STUDIA MATHEMATICA, T. XXIV. (1964)

Intermediate spaces and interpolation, the complex method

by

A. P. CALDERÓN (Chicago, Ill.)*

Introduction. In this paper we discuss in detail the so called *complex method of interpolation* which was originally introduced by the author in [1], and later, independently, by J. L. Lions in [5].

The paper consists mainly of two parts. The first one comprises paragraphs 1 to 12 and is devoted to those properties of the intermediate spaces which are consequences of properties of the interpolating spaces. In the study of duality we are let to a second complex method of interpolation yielding intermediate spaces $[B_0, B_1]^s$ (see paragraph 5). This method, which is closely related to but not identical with the first, is also discussed in some detail in the first part. The rest is devoted to the determination of the spaces intermediate between given ones. We study extensively the spaces between Banach lattices of functions, these functions being allowed to take values in a Banach space. We also consider the problem of interpolating between various classes of Hölder continuous functions, continuously differentiable functions etc. This we accomplish by considering a general class $\Lambda(B, X)$ and obtaining a suitable representation for functions in this class (see paragraph 14).

The presentation of the material is arranged as follows: paragraphs 1 to 14 contain the most important definitions and the statement of results. The remaining paragraphs contain the proofs. Thus the statements of paragraph x, $1 \leqslant x \leqslant 14$, are proved in paragraph $x+20$.

Throughout this paper we make systematic use of functions with values in a Banach space. We refer the reader to [2] for the general analytical facts about such functions.

The reader interested in other methods of interpolation can consult the work of N. Aronszajn, E. Gagliardo, C. Foias, S. Krein, J. L. Lions and J. Peetre. The method of S. G. Krein (see [3] and [4]) is closely related to the one discussed in this paper.

* This work was partly supported by the N. S. F. grant. GP.-574.

The spaces introduced in 13.4 generalize a class originally introduced by G. G. Lorentz (see [6]).

The spaces Λ in paragraph 14 include those studied by M. H. Taibleson in his dissertation, where, among other things, he solves the problem of interpolation between his spaces.

The reader may also be interested in the paper of E. M. Stein [7], where he generalized the classical convexity theorem of M. Riesz to operators depending on a complex parameter, and the paper of E. M. Stein and G. Weiss [8] concerning interpolation between L^p spaces with respect to different measures.

1. An *interpolation pair* (B^0, B^1) is a pair of complex Banach spaces B^0, B^1, continuously embedded in a complex topological vector space V. If x is an element of B^i, $i=0,1$, we denote its norm by $\|x\|_i$ or $\|x\|_{B^i}$. If in $B^0\cap B^1$ we introduce the norm

$$\|x\|_{B^0\cap B^1}=\max(\|x\|_0, \|x\|_1),$$

then $B^0\cap B^1$ becomes a Banach space.

Similarly, if we consider the space B^0+B^1 and introduce in it the norm

$$\|x\|_{B^0+B^1}=\inf[\|y\|_0+\|z\|_1],$$

where the infimum is taken over all pairs y, z, $y\in B^0$, $z\in B^1$, such that $y+z=x$, then B^0+B^1 also becomes a Banach space.

Since B^0 and B^1 are continuously embedded in V, it is evident that $B^0\cap B^1$ and B^0+B^1 are also continuously embedded in V.

2. Given an interpolation pair we consider functions $f(\xi)$, $\xi= =s+it$ defined in the strip $0\leqslant s\leqslant 1$ of the ξ-plane, with values in B^0+B^1 continuous and bounded with respect to the norm of B^0+B^1 in $0\leqslant s\leqslant 1$ and analytic in $0<s<1$, and such that $f(it)\in B^0$ is B^0-continuous and tends to zero as $|t|\to\infty$, $f(1+it)\in B^1$ is B^1-continuous and tends to zero as $|t|\to\infty$. In this linear space of functions which we denote by $\mathscr{F}(B^0, B^1)$ we introduce the norm.

$$\|f\|_{\mathscr{F}}=\max[\sup_t\|f(it)\|_0, \sup_t\|f(1+it)\|_1].$$

Then $\mathscr{F}$ becomes a Banach space.

3. Given a real number s, $0\leqslant s\leqslant 1$, we consider the subspace $B_s=[B^0, B^1]_s$ of B^0+B^1 defined by $B_s=\{x \mid x=f(s), f\in\mathscr{F}(B^0, B^1)\}$ and introduce in it the norm

$$\|x\|_s=\|x\|_{B_s}=\inf\|f\|_{\mathscr{F}}, \quad f(s)=x.$$

Then B_s is a Banach space continuously embedded in B^0+B^1. Furthermore, if $\mathscr{N}_s$ denotes the subspace of $\mathscr{F}(B^0, B^1)$ consisting of all functions $f(\xi)$ which vanish at $\xi=s$, then $\mathscr{N}_s$ is closed and the quotient space $\mathscr{F}(B^0, B^1)/\mathscr{N}_s$ is isomorphic and isometric with B_s. This isometry is effected by the mapping $\mathscr{F}(B^0, B^1)\to B_s$ defined by $f\to f(s)$.

If $x\in B^0\cap B^1$, then $\|x\|_s\leqslant\|x\|_{B^0+B^1}$. This is seen at once by taking $f(\xi)=x$.

4. Later we will state a more general theorem but this one is an immediate consequence of the definitions. Let (B^0, B^1) and (C^0, C^1) be two interpolation couples. Let L be a linear mapping from B^0+B^1 to C^0+C^1 such that $x\in B^i$ implies $L(x)\in C^1$ and

$$\|L(x)\|_{C^i}\leqslant M_1\|x\|_{B^i}, \quad i=0,1.$$

Then $x\in B_s=[B^0, B^1]_s$ implies $L(x)\in C_s=[C^0, C^1]_s$ and

$$\|Lx\|_{C_s}\leqslant M_0^{1-s}M_1^s\|x\|_{B_s}.$$

5. Now we introduce a new space $\overline{\mathscr{F}}$ of analytic functions. It consists of functions $f(\xi)$ defined in $0\leqslant s\leqslant 1$, with values in B^0+B^1 with the following properties:

i) $\|f(\xi)\|_{B^0+B^1}\leqslant c(1+|\xi|)$,

ii) $f(\xi)$ is continuous in $0\leqslant s\leqslant 1$,

iii) $f(\xi)$ is analytic in $0<s<1$,

iv) $f(it_1)-f(it_2)$ has values in B^0 and $f(1+it_1)-f(1+it_2)$ in B^1 for any $-\infty<t_1<t_2<\infty$ and

$$\max\left[\sup_{t_2,t_1}\left\|\frac{f(it_2)-f(it_1)}{t_2-t_1}\right\|_{B^0}, \quad \sup_{t_1,t_2}\left\|\frac{f(1+it_2)-f(1+it_1)}{t_2-t_1}\right\|_{B^1}\right]=\|f\|_{\overline{\mathscr{F}}}<\infty.$$

With the norm introduced in iv) $\overline{\mathscr{F}}$ reduced modulo constant functions becomes a Banach space.

6. Given a real number s, $0<s<1$, we consider the subspace $B^s=[B^0, B^1]^s$ of B^0+B^1 defined by

$$B^s=\{x \mid x=\frac{df}{d\xi}(s), f\in\overline{\mathscr{F}}(B^0, B^1)\}$$

with the norm

$$\|x\|_{B^s}=\inf\|f\|_{\overline{\mathscr{F}}}, \quad \frac{df}{d\xi}(s)=x.$$

Then B^s is a Banach space continuously embedded in B^0+B^1. If $\overline{\mathscr{N}}_s$ is the subspace of $\overline{\mathscr{F}}$ consisting of all functions $f\in\overline{\mathscr{F}}$ such that $\frac{df}{d\xi}(s)=0$, then $\overline{\mathscr{N}}_s$ is a closed subspace of $\overline{\mathscr{F}}$ and B^s is isometric with $\overline{\mathscr{F}}/\overline{\mathscr{N}}_s$.

The isometry is effected by the mapping $\overline{\mathscr{F}} \to B^0+B^1$ defined by $f \to \dfrac{df}{d\xi}(s)$.

7. If L is a linear mapping from B^0+B^1 to C^0+C^1 which maps B^i continuously into C^i $(i = 0, 1)$, then it also maps B^s continuously into C^s. More precisely, if M_i is the norm of L as a linear mapping from B^i to C^i, then its norm as a linear mapping from B^s to C^s does not exceed $M_0^{1-s}M_1^s$.

8. The interpolation methods yielding B^s and B_s, although closely related, are not identical as will be seen in 13.6.

Nevertheless we have the following relations between these spaces and their norms (see 9.5):

$$B_s \subset B^s, \qquad \|x\|_{B_s} \geqslant \|x\|_{B^s} \quad \text{for} \quad x \in B_s.$$

9.1. Let (B^0, B^1) be an interpolation pair and let V be a topological vector space in which B^0 and B^1 are continuously embedded. Let $f(\xi)$ be a function with values in V defined in $0 \leqslant s \leqslant 1$ such that for every continuous linear functional l in a separating family of such functionals $l[f(\xi)]$ is continuous in $0 \leqslant s \leqslant 1$, analytic in $0 < s < 1$ and representable as the Poisson integral of its values on the boundary of the strip $0 \leqslant s \leqslant 1$. (For this it is sufficient that $l[f(\xi)]$ be $O[e^{\pi|\xi|(1-\varepsilon)}]$ for some $\varepsilon > 0$ as $|\xi| \to \infty$.) Suppose in addition that

i) if $f(it) \in B^0$ is continuous and tends to zero as $|t| \to \infty$ and $f(1+it) \in B^1$ is continuous and tends to zero as $|t| \to \infty$, then $f \in \mathscr{F}(B^0, B^1)$;

ii) if $[f(it_1)-f(it_2)]/(t_1-t_2)$ belongs to B^0 and has bounded norm, $[f(1+it_1)\cdot f(1+it_2)]/(t_1-t_2)$ belongs to B^1 and has bounded norm, then $f \in \overline{\mathscr{F}}(B^0, B^1)$.

9.2. Consider the functions in $\mathscr{F}(B^0, B^1)$ of the form

$$f(\xi) = e^{\delta\xi^2}\sum_{n=1}^{N} x_n e^{\lambda_n \xi},$$

where $x_n \in B^0 \cap B^1$, λ_n is real and $\delta > 0$. The set of such functions and the set of their linear combinations which we will denote by $\mathscr{G}(B^0, B')$ is dense in $\mathscr{F}(B^0, B^1)$.

