\}$ as a sequence of non-overlapping members of $\operatorname{conv}\{G^*_{n+1}, G^*_{n+2}, \ldots\}$.

THEOREM 5. *If B is a non-reflexive Banach space, then there is a continuous linear functional that does not attain its sup on the unit sphere of B.*

Proof. We shall use the sequences $\{z_i\}$, $\{\beta_i\}$, and $\{g_i\}$ of the lemma, but add the restriction that $\beta_{k+1} < 2^{-(k+4)}\beta_k$ for each k. Let Φ be a linear functional of unit norm defined on (m) and such that

$$\underline{\lim}\, x_i \leqslant \Phi(x_1, x_2, x_3, \ldots) \leqslant \overline{\lim}\, x_i.$$

For example, we can let Φ be any linear functional of unit norm such that $\Phi(x_1, x_2, \ldots) = \lim_{n\to\infty} x_n$ whenever this limit exists (or we could let Φ be a Banach limit—but we do not need the "translation invariance" of a Banach limit).

Now define a linear functional g on the space B by letting $g(z) = \Phi[g_1(z), g_2(z), \ldots]$. Then $\|g\| \leqslant 1$ and we also have

$$(21)\qquad \underline{\lim}\, g_i(z) \leqslant g(z) \leqslant \overline{\lim}\, g_i(z)$$

for all z in B. Now let G be defined by

$$G(z) = \sum_1^{\infty}\beta_i g_i(z) - g(z).$$

It follows from (B) of the lemma that, if $g_n(\xi) \leqslant \frac14+2^{-n}$ for some ξ of unit norm and some n, then there is a y of unit norm such that

$$(22)\qquad \Big[\sum_1^{n}\beta_i g_i(y)-\overline{\lim}\, g_i(y)\Big] > 2^{-(n+2)}\beta_n+\Big[\sum_1^{n}\beta_i g_i(\xi)-\underline{\lim}\, g_i(\xi)\Big].$$

Since $\beta_{k+1} < 2^{-(k+4)}\beta_k$ for each k, both $\sum_{n+1}^{\infty}|\beta_i g_i(y)|$ and $\sum_{n+1}^{\infty}|\beta_i g_i(\xi)|$ are less than $2^{-(n+3)}\beta_n$. Therefore

$$\Big[\sum_1^{\infty}\beta_i g_i(y)-\overline{\lim}\, g_i(y)\Big] > \Big[\sum_1^{\infty}\beta_i g_i(\xi)-\underline{\lim}\, g_i(\xi)\Big],$$
$$\Big[\sum_1^{\infty}\beta_i g_i(y)-g(y)\Big] > \Big[\sum_1^{\infty}\beta_i g_i(\xi)-g(\xi)\Big],$$

or $G(y) > G(\xi)$. Therefore if G attains its sup on the unit sphere at u, then $g_n(u) > \frac14+2^{-n} > \frac14$ for all n and therefore $g(u) \geqslant \frac14$. We would then have

$$G(u) \leqslant \sum_1^{\infty}\beta_i - \tfrac14,$$

but it follows from (A) of the lemma and $g(z_n) = 0$ that

$$G(z_n) = \sum_{1}^{\infty} \beta_i g_i(z_n) > \frac{79}{80} \sum_{1}^{n} \beta_i - \sum_{n+1}^{\infty} \beta_i \to \frac{79}{80} \sum_{1}^{\infty} \beta_i \quad \text{as} \quad n \to \infty.$$

Therefore

$$\frac{79}{80} \sum_{1}^{\infty} \beta_i \leqslant G(u) \leqslant \sum_{1}^{\infty} \beta_i - \frac{1}{4},$$

which implies that $\sum_{1}^{\infty} \beta_i \geqslant 20$. However, $\sum_{1}^{\infty} \beta_i < 10$.

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HARVEY MUDD COLLEGE, CLAREMONT, CALIFORNIA
THE INSTITUTE FOR ADVANCED STUDY, PRINCETON, N. J.

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Generalized convolutions

by

K. URBANIK (Wrocław)

Introduction

Let $\mathfrak{P}$ be the class of all probability measures defined on Borel subsets of the positive half-line. By E_a $(a \geqslant 0)$ we shall denote the probability measure concentrated at the point a. For any positive number a we define a transformation T_a of $\mathfrak{P}$ onto itself by means of the formula $(T_a P)(\mathscr{A}) = P(a^{-1}\mathscr{A})$, where $P \in \mathfrak{P}$, $\mathscr{A}$ is a Borel set and $a^{-1}\mathscr{A} = \{a^{-1}x\colon x \in \mathscr{A}\}$. Of course, the family T_a $(a > 0)$ forms a group under composition and $T_a T_b = T_{ab}$ $(a, b > 0)$. Further, we define the transformation T_0 by assuming $T_0 P = E_0$ for all P from $\mathfrak{P}$. It is very easy to verify that for every bounded continuous function f the equation

$$\int_{0}^{\infty} f(x)(T_a P)(dx) = \int_{0}^{\infty} f(ax) P(dx) \qquad (a \geqslant 0,\ P \in \mathfrak{P}) \tag{1}$$

holds.

We say that a sequence $P_1, P_2, \ldots$ of probability measures is *weakly convergent* to a probability measure P, in symbols $P_n \to P$, if for every bounded continuous function f the equation

$$\lim_{n \to \infty} \int_{0}^{\infty} f(x) P_n(dx) = \int_{0}^{\infty} f(x) P(dx)$$

holds. From this definition of weak convergence and from (1) it follows that

(*) $T_{a_n} P_n \to T_a P$ whenever $a_n \to a$ and $P_n \to P$.

In particular,

(**) if $a_n \to 0$ and $P_n \to P$, then $T_{a_n} P_n \to E_0$.

A commutative and associative $\mathfrak{P}$-valued binary operation $\circ$ defined on $\mathfrak{P}$ is called a *generalized convolution* if it satisfies the following conditions:

(i) the measure E_0 is a unit element, i.e. $E_0 \circ P = P$ for all $P \in \mathfrak{P}$;