9.3. An immediate consequence of the preceding statement is the following: $B^0 \cap B^1$ is dense in $B_s = [B^0, B^1]_s$, $0 \leqslant s \leqslant 1$. Furthermore, $B_0 = [B^0, B^1]_0$ is a closed subspace of B^0 and its norm coincides with that of B^0. We also have $\mathscr{F}(B^0, B^1) = \mathscr{F}(B_0, B_1)$, $[B^0, B^1]_s = [B_0, B_1]_s$, $0 \leqslant s \leqslant 1$, and $\overline{\mathscr{F}}(B^0, B^1) = \overline{\mathscr{F}}(B_0, B_1)$, $[B^0, B^1]^s = [B_0, B_1]^s$.

9.4. Functions in $\mathscr{F}$ satisfy certain inequalities which have various applications in this theory. Let us denote by $\mu_0(\xi, t)$, $\mu_1(\xi, t)$ the Poisson kernels for the strip $0 \leqslant s \leqslant 1$. They can be obtained readily from the Poisson kernel for the halfplane by mapping conformally the halfplane onto the strip. Explicitly these kernels are

$$\mu_0(\xi, \tau) = \frac{e^{-\pi(\tau-t)}\sin\pi s}{\sin^2\pi s+[\cos\pi s - e^{-\pi(\tau-t)}]^2}, \qquad \xi = s+it,$$

$$\mu_1(\xi, \tau) = \frac{e^{-\pi(\tau-t)}\sin\pi s}{\sin^2\pi s+[\cos\pi s + e^{-\pi(\tau-t)}]^2}, \qquad \xi = s+it.$$

Then for every function f in $\mathscr{F}(B^0, B^1)$ we have the following inequalities:

$$\text{i)}\ \log\|f(s)\|_{B_s} \leqslant \int_{-\infty}^{+\infty}[\log\|f(it)\|_{B^0}]\mu_0(s, t)\,dt + \int_{-\infty}^{+\infty}[\log\|f(1+it)\|_{B^1}]\mu_1(s, t)\,dt, \qquad 0 < s < 1;$$

$$\text{ii)}\ \|f(s)\|_{B_s} \leqslant \left[\frac{1}{1-s}\int_{-\infty}^{+\infty}\|f(it)\|_{B^0}\mu_0(s, t)\,dt\right]^{1-s} \times \left[\frac{1}{s}\int_{-\infty}^{+\infty}\|f(1+it)\|_{B^1}\mu_1(s, t)\,dt\right]^{s};$$

$$\text{iii)}\ \|f(s)\|_{B_s} \leqslant \int_{-\infty}^{+\infty}\|f(it)\|_{B^0}\mu_0(s, t)\,dt + \int_{-\infty}^{+\infty}\|f(1+it)\|_{B^1}\mu_1(s, t)\,dt.$$

The following is a consequence of i).

Let $f_n \in \mathscr{F}(B^0, B^1)$ be such that

$$\int_{-\infty}^{+\infty}[\log^+\|f_n(it)\|_{B^0} + \log^+\|f_n(1+it\|_{B^1}]e^{-\pi|t|}\,dt$$

is bounded and suppose that $f_n(it)$ tends to zero in B^0 for all t in a set of positive measure; then $\|f_n(s)\|_s \to 0$ for $0 < s < 1$.

9.5. Let $f \in \overline{\mathscr{F}}(B^0, B^1)$ have the property that for all t in set of positive measure

$$\frac{1}{h}[f(it+ih)-f(it)]$$

converges in B^0 as h tends to zero. Then

$$f'(s) \in B_s = [B^0, B^1]_s, \qquad 0 < s < 1.$$

In particular, if B^0 is reflexive this limit exists for almost all t and any $f \in \overline{\mathscr{F}}(B^0, B^1)$ and consequently $[B^0, B^1]_s = [B^0, B^1]^s$. Furthermore, the norms of these spaces coincide. In the general case we have $[B^0, B^1]_s \subset [B^0, B^1]^s$ and if $x \in [B^0, B^1]_s$, then $\|x\|_{B_s} \geqslant \|x\|_{B^s}$.

9.6. The following property of interpolation pairs is useful in establishing complete continuity of certain operators. Suppose there exists a directed set of operators π_λ on B^0+B^1 such that $\pi_\lambda(B^i) \subset B^i$ and the norms of π_λ as operators on B^i are bounded, $i = 0, 1$; $\pi_\lambda(B^0)$ is finite dimensional and, for every $x \in B^0$, $\|\pi_\lambda x - x\|_{B^0} \to 0$.

Let K be a compact subset of B^0, c a constant and E a set of positive measure of the line. Then for each s, $0 < s < 1$, there exists a compact subset K_s of B_s such that $f \in \mathscr{F}(B^0, B^1)$, $|||f||| \leqslant c$ and $f(it) \in K$ for $t \in E$ imply that $f(s) \in K_s$.

10.1. Let (A_i, B_i), $i = 1, 2, \ldots, n$, and (A, B) be interpolation pairs. Let $L(x_1, \ldots, x_n)$, $x_i \in A_i \cap B_i$, be a multilinear operation defined in the direct sum $\bigoplus_{i=1}^{n} (A_i \cap B_i)$ with values in $A \cap B$ and such that

$$\|L(x_1, x_2, \ldots, x_n)\|_A \leqslant M_0 \prod_1^n \|x_i\|_{A_i}, \qquad \|L(x_1, x_2, \ldots, x_n)\|_B \leqslant M_1 \prod_1^n \|x_i\|_{B_i}.$$

Then if $C = [A, B]_s$ and $C_i = [A_i, B_i]_s$ we have

$$\|L(x_1, x_2, \ldots, x_n)\|_C \leqslant M_0^{1-s} M_1 \prod_2^n \|x_i\|_{C_i}, \qquad 0 \leqslant s \leqslant 1,$$

and thus L can be extended continuously to a multilinear mapping of $\bigoplus_1^n C_i$ into C.

10.2. The preceding result can be extended in the following fashion. Let $\mathscr{M}$ be the space of multilinear mappings L of $\bigoplus_1^n A_i \cap B_i$ into $A+B$ with the norm

$$\|L\| = \sup \|L(x_1, \ldots, x_n)\|_{A+B}, \qquad \|x_i\|_{A_i \cap B_i} \leqslant 1,$$

and $\mathscr{M}_0$ and $\mathscr{M}_1$, the spaces of multilinear mappings of $\bigoplus_1^n (A_i \cap B_i)$ into A and B respectively with the norms

$$\|L\|_0 = \sup \|L(x_1, \ldots, x_n)\|_A, \qquad \|x_i\|_{A_i} \leqslant 1,$$

and

$$\|L\|_1 = \sup \|L(x_1, \ldots, x_n)\|_B, \qquad \|x_i\|_{B_i} \leqslant 1.$$

Then $\mathscr{M}$, $\mathscr{M}_0$ and $\mathscr{M}_1$ are Banach spaces and we have $\mathscr{M}_1 \subset \mathscr{M}$ and $\|L\|_i \geqslant \|L\|$, that is, the $\mathscr{M}_i$ are embedded continuously in $\mathscr{M}$. Consider now the multilinear mapping $\mathscr{L}(L, x_1, \ldots, x_n)$, $L \in \mathscr{M}$, $x_i \in A_i \cap B_i$, into $A+B$, defined by

$$\mathscr{L}(L, x_1, \ldots, x_n) = L(x_1, \ldots, x_n).$$

Then if $L \in \mathscr{N} = [\mathscr{M}_0, \mathscr{M}_1]_s$, we have $L(x_1, \ldots, x_n) \in C = [A, B]_s$ and

$$\|L(x_1, \ldots, x_n)\|_C \leqslant \|L\|_{\mathscr{N}} \prod_1^n \|x_i\|_{C_i}$$

and L can be extended to a bounded multilinear mapping of $\bigoplus_1^n C_i$ into C. In other words, $[\mathscr{M}_0, \mathscr{M}_1]_s$ is contained in the space of bounded multilinear mappings of $\bigoplus_1^n [A, B]_s$ into $[A, B]_s$ and by the preceding inequality the embedding is continuous.

10.3. Sometimes it is desirable to establish the boundedness of a multilinear mapping under less restrictive assumptions than those of the preceding paragraph. Suppose L is a multilinear mapping defined on $\bigoplus_1^n (A_i \cap B_i)$ with values in a locally convex topological vector space V in which A and B are continuously embedded. Suppose there exists a sequence of functions $\{F_k\}$, $F_k \in \mathscr{F}(\mathscr{M}_0, \mathscr{M}_1)$, with the following properties:

i) for a given s, $0 < s < 1$, every continuous linear functional l on V and $x_i \in A_i \cap B_i$, we have,

$$\lim_{k\to\infty} l\,[F_k(s)(x_1, \ldots, x_n) - L(x_1, \ldots, x_n)] = 0.$$

ii) $$\int_{-\infty}^{+\infty} \log^+ \|F_k(it)\|_{\mathscr{M}_0} \mu_0(s, t)\, dt + \int_{-\infty}^{+\infty} \log^+ \|F_k(1+it)\|_{\mathscr{M}_1} \mu_1(s, t)\, dt \leqslant c$$

where μ_0 and μ_1 are defined as in 9.4.

iii) for t in a set of positive measure $F_k(it)$ converges in $\mathscr{M}_0$. Then $L(x_1, \ldots, x_n) \in C = [A, B]_s$ and

$$\|L(x_1, \ldots, x_n)\|_C \leqslant e^c \prod_1^n \|x_i\|_{C_i}, \qquad C_i = [A_i, B_i]_s.$$

10.4. Let L be a bounded multilinear mapping of $\bigoplus_1^n A_i$ into A. We will say that L is *completely continuous* if the image of the set $\|x_i\|_{A_i} \leqslant 1$ is a totally bounded subset of A. As in the case of completely continuous linear operations, the set of completely continuous multilinear operations from $\bigoplus_1^n A_i$ to A is a closed subspace of the space of all bounded multilinear operations from $\bigoplus_1^n A_i$ to A.

Referring to 10.2, let $F(\xi)\in\mathscr{F}(\mathscr{M}_0, \mathscr{M}_1)$ be such that $F(it)$ is a completely continuous multilinear mapping of $\bigoplus_1^n A_i$ into A for all t in a set E of positive measure of the line. Suppose in addition that the pair (A, B) has the property 9.6. Then for each s, $0<s<1$, $F(s)$ is a completely continuous multilinear map from $\bigoplus_1^n [A_i, B_i]_s$ into $[A, B]_s$.

10.5. Let (A, B) be an interpolation pair and suppose that A and B are Banach algebras with the property that multiplications in A and B coincide in $A\cap B$. Then $A\cap B$ is a subalgebra of both A and B and for each s, $0<s<1$, and $x, y\in A\cap B$ we have

$$\|xy\|_C \leqslant \|x\|_C\|y\|_C, \qquad C=[A, B]_s,$$

so that multiplication can be extended continuously to C, which then becomes a Banach algebra. This statement is a special case of 10.1.

11.1. Let $L(x_1, \ldots, x_n)$ be a multilinear mapping defined for $x_1 = A_1+B_1$, $x_i = A_i\cap B_i$ $(i=2,3,\ldots,n)$ with values in $A+B$. Suppose that for $x_1\in A_1$, $x_1\in B_1$ we have $L(x_1, \ldots, x_n)\in A$, $L(x_1, \ldots, x_n)\in B$ and

1) $\|L(x_1, \ldots, x_n)\|_A \leqslant M_0\prod_1^n \|x_i\|_{A_i}$,

2) $\|L(x_1, \ldots, x_n)\|_B \leqslant M_1\prod_1^n \|x_i\|_{B_i}$

respectively. Then for $x_1\in C_1 = [A_1, B_1]^s$, $0<s<1$, we have $L(x_1, \ldots, x_n)\in C = [A, B]^s$ and

$$\|L(x_1, \ldots, x_n)\|_C \leqslant M_0^{1-s}M_1^s\prod_1^n \|x_i\|_{C_i}$$

where again $C_1 = [A_1, B_1]^s$ and $C_i = [A_i, B_i]_s$, $i=2,\ldots,n$. Thus L can be extended continuously to a multilinear mapping of $\bigoplus_1^n C_i$ into $C=[A, B]^s$ with norm not exceeding $M_0^{1-s}M_1^s$.

11.2. Given the interpolation pairs (A, B) and (A_j, B_j), $j=1, 2, \ldots, n$, let $\mathscr{M}$ be the space of bounded multilinear mappings L of $\bigoplus_1^n (A_i\cap B_i)$ into $A+B$ and $\mathscr{M}_0$ and $\mathscr{M}_1$ the subspaces of $\mathscr{M}$ consisting of the multilinear mappings in $\mathscr{M}$ which map $\bigoplus_1^n (A_i\cap B_i)$ boundedly into A and B respectively with the corresponding norms. Evidently $\mathscr{M}_0$ and $\mathscr{M}_1$ are continuously embedded in $\mathscr{M}$ and are an interpolation pair. Then the space $\mathscr{N} = [\mathscr{M}_0, \mathscr{M}_1]^s$ consists of multilinear mappings L mapping $\bigoplus_1^n (A_i\cap B_i)$ into $C=[A, B]^s$ and such that

$$\|L(x_1, \ldots, x_n)\|_C \leqslant \|L\|_{\mathscr{N}}\prod_1^n \|x_j\|_{C_j}$$

where $C_j = [A_j, B_j]_s$. Evidently each such L can be extended uniquely to a multilinear bounded mapping of $\bigoplus_1^n [A_i, B_i]_s$ into $[A, B]^s$.

12.1. In this paragraph we discuss the dual of $[A, B]_s$. Without loss of generality we may assume that $A\cap B$ is dense in both A and B. We denote by $A^*, B^*, \ldots$ the duals of $A, B, \ldots$ If y is an element of A^*, its value at x, $x\in A$, will be denoted by $\langle x, y\rangle$. By restricting bounded linear functionals on A and B to $A\cap B$ we obtain continuous embedding of A^* and B^* in $(A\cap B)^*$. Let $C=[A, B]_s$ and $C' = [A^*, B^*]^s \subset (A\cap B)^*$. Then for $y\in C'$ and $x\in A\cap B$ we have

$$\langle x, y\rangle \leqslant \|x\|_C\|y\|_{C'},$$

that is, y is continuous with respect to the norm of C on $(A\cap B)^*$. Since $A\cap B$ is dense in C, this linear functional can be extended uniquely to a continuous linear functional l_y on C, with $\|l_y\|_{C^*}\leqslant\|y\|_{C'}$. Conversely every continuous linear functional l on C is of this form, that is $l(x) = l_y(x)$ for some $y\in C'$ with $\|y\|_{C'}\leqslant\|l\|_{C^*}$. This y is uniquely determined by l and thus $C' = [A^*, B^*]^s$ is isomorphic and isometric with C^*.

Given $y\in C'$ and $x\in C$ the value of $l_y(x)$ can be calculated as follows: let $f\in\mathscr{F}(A, B)$, $f(s)=x$ and $g\in\overline{\mathscr{F}}(A^*, B^*)$, $g'(s)=y$; then

$$l_y(x) = -i\int_{-\infty}^{+\infty}\langle f(it)\mu_0(s, t), dg(it)\rangle - i\int_{-\infty}^{+\infty}\langle f(1+it)\mu_1(s, t)\, dg(1+it)\rangle$$

where $\mu_0(s, t)$ and $\mu_1(s, t)$ are the Poisson kernels in 9.4 and the integrals are to be interpreted as explained in 32.1.

12.2. If one of the two spaces A, B is reflexive, the same is true for $C=[A, B]_s$, $0<s<1$.

12.3. Let $A_1 = [A, B]_\alpha$, $B_1 = [A, B]_\beta$, $0\leqslant\alpha\leqslant\beta\leqslant 1$, and let $(1-\sigma)\alpha+\sigma\beta = s$, $0\leqslant\sigma\leqslant 1$. Then $[A, B]_s$ and $[A_1, B_1]_\sigma$ and their norms coincide provided that $(A\cap B)^*$ is dense in $A_1\cap B_1$ with respect to the norm of $A_1\cap B_1$. This condition is automatically satisfied if $A\supset B$ or $B\supset A$.

In the following paragraphs we discuss interpolation in spaces constructed by means of lattices of measurable functions. These spaces include the complex Koethe-Banach spaces and in particular the Lebesgue, Orlicz and Lorentz spaces and many of their generalizations. We will first develop that part of the theory of Banach lattices which is necessary for the formulation and derivation of our results.

13.1. Consider a totally σ-finite measure space $\mathscr{M}$ and the class of real valued measurable functions on $\mathscr{M}$. Two functions coinciding almost everywhere will be identified, and relationships between functions will mean relationships between the values of the functions which are valid almost everywhere. A subclass X of measurable functions is a *Banach lattice* on $\mathscr{M}$ if it has the following properties:

i) X is linear;

ii) there is a norm defined in X with respect to which it is complete;

iii) $f \in X$ and $|g| \leqslant |f|$ imply that $g \in X$ and $\|g\|_X \leqslant \|f\|_X$. Evidently $\|f\|_X = 0$ implies that f equals the zero functions or $f = 0$, where 0 denotes the zero function.

13.2. Let $\mu(x)$ be a positive integrable function on $\mathscr{M}$ and for any two measurable functions f and g on $\mathscr{M}$ let

$$d(f, g) = \int_{\mathscr{M}} \frac{|f-g|}{1+|f-g|}\mu(x)\,dx.$$

Then, if we introduce the distance function $d(f, g)$, the space of all measurable functions on $\mathscr{M}$ becomes a complete metric vector space V, in which X is continuously embedded; in other words, $\|f_n - f\|_X \to 0$ implies $d(f_n, f) \to 0$.

Let $f_n \in X$ be such that $\sum \|f_n\|_X < \infty$. Then there exists an element f of X such that

$$\lim_{N\to\infty} \Big\| \sum_1^N f_n - f \Big\|_X = 0,$$

the series $\sum f_n(x)$ is absolutely convergent and its sum is $f(x)$ for almost all $x \in \mathscr{M}$.

13.3. Given a Banach lattice X, others can be constructed by various methods.

Let $\varphi(x, t)$ be a function which for each $x \in \mathscr{M}$ is a concave increasing function of t in $0 \leqslant t < \infty$ vanishing at $t = 0$ (no measurability assumptions on $\varphi(x, t)$ are necessary here), and consider the class, which we will denote by $\varphi(X)$, consisting of all measurable functions $g(x)$ on $\mathscr{M}$ such that $|g(x)| \leqslant \lambda\varphi[x, f(x)]$ for some $f(x) \in X$, $\|f\|_X \leqslant 1$ and $\lambda > 0$. Define the norm $\|g\|_{\varphi(X)}$ of g as the greatest lower bound of the values of λ for which an inequality like the preceding holds. Then $\varphi(X)$ becomes a Banach lattice.

For example if X is $L^1(\mathscr{M})$ and $\varphi(x, t) = \big(\mu(x)^{-1/p}\big)(t^{1/p})$, where $\mu(x)$ is a positive measurable function of x, then, as readily verified $\varphi(X)$ is the $L^p(\mathscr{M})$ with respect to the weight function $\mu(x)$.

Suppose on the other hand that $\varphi(x, t) = \varphi(t)$ where $\varphi(t)$ is the inverse of the convex increasing function $\Phi(t)$. Then, if again $X = L^1(\mathscr{M})$, $\varphi(X)$ is the Orlicz space L_Φ (see the definition of L_Φ in [10], (I), p. 173).

13.4. Given a measurable function f on $\mathscr{M}$ which is integrable on sets of finite measure, we associate with it the function $f^{**}(t)$, $0 < t < \infty$, defined by

$$f^{**}(t) = \frac{1}{t}\sup\int_E |f|\,dx,$$

where the supremum is taken over all measurable sets E in $\mathscr{M}$ of measure $\leqslant t$. Then we have

i) if λ is a constant, then $(\lambda f)^{**} = |\lambda| f^{**}$;

ii) $(f+g)^{**} \leqslant f^{**} + g^{**}$;

iii) if $f \geqslant 0$, $g \geqslant 0$ and $0 < s < 1$, then $(f^s g^{1-s})^{**} \leqslant (f^{**})^s (g^{**})^{1-s}$;

iv) if $f \geqslant 0$, and φ is a concave non-negative function in $0 \leqslant t < \infty$, $\varphi(0) = 0$, then $\varphi(f)^{**} \leqslant \varphi(f^{**})$.

Let now X be a Banach lattice on the halfline $0 < t < \infty$; we denote by X^* the class of measurable functions f on $\mathscr{M}$ such that $f^{**} \in X$ and introduce in X^* the norm $\|f\|_{X^*} = \|f^{**}\|_X$. Then X^* becomes a Banach lattice. For example, if X is the space of functions $g(t)$, $0 < t < \infty$, such that

$$\|g\|_X = \Big[\int_0^\infty |g(t)|^q t^{q/p-1} dt\Big]^{1/q} < \infty, \qquad 1 < p < \infty, 1 \leqslant q \leqslant \infty,$$

then X^* is the Lorentz space $L_{p,q}$.

13.5. Let now X_1 and X_2 be two Banach lattices on $\mathscr{M}$; they are both continuously embedded in the space of all measurable functions on $\mathscr{M}$ with the metric introduced in 13.2, and consequently $X_1 + X_2$ and $X_1 \cap X_2$ with the norms introduced in section 1 are also Banach spaces. Furthermore, they are Banach lattices on $\mathscr{M}$.

Let $0 < s < 1$ and consider the class X of functions f such that $|f(x)| \leqslant \lambda |g(x)|^{1-s} |h(x)|^s$ for some constant $\lambda > 0$ and some $g \in X_1$, $h \in X_2$ with $\|g\|_{X_1} \leqslant 1$ and $\|h\|_{X_2} \leqslant 1$, and let $\|f\|_X$ be the greatest lower bound of the values of λ for which such an inequality holds. Then X, which we will denote also by $X_1^{1-s} X_2^s$, is a Banach lattice on $\mathscr{M}$. This construction arises naturally in interpolation and for this reason it may be of interest to obtain more direct descriptions of $X_1^{1-s} X_2^s$. This can be done for example in the following cases.

Let X be a Banach lattice, $\varphi_1(x,t)$ and $\varphi_2(x,t)$ two concave functions as described in 13.3. Then $\varphi_1(X)^{1-s}\varphi_2(X)^s$ is equivalent to $\varphi(X)$ where $\varphi(x,t)$ is the concave function $\varphi(x,t)=\varphi_1^{1-s}(x,t)\varphi_2^s(x,t)$. Here equivalence means that the two Banach lattices consist of the same elements but have merely equivalent norms.

In what follows we shall assume for simplicity that $\mathscr{M}$ is non-atomic. Let X_1 and X_2 be two Banach lattices on the halfline $0<t<\infty$, and let X_1^* and X_2^* be the lattices associated with X_1 and X_2 as in 13.4. Then $(X_1^*)^{1-s}(X_2^*)^s$ is contained in $(X_1^{1-s}X_2^s)^*$ and the inclusion map is continuous. Under some additional conditions the two lattices coincide and their norms are equivalent. One such condition is the following: consider the group of operators $H^sf(t)=e^{s/2}f(te^s)$, $-\infty<s<\infty$; this group should be strongly continuous in both X_1 and X_2 and satisfy the inequality $\|H^s\|\leqslant ce^{a|s|}$, $a<\frac{1}{2}$.

A second condition, which is a consequence of the preceding, is this: functions $f(t)$ in X_1 or X_2 should be integrable on finite intervals and the operators

$$\text{i)}\qquad S_1f=\frac{1}{t}\int_0^t f(s)\,ds,\qquad S_2f=\int_0^\infty \frac{f(s)}{s}\,ds$$

should be bounded in both X_1 and X_2.

For example, consider the lattice X of functions $f(x)$ in $0<x<\infty$ such that

$$\Big[\int_0^\infty |f(x)|^q x^{q/p-1}dx\Big]^{1/q}=\|f\|_X<\infty,\qquad 1<p<\infty,1\leqslant q\leqslant\infty.$$

Then if $\varphi(x,t)=x^{1/q-1/p}t^{1/q}$, according to 13.3 we have $X=\varphi(L^1)$ where L^1 is the space of functions integrable in $(0,\infty)$. We also have $\|H^sf\|_X=e^{(1/2-1/p)s}\|f\|_X$, that is X satisfies the first condition above. Let now X_1 and X_2 be two such spaces with indices p_1,q_1 and p_2,q_2, respectively. Then if $0<s<1$, $X_1^{1-s}X_2^s$ is also a space of the same kind with indices

$$\frac{1}{p}=\frac{1-s}{p_1}+\frac{s}{q_1},\qquad \frac{1}{q}=\frac{1-s}{q_1}+\frac{s}{q_2},$$

respectively.

Consequently $(X_1^*)^{1-s}(X_2^*)^s$ and $(X_1^{1-s}X_2^s)^*$ coincide and have equivalent norms. The lattices X_1^*, X_2^* and $(X_1^{1-s}X_2^s)^*$ are respectively the Lorentz spaces L_{p_1,q_1}, L_{p_2,q_2} and $L_{p,q}$. Thus $(L_{p_1,q_1})^{1-s}(L_{p_2,q_2})^s$ and $L_{p,q}$ coincide and have equivalent norms.

13.6. Let now B be a complex Banach space. A function on $\mathscr{M}$ with values in B is said to be *measurable* if it is the limit almost everywhere of a sequence of simple B-valued functions. A function with values in B is said to be *simple* if it takes finitely many values, each on a measurable subset of $\mathscr{M}$. Evidently, if $f(x)$ is B-valued measurable function, then $\|f(x)\|_B$ is a real valued measurable function.

Given a Banach lattice X on $\mathscr{M}$ we denote by $X(B)$ the class of B-valued measurable functions $f(x)$ such that $\|f(x)\|_B\in X$ and define $\|f\|_{X(B)}=\big\|(\|f(x)\|_B)\big\|_X$. With this norm $X(B)$ is a Banach space.

Let now (B_0,B_1) be an interpolation pair and X_0 and X_1 two Banach lattices on $\mathscr{M}$. Then $X_0(B_0)$ and $X_1(B_1)$ are continuously embedded in $(X_0+X_1)(B_0+B_1)$ and thus they also are an interpolation pair. Setting $X=X_0^{1-s}X_1^s$ and $B=[B_0,B_1]_s$, $0<s<1$, we have the following results:

i) $[X_0(B_0),X_1(B_1)]_s\subset X(B)$ and the inclusion is norm decreasing. These spaces and their norms coincide if X has the property that $f\in X$, $|f_n|\leqslant|f|$ and $f_n\to 0$ almost everywhere imply $\|f_n\|_X\to 0$.

ii) $X(B)\subset[X_0(B_0),X_1(B_1)]^s$ and the inclusion is norm decreasing. If the unit sphere of $X(B)$ is closed in $X_0(B_0)+X_1(B_1)$, these spaces and their norms coincide.

Combining i) and ii) with 9.5, we obtain the following result: if $X_0(B_0)$ is reflexive, then $[X_0(B_0),X_1(B_1)]_s=X(B)=[X_0(B_0),X_1(B_1)]^s$ and the norms of these spaces coincide.

With ii) we can construct an example showing that in general $[A,B]_s\neq[A,B]^s$. Let $B_0=B=B_1$ be the complex numbers, X_0 the bounded functions in $(0,\infty)$ and X_s the class of functions $f(x)$, $0<x<\infty$, such that $x^s|f(x)|$ is bounded, and let $\|f\|_{X_s}=\operatorname{ess\,sup}x^s|f(x)|$. Then one verifies readily that

$$X_0^{1-s}X_1^s=X_s$$

and that the unit sphere of $X_s(B)$ is closed in $X_0(B)+X_1(B)$. Consequently by ii) we have

$$X_s(B)=[X_0(B),X_1(B)]^s.$$

But, as easily seen, $X_0(B)\cap X_1(B)$ is not dense in $X_s(B)$ and by 9.3 this implies that

$$X_s(B)\neq[X_0(B),X_1(B)]_s.$$

14. This paragraph is devoted to interpolation of function spaces related to the spaces of Lipschitz functions in Euclidean space. For this purpose we study the spaces $\Lambda(B,X)$ below and establish some general results on them by which the interpolation problem is reduced to that of interpolation between the spaces discussed in 13.

14.1. Let B be a complex Banach space and τ_y, $y \in R^n$, a strongly continuous representation of R^n into a group of isometries of B, that is, such that for every $u \in B$, $\tau_y u$ is a B-valued continuous function of y.

Let X denote a Banach lattice of measurable functions on the half-line $(0, \infty)$ such that

i) functions in X are integrable on closed intervals contained in the open half-line $(0, \infty)$,

ii) there is a positive integer k associated with X such that the integrals

$$\int_0^t g(s)\frac{ds}{s}, \qquad \int_t^\infty g(s)\left(\frac{t}{s}\right)^k \frac{ds}{s}$$

are absolutely convergent for $g \in X$ and represent bounded operators on X.

Let now $\varphi(y)$ be an infinitely differentiable, spherically symmetric function in R^n with moments of orders less than k equal to zero. For $u \in B$ let $F(t) = Tu$ be the B-valued function of t, $0 < t < \infty$, defined by

$$Tu = t^{-n}\int (\tau_y u)\varphi\left(\frac{1}{t}y\right)dy.$$

The integral here is to be interpreted as Riemann vector valued integral, and Tu is then a continuous B-valued function of t. Now we introduce the space $\Lambda(B, X)$ as follows:

$$\Lambda(B, X) = \{u \mid u \in B, \|Tu\|_B = \|F(t)\|_B \in X\};$$

and define a norm in $\Lambda(B, X)$ by

$$\|u\|_\Lambda = \|u\|_B + \|(\|Tu\|_B)\|_X.$$

Then we have the following result: the space $\Lambda(B, X)$ is complete and its embedding in B is continuous. Furthermore, up to equivalence of norm $\Lambda(B, X)$ is independent of the choice of the function φ.

These spaces include several "classical" spaces as shown by the following result and examples.

iii) Suppose that X, in addition to i) and ii), has the property that the integral

$$t^r \int_0^t g(s)\frac{ds}{s^{\gamma+1}},$$

where r is an integer $0 \leqslant r < k$, is absolutely convergent for $g \in X$ and represents a bounded operator on X. Then $\Lambda(B, X)$ consists precisely of those elements u of B which have the following properties:

$\tau_y u$ is a bounded B-valued function of y with bounded continuous derivatives up to order r; if $m \geqslant k-r$ and

$$\Delta_y u = \sum_0^m \binom{m}{j}(-1)^j \tau_{jy} u,$$

then $t^r\|\Delta_{tz} w\|_B \in X$, where w is any derivative of $\tau_y u$ of order r at $y = 0$. Furthermore, the norm of $\Lambda(B, X)$ is equivalent to the smallest upper bound for the norms in B of $\tau_y u$ and its derivatives up to order r, and for $\|t^r\|\Delta_{tz} w\|_B\|_X$, where $|z| = 1$ and w is a derivative of order r of $\tau_y u$ at $y = 0$.

For example, let $B = L^p(R^n)$, $1 < p \leqslant \infty$, so that the elements of B are functions $u(x)$ on R^n. Let $\tau_y u = u(x-y)$. Let $k \geqslant 1$ be an integer and X the class of functions $g(t)$ such that $t^{-k+\alpha}g(t)$ is bounded, where α is fixed and $0 < \alpha < 1$, and let $\|g\|_X = \operatorname{ess\,sup}|(t^{-k+\alpha}g(t)|$. Then X satisfies the conditions of iii) with $r = k-1$. Consequently by setting $m = 1$ in iii) we find that $\Lambda(B, X)$ consists of functions in $L^p(R^n)$ with derivatives up to order $k-1$ in $L^p(R^n)$, and with derivatives w of order $k-1$ satisfying the condition that

$$t^{\alpha-1}\|w(x-tz) - w(x)\|_p$$

is uniformly bounded in t and z, $|z| = 1$.

If $\alpha = 1$ in the preceding example, then the conditions of iii) are satisfied with $k \geqslant 2$, $r = k-2$, $m \geqslant 2$, and

$$t^{-1}\|w(x-2tz) - 2w(x-tz) + w(x)\|_p$$

is uniformly bounded in t and z, $|z| = 1$.

Further examples can be constructed by replacing $L^p(R^n)$ by the class of continuous bounded functions in R^n. This yields classes of Lipschitzian functions or functions with Lipschitzian derivatives. Or one may use different spaces X; for example, the spaces studied by Taibleson in his dissertation [9] can be obtained by suitable choice of B and X.

14.2. As we have seen above, $\Lambda(B, X)$ is defined as the inverse image of $X(B)$ under the mapping T. Now we construct a mapping of the direct sum $X(B) \oplus B$ of $X(B)$ and B into $\Lambda(B, X)$ as follows.

Let $\psi_1(y)$ and $\psi_2(y)$ be two infinitely differentiable spherically symmetrically functions in R^n with compact support. Given $F \in X(B)$ and $u \in B$ let

$$\text{i)}\quad \mathcal{S}(F, u) = \int (\tau_y\, u)\psi_2(y)\,dy + \int_1^\infty t^{n-1}\left\{\int\left[\tau_y F\left(\frac{1}{t}\right)\psi_1(ty)\right]dy\right\}dt.$$

Then the integral on the right is absolutely convergent in B and $\mathscr{S}$ maps $X(B)\oplus X$ continuously into $\Lambda(B,X)$. Furthermore, given the function φ in 14.1, it is possible to select ψ_1 and ψ_2 in such a way that

$$\mathscr{S}(Tu,u)=u,$$

that is, if for $u\in\Lambda(B,X)$ we define $\mathscr{I}u=(Tu,u)$, then $\mathscr{I}$ maps $\Lambda(B,X)$ isometrically into $X(B)\oplus B$, $\mathscr{S}$ maps $X(B)\oplus B$ onto $\Lambda(B,X)$ and $\mathscr{S}$ is a left inverse of $\mathscr{I}$. The existence of such an operator $\mathscr{S}$ will permit to reduce interpolation between spaces $\Lambda(B,X)$ to interpolation between spaces $X(B)$.

An additional consequence of the existence of $\mathscr{S}$ is this. Consider the operator $\mathscr{I}\mathscr{S}$; since $\mathscr{S}$ is onto, it maps $X(B)\oplus B$ onto the range of $\mathscr{I}$, since $(\mathscr{I}\mathscr{S})^2=\mathscr{I}(\mathscr{S}\mathscr{I})\mathscr{S}=\mathscr{I}\mathscr{S}$, $\mathscr{I}\mathscr{S}$ is a projection and thus the range of $\mathscr{I}\mathscr{S}$, which coincides with the range of $\mathscr{I}$, is a complemented subspace of $X(B)\oplus B$. Consequently $\Lambda(B,X)$ is isometric with a complemented subspace of $X(B)\oplus B$.

14.3. Let now (B_0,B_1) be an interpolation pair and τ_y a representation of R^n in a group of linear transformations of B_0+B_1, such that τ_y restricted to B_i is a strongly continuous group of isometries of B_i, $i=0,1$. Let X_0,X_1 be two lattices of measurable functions on the halfline $(0,\infty)$ satisfying conditions i) and ii) of 14.1. Then, since $\Lambda(B_0,X_0)$ and $\Lambda(B_1,X_1)$ are continuously embedded in B_0 and B_1 respectively, and these are in turn continuously embedded in B_0+B_1, the pair $\Lambda(B_0,X_0)$, $\Lambda(B_1,X_1)$ is an interpolation pair and we have the following result:

The linear transformation τ_y restricted to $B=[B_0,B_1]_s$ is a strongly continuous representation of R^n into a group of isometries of B; the space $X=X_0^{1-s}X_1^s$, satisfies conditions i) *and* ii) *of* 14.1. *If $X(B)=[X_0(B_0),X_1(B_1)]_s$, then $\Lambda(B,X)=[\Lambda(B_0,X_0),\Lambda(B_1,X_1)]_s$ up to equivalence of norms. If $[B_0,B_1]^s=[B_0,B_1]_s$* and *$X(B)=[X_0(B_0),X_1(B_1)]^s$, then $\Lambda(B,X)=[\Lambda(B_0,X_0),\Lambda(B_1,X_1)]^s$ up to equivalence of norms.*

21. One merely has to verify completeness of $B^0\cap B^1$ under the norm described, all other required properties being evident. Suppose that $\{x_n\}$ is such that $\|x_n-x_m\|_{B^0\cap B^1}\to 0$ as $n,m\to\infty$. Then $\|x_n-x_m\|_0\to 0$ and $\|x_n-x_m\|_1\to 0$, i. e. x_n is a Cauchy sequence in both B^0 and B^1 and therefore converges to a limit in each of these spaces. But B^0 and B^1 are continuously embedded in V and therefore these limits are also limits of the sequence in V. Since limits in V are unique, it follows that these limits coincide. If x denotes this limit we have $x\in B^0$ and $x\in B^1$ and consequently $x\in B^0\cap B^1$. Furthermore, $\|x_n-x\|_0\to 0$ and $\|x_n-x\|_1\to 0$ and consequently x_n converges to x with respect to the norm of $B^0\cap B^1$.

Concerning B^0+B^1, the homogeneity of the norm introduced and the triangle property are readily verifiable. Let us show that $\|x\|_{B^0+B^1}=0$ implies $x=0$ and completeness of B^0+B^1 with respect to its norm. If $\|x\|_{B^0+B^1}=0$ there exists two sequences $y_n\in B^0$, $z_n\in B^1$ such that $\|y_n\|_0\to 0$, $\|z_n\|_1\to 0$, $y_n+z_n=x$. Now we have also $y_1+z_1=x$ and consequently $y_n-y_1+z_n-z_1=0$, or $y_n-y_1=z_1-z_n$. Now $y_n-y_1\to -y_1$ in B^0 and thus also in V, and $y_n-y_1=z_1-z_n\to z_1$ in B^1 and thus also in V. Consequently $-y_1=\lim(y_n-y_1)=z_1$ whence $x=y_1+z_1=0$. To show completeness it is enough to show that if x_n is such that $\sum_1^\infty\|x_n\|_{B^0+B^1}<\infty$, then $s_N=\sum_1^N x_n$ converges in B^0+B^1 to a limit. Let $x_n=y_n+z_n$ with $y_n\in B^0$, $z_n\in B^1$, $\|y_n\|_0\leqslant\|x_n\|_{B^0+B^1}+2^{-n}$, $\|z_n\|_1\leqslant\|x\|_{B^0+B^1}+2^{-n}$. Then $\sum_1^\infty\|y_n\|_0<\infty$ and $\sum_1^\infty\|z_n\|<\infty$, and this implies the desired conclusion.

22. Evidently the quantity introduced has all required properties, and again we merely have to verify that $\mathscr{F}$ is complete. Suppose that f_n is such that $\|f_n-f_m\|_{\mathscr{F}}\to 0$. Then for each $\xi=s+it$ $(0\leqslant s\leqslant 1)$ we have

$$\begin{aligned}\|f_n(\xi)-f_m(\xi)\|_{B^0+B^1}&\leqslant\max[\sup_t\|f(it)\|_{B^0+B^1},\sup\|f(1+it)\|_{B^0+B^1}]\\&\leqslant\|f_n-f_m\|_{\mathscr{F}}\to 0\end{aligned}$$

and consequently $f_n(\xi)$ converges in B^0+B' to a limit function $f(\xi)$ which is continuous and bounded in $0\leqslant s\leqslant 1$ and analytic in $0<s<1$. Furthermore, we have $\|f_n(it)-f_m(it)\|_{B^0}\leqslant\|f_n-f_m\|_{\mathscr{F}}$ and consequently $f_n(it)$ converges to a limit in B^0 which must coincide with its limit in B^0+B^1. Consequently $f(it)\in B^0$ and $\|f_n(it)-f(it)\|_{B^0}\to 0$ uniformly in t which implies that $f(it)$ is continuously B^0-valued and tends to zero as $t\to\infty$. The corresponding conclusion also holds for $f(1+it)$ which shows that $f(\xi)\in\mathscr{F}(B^0,B^1)$. Since $\|f_n(it)-f(it)\|_{B^0}\to 0$ and $\|f_n(1+it)-f(1+it)\|_{B^1}\to 0$ uniformly in t it follows that $\|f_n-f\|_{\mathscr{F}}\to 0$ and our assertion is established.

23. Evidently B_s is the image of $\mathscr{F}(B^0,B^1)$ under the linear mapping $\mathscr{F}(B^0,B^1)\to B^0+B^1$ defined by $f\to f(s)$. That this mapping is continuous follows from

$$\|f(s)\|_{B^0+B^1}\leqslant\max\sup_t\|f(it)\|_{B^0+B^1}\sup_t\|f(1+it)\|_{B^0+B^1}\leqslant\|f\|_{\mathscr{F}};$$

$\mathscr{N}_s$ is then the kernel of the mapping and B_s is simply given the norm of $\mathscr{F}(B^0,B^1)/\mathscr{N}_s$.

24. In the first place it is clear that L is a continuous map from B^0+B^1 to C^0+C^1. Given $x\in B_s$ and $\varepsilon>0$, there exists $f\in\mathscr{F}(B^0,B^1)$ such that $f(s)=x$ and $\|f\|_{\mathscr{F}}\leqslant\|x\|_{B_s}+\varepsilon$. The function $g(\xi)=M_0^{\xi-1}M_1^{-\xi}L[f(\xi)]$ evidently belongs to $\mathscr{F}(C^0,C^1)$ and $\|g(\xi)\|_{\mathscr{F}}\leqslant\|x\|_{B_s}+\varepsilon$. Thus

$$\|x\|_{B_s}+\varepsilon\geqslant\|g\|_{\mathscr{F}}\geqslant\|g(s)\|_{C_s}=\|M_0^{s-1}M_1^{-s}Lf(s)\|_{C_s}=M_0^{s-1}M_1^{-s}\|Lx\|_{C_s},$$

whence

$$\|Lx\|_{C_s} \leqslant M_0^{1-s} M_1^s (\|x\|_{B_s} + \varepsilon).$$

Since ε is arbitrary, the desired conclusion follows.

25. It is clear that $\overline{\mathcal{F}}$ is a linear space. Furthermore, if $\|f\|_{\overline{\mathcal{F}}} = 0$, then $f(it)$ and $f(1+it)$ are constant, which implies that $f(\xi)$ is constant. Let now f_n be such that $\|f_n - f_m\|_{\overline{\mathcal{F}}} \to 0$. Then from the inequality

$$\left\|\frac{f(\xi+ih)-f(\xi)}{ih}\right\|_{B^0+B^1}$$

$$\leqslant \max\left[\sup_{t_1,t_2}\left\|\frac{f(it_2)-f(it_1)}{t_2-t_1}\right\|_{B^0+B^1}, \sup_{t_2,t_1}\left\|\frac{f(1+it_2)-f(1+it_2)}{t_2-t_1}\right\|_{B^0+B^1}\right] \leqslant \|f\|_{\overline{\mathcal{F}}}$$

we obtain

$$\left\|\frac{df}{d\xi}\right\|_{B^0+B^1} \leqslant \|f\|_{\overline{\mathcal{F}}}$$

and consequently

$$\|f(\xi)-f(0)\|_{B^0+B^1} \leqslant |\xi|\,\|f\|_{\overline{\mathcal{F}}}.$$

Thus $f_n(\xi)-f(0)$ converges uniformly on every compact subset of $0 \leqslant s \leqslant 1$. The limit function $f(\xi)$ obviously satisfies conditions i), ii), iii). Furthermore, for every pair t_1, t_2, $f_n(it_2)-f_n(it_1)$ converges in B^0, consequently $f(it_2)-f(it_1)$ belongs to B^0 and

$$\left\|\frac{f(it_2)-f(it_1)}{t_2-t_1} - \frac{f_n(it_2)-f_n(it_1)}{t_2-t_1}\right\|_{B^0} \to 0.$$

A similar conclusion holds for $f(1+it)$. Consequently $f \in \overline{\mathcal{F}}$ and $\|f-f_n\|_{\overline{\mathcal{F}}} \to 0$.

26. From the inequality

$$\left\|\frac{df}{d\xi}(s)\right\|_{B^0+B'} \leqslant \|f\|_{\overline{\mathcal{F}}}$$

it follows that the mapping $\overline{\mathcal{F}} \to B^0+B^1$ defined by

$$f \to \frac{df}{d\xi}(s)$$

is continuous. The kernel $\overline{\mathcal{N}}_s$ of the mapping is closed and the range is B^s. The norm we have introduced in B^s is precisely the norm of $\overline{\mathcal{F}}/\overline{\mathcal{N}}_s$ and B^s is therefore complete.

27. Let (B^0, B^1) and (C^0, C^1) be two interpolation pairs. Let L be a linear mapping from B^0+B^1 to C^0+C^1 which maps B^i into C^i with norm M_i, $i = 0, 1$. Let $x \in B^s$ and $f \in \overline{\mathcal{F}}$ such that $\frac{df}{d\xi}(s) = x$, $\|f\|_{\overline{\mathcal{F}}} \leqslant \|x\|_{B^s} + \varepsilon$. Consider the function

$$g(\xi) = M_0^{\eta-1} M_1 L[f(\eta)]\Big|_0^\xi - \int_0^\xi (\log M_0 - \log M_1)\, M_0^{\eta-1} M_1^{-\eta} L[f(\eta)]\,d\eta,$$

where the integral is taken along any path joining the points 0 and ξ and contained in $0 \leqslant s \leqslant 1$. If the path has all its points except 0 and perhaps ξ in $0 < s < 1$, since $\frac{d}{d\xi}L(f) = L\frac{df}{d\xi}$ and $\frac{df}{d\xi}$ is continuous and has bounded norm in B^0+B^1 in the strip $0 < s < 1$, and L is a bounded linear mapping from B^0+B^1 into C^0+C^1, we find that $\frac{d}{d\xi}L(f)$ is continuous and has bounded norm in C^0+C^1. Consequently we may integrate by parts the integral in the preceding expression obtaining

$$g(\xi) = \int_0^\xi M_0^{\eta-1} M_1^{-\eta}\, dL[f(\eta)],$$

where the integral is to be interpreted as a vector valued Stieltjes integral. From this it follows that $\|g(\xi)\|_{C^0+C^1} \leqslant c|\xi|$. Furthermore, since $L[f(it)]$ has values in C^0 and is a C^0-Lipschitz function, it follows that $g(it_2)-g(it_1) \in C^0$ and $\|g(it_2)-g(it_1)\|_{C^0}$ does not exceed M_0^{-1} times the total C^0-variation of $L[f(it)]$ in the interval (t_1, t_2). Now as readily seen this in turn does not exceed M_0 times the total B^0-variation of $f(it)$ in (t_2, t_1) which is less than or equal to $|t_2-t_1|\,\|f\|_{\overline{\mathcal{F}}} \leqslant |t_2-t_1|(\|x\|_{B^s}+\varepsilon)$. Hence

$$\|g(it_2)-g(it_1)\|_{C^0} \leqslant |t_2-t_1|(\|x\|_{B^s}+\varepsilon).$$

In a similar fashion one obtains

$$\|g(1+it_2)-g(1+it_1)\|_{C^1} \leqslant |t_2-t_1|(\|x\|_{B^s}+\varepsilon).$$

Thus we have proved that $g \in \overline{\mathcal{F}}(C^0, C^1)$ and that $\|g\|_{\overline{\mathcal{F}}} \leqslant \|x\|_{B^s}+\varepsilon$. Now

$$g'(s) = M_0^{s-1} M_1^{-s} \frac{d}{d\xi} L(f)]_{\xi=s} = M_0^{s-1} M_1^{-s} L[f(s)] = M_0^{s-1} M_1^{-s} L(x)$$

which shows that $L(x) = M_0^{1-s} M_1^s g'(s)$ belongs to C^s and that

$$\|L(x)\|_{C^s} \leqslant M_0^{1-s} M_1^s (\|x\|_{B^s}+\varepsilon).$$

Since ε is arbitrary, the desired conclusion follows.

29.1. Let $\mu_0(\xi, t)$ and $\mu_1(\xi, t)$ be the Poisson kernels associated with the strip $0 \leqslant s \leqslant 1$. Consider the function

$$g(\xi) = \int_{-\infty}^{+\infty} f(it)\mu_0(\xi, t)\,dt + \int_{-\infty}^{+\infty} f(1+it)\mu_1(\xi, t)\,dt;$$

then $g(\xi)$ has values in B^0+B^1 and is continuous there. Furthermore, if l is a continuous linear functional in the family we have

$$l[g(\xi)] = \int_{-\infty}^{+\infty} l[f(it)]\mu_0(\xi, t)\,dt + \int_{-\infty}^{+\infty} l[f(1+it)]\mu_1(\xi, t)\,dt = l[f(\xi)]$$

and since the linear functionals l form a separating family, it follows that $f(\xi) = g(\xi)$. Now, from the definition of $g(\xi)$, it follows that $g(\xi)$ is bounded in the case i) and that $\|g(\xi)\|_{B^0+B^1} \leqslant c(1+|\xi|)$ in the case ii). Finally, since $f(\xi)$ is continuous in B^0+B^1, may form the expression

$$f(\xi) - \frac{1}{2\pi i}\int \frac{f(\eta)}{\eta-\xi}\,d\eta,$$

where the integral is taken over a circle contained in $0 < s < 1$, and ξ is a point interior to this circle. Then

$$l\left[f(\xi) - \frac{1}{2\pi i}\int \frac{f(\eta)}{\eta-\xi}\,d\xi\right] = l[f(\xi)] - \frac{1}{2\pi i}\int \frac{l[f(\eta)]}{\eta-\xi}\,d\eta = 0$$

whence the expression above is zero which shows that $f(\xi)$ is analytic.

29.2. Since $\|e^{\delta\xi^2}f(\xi) - f(\xi)\|_{\mathscr{F}} \to 0$ as $\delta \to +0$ for every $f \in \mathscr{F}(B^0, B^1)$, it will be enough to show that every $g(\xi)$ of the form $g(\xi) = e^{\delta\xi^2}f(\xi)$, $f \in \mathscr{F}$, can be approximated by functions in the class described above. Let such a $g(\xi)$ be given and let

$$g_n(\xi) = \sum_{j=-\infty}^{+\infty} g(\xi + 2\pi ijn), \qquad n \geqslant 1.$$

Then clearly $g_n(\xi)$ has values in B^0+B^1, is analytic in $0 < R(\xi) < 1$ and continuous in $0 \leqslant R(\xi) \leqslant 1$, and is periodic with period $2\pi in$. Furthermore, $g_n(it) \in B^0$, $g_n(1+it) \in B^1$, $\|g_n(it) - g(it)\|_0 \to 0$, $\|g_n(1+it) - g(1+it)\|_1 \to 0$ as $n \to \infty$, uniformly on every bounded set of values of t, and $\|g_n(it)\|_0$ and $\|g_n(1+it)\|_1$ are bounded, uniformly in n. Now it is easy to see that these properties imply that $e^{s\xi^2}g_n(\xi) \in \mathscr{F}$ for every $s > 0$, and that given $\varepsilon > 0$, a proper choice of s and n will yield

$$\|e^{s\xi^2}g_n(\xi) - g(\xi)\|_{\mathscr{F}} < \varepsilon/2.$$

Now $g_n(\xi)$ has a Fourier series representation

$$(1) \qquad g_n(\xi) = \sum_{k=-\infty}^{+\infty} a_{kn}e^{k\xi/n}, \qquad \xi = s+it,$$

where

$$a_{kn} = \frac{1}{2\pi mn}\int_{-\pi nm}^{\pi nm} g_n(s+it)e^{-k(s+it)/n}\,dt, \qquad m = 1, 2, \ldots$$

Because of periodicity, the value of the right-hand side is independent of m, and on account of the analyticity and boundedness of the integrand, its dependence on s is arbitrarily small for sufficiently large m. Consequently a_{kn} is independent of s and

$$a_{kn} = \frac{1}{2\pi n}\int_{-\pi n}^{\pi n} g_n(it)e^{-ikt/n}\,dt = \frac{1}{2\pi n}\int_{-\pi n}^{\pi n} g_n(1+it)e^{-k(1+it)/n}\,dt.$$

Now $g_n(it) \in B^0$ and is B^0-continuous, which implies that the value of the first integral is in B^0 that is $a_{kn} \in B^0$. Similarly, the second expression of a_{kn} shows that $a_{kn} \in B^1$ and we conclude that $a_{kn} \in B^0 \frown B^1$. Consider now the $(C, 1)$ means of the series (1). We have

$$\sigma_m(g_n, \xi) = \sum_{-m}^{m}\left(1 - \frac{|k|}{m+1}\right)a_{kn}e^{k\xi/n} = \int_{-\pi n}^{\pi n} g_n(\xi - it)K_m\left(\frac{t}{n}\right)\frac{dt}{n},$$

where $K_m(s)$ is the Fejér kernel. From the B^0-continuity of $g_n(it)$ it follows that $\|\sigma_m(g_n, it) - g_n(it)\|_0 \to 0$ as $m \to \infty$, uniformly in t for each n. Similarly, we find that $\|\sigma_m(g_n, 1+it) - g_n(1+it)\|_1 \to 0$ as $m \to \infty$, uniformly in t for each n. Consequently for each $s > 0$, n, we have $\|e^{s\xi^2}[\sigma_m(g_n, \xi) - g_n(\xi)]\|_{\mathscr{F}} \to 0$ as $m \to \infty$. Thus for suitable s, m and n we have

$$\|e^{s\xi^2}\sigma_m(g_n, \xi) - g(\xi)\|_{\mathscr{F}} < \varepsilon.$$

Now $e^{s\xi^2}\sigma_m(g_n, \xi)$ is a function of the desired form and 9.2 is established.

29.3. Let $x \in B_s$ be given and let $\varepsilon > 0$. Then there exists $f \in \mathscr{F}(B^0, B^1)$ such that $f(s) = x$. By 9.2 we can find a function $g(\xi)$ with values in $B^0 \frown B^1$ such that $\|f - g\|_{\mathscr{F}} < \varepsilon$. Consequently, we have $\|f(s) - g(s)\|_{B_s} = \|x - g(s)\|_{B_s} \leqslant \|f - g\|_{\mathscr{F}} < \varepsilon$, since $g(s) \in B^0 \frown B^1$ our first assertion is established. That $B_0 \subset B^0$ is evident. Let us show that the norm of an element $x \in B_0$ coincides with its norm as an element of B^0. Given $\varepsilon > 0$ we can find $x_1 \in B^0 \frown B^1$ such that $\|x - x_1\|_{B_0} < \varepsilon$. Consider $f_n(\xi) = x_1 e^{\xi^2 - n\xi} \in \mathscr{F}$. Then clearly $f(0) = x_1$ and $\|f\|_{\mathscr{F}} \leqslant \|x_1\|_{B^0} + e^{1-n}\|x_1\|_{B^1}$. Consequently $\|x_1\|_{B_0} \leqslant \|x_1\|_{B^0} + e^{1-n}\|x_1\|_{B^1}$ and letting $n \to \infty$ we find $\|x_1\|_{B_0} \leqslant \|x_1\|_{B^0}$. Now from the definition of the norm of B_0 we have $\|x\|_{B^0} \leqslant \|x\|_{B_0}$, $\|x-x_1\|_{B^0} \leqslant \|x-x_1\|_{B_0} < \varepsilon$. Thus $\|x\|_{B_0} \leqslant \|x_1\|_{B_0}+\varepsilon \leqslant \|x_1\|_{B^0}+\varepsilon \leqslant \|x\|_{B^0}+2\varepsilon$. Since ε is arbitrary we obtain $\|x\|_{B_0} \leqslant \|x\|_{B^0}$ which combined with the reverse inequality above gives $\|x\|_{B_0} = \|x\|_{B^0}$. To prove the last statement we observe that if $f(\xi)\in\mathscr{F}(B^0, B^1)$, then $f(it)\in B_0$ and $f(1+it)\in B_1$, and therefore $f(\xi)\in\mathscr{F}(B_0, B_1)$. Thus $\mathscr{F}(B^0, B^1) \subset \mathscr{F}(B_0, B_1)$. Now the reverse inclusion is evident whence $\mathscr{F}(B^0, B^1) = \mathscr{F}(B_0, B_1)$. That the norms of these spaces coincide follows from the equality of the norms of B^i and B_i, $i=0,1$.

29.4. Let $\varphi_0(t)$ and $\varphi_1(t)$ be bounded infinitely differentiable and such that $\varphi_0(t) \geqslant \log\|f(it)\|_{B^0}$, $\varphi_1(t) \geqslant \log\|f(1+it)\|_{B^1}$. Let $\Phi(\xi)$ be an analytic function whose real part is bounded and continuous in $0 \leqslant s \leqslant 1$ and such that $\mathrm{R}[\Phi(it)] = \varphi_0(t)$, $\mathrm{R}[\Phi(1+it)] = \varphi_1(t)$. This function exists and

$$\mathrm{R}[\Phi(\xi)] = \int_{-\infty}^{+\infty} \varphi_0(t)\mu_0(\xi, t)\,dt + \int_{-\infty}^{+\infty} \varphi_1(t)\mu_1(\xi, t)\,dt.$$

Furthermore, the differentiability of φ_0 and φ_1 implies that $\Phi(\xi)$ is continuous in $0 \leqslant s \leqslant 1$. Consequently $e^{-\Phi(\xi)}f(\xi)\in\mathscr{F}$ and since

$$\|e^{-\Phi(it)}f(it)\|_{B^0} \leqslant e^{-\varphi_0(t)}\|f(it)\|_{B^0} \leqslant 1,$$
$$\|e^{-\Phi(1+it)}f(1+it)\|_{B^1} \leqslant e^{-\varphi_1(t)}\|f(1+it)\|_{B^1} \leqslant 1,$$

it follows that $\|e^{-\Phi(\xi)}f(\xi)\|_{\mathscr{F}} \leqslant 1$. Consequently

$$\|e^{-\Phi(s)}f(s)\|_{B_s} \leqslant 1$$

and

$$\log\|f(s)\|_{B_s} \leqslant \mathrm{R}[\Phi(s)] = \int_{-\infty}^{+\infty} \varphi_0(t)\mu_0(s, t)\,dt + \int_{-\infty}^{+\infty} \varphi_1(t)\mu_1(s, t)\,dt.$$

Taking now a decreasing sequence of functions φ_0 and φ_1 converging to $\log\|f(it)\|_{B^0}$ and $\log\|f(1+it)\|_{B^1}$ respectively and passing to the limit, we obtain i). To obtain ii) we observe that

$$\int_{-\infty}^{+\infty} \mu_0(s, t)\,dt = 1-s \quad\text{and}\quad \int_{-\infty}^{+\infty} \mu_1(s, t)\,dt = s$$

and from this from Jensen's inequality it follows that

$$\exp\left[\frac{1}{1-s}\int_{-\infty}^{+\infty} \log\|f(it)\|_{B^0}\,\mu_0(s, t)\,dt\right] \leqslant \frac{1}{1-s}\int_{-\infty}^{+\infty} \|f(it)\|_{B^0}\,\mu_0(s, t)\,dt,$$
(1)
$$\exp\left[\frac{1}{s}\int_{-\infty}^{+\infty} \log\|f(1+it)\|_{B^1}\,\mu_1(s, t)\,dt\right] \leqslant \frac{1}{s}\int_{-\infty}^{+\infty} \|f(1+it)\|_{B^1}\,\mu_1(s, t)\,dt.$$

Multiplying and dividing the first and second terms of the right-hand side of i) by $1-s$ and s respectively, taking exponentials and using the preceding inequalities, ii) follows.

To obtain iii) we use the inequality

$$e^{a+b} \leqslant (1-s)e^{a/(1-s)} + se^{b/s}, \qquad 0 < s < 1,$$

which is a consequence of the convexity of the exponential function. Replacing a and b by the first and second terms of the right-hand side of i) and applying (1) again we obtain iii).

To prove the last assertion in 9.4, we merely have to observe that for $0 < s < 1$ we have

$$\mu_0(s, t) \leqslant ce^{-\pi|t|}, \qquad \mu_1(s, t) \leqslant ce^{-\pi|t|},$$

where c depends on s. Thus if E is the set where $\|f_n(it)\|_{B_0}$ tends to zero i) yields

$$\log\|f_n(s)\|_{B_s} \leqslant c\int_{-\infty}^{+\infty} [\log^+\|f_n(it)\|_{B_0} + \log^+\|f_n(1+it)\|_{B_1}]e^{-\pi|t|}dt + \int_E \log\|f_n(it)\|_{B_0}\mu_0(s, t)\,dt$$

and the right-hand side tends to $-\infty$ as $n\to\infty$. Thus $\|f_n(s)\|_{B_s}\to 0$.

29.5. Let

$$f_n(\xi) = \left[f\left(\xi+\frac{i}{n}\right) - f(\xi)\right]\frac{n}{i}.$$

Then $\|f_n(it)-f_m(it)\|_{B^0}\to 0$ as n and m tend to infinity for $t\in E$. Further, we have $e^{\varepsilon\xi^2}f_n(\xi)\in\mathscr{F}$ for every $\varepsilon > 0$. From inequality 9.4, i) we obtain

$$\log\|e^{\varepsilon s^2}[f_n(s)-f_m(s)]\|_{B_s} \leqslant \int_{-\infty}^{+\infty} \log\|e^{-\varepsilon t^2}[f_n(it)-f_m(it)]\|_{B^0}\,\mu_0(s, t)\,dt + \int_{-\infty}^{+\infty} \log\|e^{-\varepsilon(1+it)^2}[f_n(1+it)-f_m(1+it)]\|_{B^1}\,\mu_1(s, t)\,dt.$$

Since $\|f_n(it)-f_m(it)\|_{B^0} \leqslant 2\|f\|_{\mathscr{F}}$ and

$$\|f_n(1+it)-f_m(1+it)\|_{B^1} \leqslant 2\,\|f\|_{\mathscr{F}}$$

and since $\|f_n(it)-f_m(it)\|_{B^0}\to 0$ for $t\in E$, and $\mu_0(s, t) > 0$, it follows that the right-hand side of the inequality above tends to $-\infty$ as $n, m\to\infty$. Consequently $\log\|e^{\varepsilon s^2}[f_n(s)-f_m(s)]\|_{B_s}\to -\infty$ and $\|f_n(s)-f_m(s)\|_{B_s}\to 0$ which means that $f_n(s)$, which belongs to B_s, converges in B_s as $n\to\infty$. But $f_n(s)$ also converges to $f'(s)$ in B^0+B^1, whence $f'(s)\in B_s = [B^0, B^1]_s$.

Suppose now that B^0 is reflexive. Then $f(it)$ is a continuous B^0-valued function and therefore its range lies in a separable subspace M of B^0. For each t, let $S_m(t) \subset M$ be the weak closure in B^0 of the set $\{f_k(it)\}$, $k \geqslant m$, and $S(t) = \bigcap_n S_n(t)$. Since the $S_n(t)$ are bounded and weakly closed and the unit sphere of B^0 is weakly compact, the $S_n(t)$ are weakly compact and $S(t)$ is non-void. Let $g(t)$ be a function such that $g(t) \in S(t)$ for all t. Since $S(t) \subset M$, $g(t)$ is separably valued. Furthermore, if l is a continuous linear functional in B^0, we infer that $l[f(it)] = i\varphi(t)$ is a Lipschitz function of t, and

$$l[f_n(it)] = n\left[\varphi\left(t+\frac{1}{n}\right) - \varphi(t)\right].$$

Now, the image of $S_n(t)$ under l is the closure of the set

$$\left\{k\left[\varphi\left(t+\frac{1}{k}\right) - \varphi(t)\right]\right\}, \qquad k \geqslant n,$$

and the image of $S(t)$ is contained in the intersection of all these sets. If t is a point where $\varphi(t)$ is differentiable, then this intersection reduces to a single number namely $\varphi'(t)$ and consequently the image of $S(t)$ under l is precisely $\varphi'(t)$, and in particular $l[g(t)] = \varphi'(t)$ wherever $\varphi'(t)$ exists, that is, almost everywhere. Since this is valid for every l, we conclude that $g(t)$ is weakly measurable, and since it is separably valued, it is also a strongly measurable B^0-valued function. Furthermore, since the sets $S(t)$ are uniformly bounded, $g(t)$ is bounded and

$$l[f(it)] = i\varphi(t) = i\varphi(0) + i\int_0^t \varphi'(\tau)\,d\tau = l[f(0)] + i\int_0^t l[g(\tau)]\,d\tau,$$

whence

$$f(it) = f(0) + i\int_0^t g(\tau)\,d\tau,$$

which means that $f(it)$ has a strong derivative in B^0 for almost all t, and the first statement in 9.5 asserts that $f'(s) \in [B^0, B^1]_s$. Since this holds for every $f \in \mathscr{F}$, it follows that $[B^0, B^1]^s \subset [B^0, B^1]_s$. Let us consider now the norms of these spaces. If $x \in [B^0, B^1]^s$, there exists an $f(\xi) \in \overline{\mathscr{F}}[B^0, B^1]$ such that $f'(s) = x$, $\|f\|_{\overline{\mathscr{F}}} \leqslant \|x\|_{B^s} + \varepsilon$. Consider as before the function

$$h_n(\xi) = e^{\varepsilon\xi^2}\left[f\left(\xi + \frac{i}{n}\right) - f(\xi)\right]\frac{n}{i}.$$

This function belongs to $\mathscr{F}(B^0, B^1)$ and its norm as an element of $\mathscr{F}$ does not exceed $e^{\varepsilon}\|f\|_{\overline{\mathscr{F}}}$, which implies that $\|h_n(s)\|_{B_s} \leqslant e^{\varepsilon}\|f\|_{\overline{\mathscr{F}}} \leqslant e^{\varepsilon}[\|x\|_{B^s} + \varepsilon]$.

But, as we saw previously, $\|h_n(s) - e^{\varepsilon s^2}x\|_{B_s} \to 0$ as $n \to \infty$, and thus we obtain $\|x\|_{B_s} \leqslant e^{\varepsilon}[\|x\|_{B^s} + \varepsilon]$. Since ε is arbitrary, it follows that $\|x\|_{B_s} \leqslant e^{\varepsilon s^2}\|x\|_{B^s}$. If we now drop the assumption on the reflexivity of B^0, given $x \in [B^0, B^1]_s$ and $f \in \mathscr{F}(B^0, B^1)$ such that $f(s) = x$ and $\|f\|_{\mathscr{F}} \leqslant \|x\|_{B_s} + \varepsilon$, we set $g(\xi) = \int_0^\xi f(\eta)\,d\eta$. Then it is readily seen that $g \in \overline{\mathscr{F}}$, $\|g\|_{\overline{\mathscr{F}}} = \|f\|_{\mathscr{F}}$ and $g'(s) = x$. Consequently, $x \in [B^0, B^1]^s$, and $\|x\|_{B^s} \leqslant \|g\|_{\overline{\mathscr{F}}} = \|f\|_{\mathscr{F}} \leqslant \|x\|_{B_s} + \varepsilon$, and since ε is arbitrary, we find that $\|x\|_{B^s} \leqslant \|x\|_{B_s}$. This, combined with the opposite inequality obtained above, yields the desired result.

29.6. First, let us observe that if K is a compact subset of B^0, given $\sigma > 0$, there exists μ such that for $\lambda > \mu$ the image of K under $I - \pi_\lambda$ (where I is the identity operator) is contained in a sphere of radius σ of B^0. For let N be a bound for the norms of π_λ as an operator on B^i, $i = 0, 1$. Let $(x_1, \ldots, x_n)$ be a finite subset of K such that every point of K is at distance less then $\sigma/2(N+1)$ from it. Let λ be such that $\|x_j - \pi_\lambda x_j\|_{B^0} < \sigma/2$, $j = 1, 2, \ldots, n$. Then, if $x \in K$ and x_j is such that $\|x - x_j\|_{B^0} < \sigma/2(N+1)$, we have

$$\|x - \pi_\lambda x\|_{B^0} \leqslant \|(x - x_j) - \pi_\lambda(x - x_j) + x_j - \pi_\lambda x_j\|_{B^0} < \frac{\sigma}{2(N+1)} + \frac{N\sigma}{2(N+1)} + \frac{\sigma}{2} = \sigma.$$

Thus if we set

$$\delta(\lambda) = \sup_{x \in K}\|x - \pi_\lambda x\|_{B^0},$$

we have $\delta(\lambda) \to 0$. Since π_λ is a bounded operator on B^i, $i = 0, 1$, with norm not exceeding N, π_λ is also a bounded operator on $B^0 + B^1$ with norm not exceeding N. Let D_λ be the range of π_λ restricted to B^0 and $D'_\lambda = D_\lambda \cap B^0 \cap B^1$; then, if $f \in \mathscr{F}(B^0, B^1)$, $\pi_\lambda f(\xi) \in D'_\lambda$. In fact, since $f(\xi)$ is a $B^0 + B^1$-valued function which is continuous in $0 \leqslant s \leqslant 1$, the same holds for $\pi_\lambda f$; now if l is a continuous linear functional on $B^0 + B^1$ which vanishes on D_λ, since $f(it) \in B^0$, we have $\pi_\lambda f(it) \in D_\lambda$ and $l[\pi_\lambda f(it)] = 0$, which implies that the analytic function $l[\pi_\lambda f(\xi)]$ vanishes identically. Since this is true for every l which vanishes on D_λ, it follows that $\pi_\lambda f(\xi) \in D_\lambda$ (1). On the other hand, since $f(1+it) \in B^1$, $\pi_\lambda f(1+it) \in B^1$ and consequently $\pi_\lambda f(1+it) \in B^1 \cap D_\lambda$, but $D_\lambda \subset B^0$ which implies $B^1 \cap D_\lambda = B^0 \cap B^1 \cap D_\lambda = D'_\lambda$. Hence $\pi_\lambda f(1+it) \in D'_\lambda$. Let l be a continuous linear functional on $B^0 + B^1$ which vanishes on D'_λ; then $l[\pi_\lambda f(1+it)] = 0$,

(1) Observe that since D_λ is finite dimensional, it is closed.

and therefore $l[\pi_\lambda f(\xi)]$ vanishes identically, which implies that $\pi_\lambda f(\xi)\in D'_\lambda$. Now suppose that $f\in\mathscr{F}(B^0,B^1)$ satisfies the hypothesis of 9.6. Then

$$\|f(it)-\pi_\lambda f(it)\|_{B^0}\leqslant c(1+N),\qquad \|f(1+it)-\pi_\lambda f(1+it)\|_{B^1}\leqslant c(1+N).$$

Furthermore, since $f(it)\in K$ for t in E,

$$\|f(it)-\pi_\lambda f(it)\|_{B^0}\leqslant \delta(\lambda)\quad\text{for}\quad t\in E.$$

Applying inequality i) of 9.4, and assuming $c(1+N)\geqslant 1$ we obtain

$$\log\|f(s)-\pi_\lambda f(s)\|_{B_s}\leqslant[\log\delta(\lambda)]\int_E\mu_0(s,t)\,dt+\log c(1+N),$$

whence

$$\|f(s)-\pi_\lambda f(s)\|_{B_s}\leqslant c(1+N)\,\delta(\lambda)^d,\qquad d>0.$$

On the other hand, $\|\pi_\lambda f(s)\|_{B_s}\leqslant cN$ and $\pi_\lambda f(s)\in D'_\lambda$ consequently

$$f(s)=[f(s)-\pi_\lambda f(s)]+\pi_\lambda f(s)\in\{x\mid\|x\|_{B_s}\leqslant c(1+N)\,\delta(\lambda)^d\}+ +\{x\mid\|x\|_{B_s}\leqslant cN\}\cap D'_\lambda$$

and since this holds for all λ, we obtain

$$f(s)\in\bigcap_\lambda[\{x\mid\|x\|_{B_s}\leqslant c(1+N)\,\delta(\lambda)^d\}+\{x\mid\|x\|_{B_s}\leqslant cN\}\cap D'_\lambda].$$

Now the set $\{x\mid\|x\|_{B_s}\leqslant cN\}\cap D'_\lambda$ is a closed bounded subset of the finite dimensional space D_λ and hence, a compact set. Since $\delta(\lambda)\to 0$, the spheres $\{x\mid\|x\|_{B_s}\leqslant c(1+N)\,\delta(\lambda)^d\}$ have arbitrarily small radii and this implies that the set in the expression displayed above is totally bounded. Thus 9.6 is established.

30.1. Let D_j $(j=1,2,\dots,n)$ be Banach spaces and M_j a dense subspace of D_j. Let $L(x_1,\dots,x_n)$, $x_j\in M_j$, be a multilinear function defined in $\bigoplus_{i=1}^n M_j$, with values in a Banach space D and such that

$$\|L(x_1,x_2,\dots,x_n)\|_D\leqslant c\prod_1^n\|x_j\|_{D_j}.$$

Then L can be extended uniquely to a multilinear function $\overline{L}$ defined on $\bigoplus_1^n D_j$ with values in D and satisfying the preceding inequality. In fact, this inequality implies uniform continuity of L on any bounded subset of $\bigoplus_1^n M_j$. If $x_j,y_j\in M_j$, $\|x_j\|_{D_j}\leqslant c_1$, $\|y_j\|_{D_j}\leqslant c_1$ we have

$$(1)\quad \|L(x_1,x_2,\dots,x_n)-L(y_1,y_2,\dots,y_n)\|_D$$
$$\leqslant\sum_{j=1}^n\|L(y_1,\dots,x_j,x_{j+1},\dots,x_n)-L(y_1,y_2,\dots,y_j,x_{j+1},\dots,x_n)\|_D$$
$$\leqslant c\sum_{j=1}^n c_1^{n-1}\|x_j-y_j\|_{D_j},$$

so that L can be extended continuously in only one way. That this extension is multilinear and satisfies the required inequality is immediate.

Consider now the spaces $\mathscr{F}_j=\mathscr{F}(A_j,B_j)$ and their dense subspaces $\mathscr{G}(A_j,B_j)$ consisting of functions in $\mathscr{F}(A_j,B_j)$ with finite dimensional range contained in $A_j\cap B_j$. Let $\mathscr{L}(f_1,\dots,f_n)$, $f_j\in\mathscr{G}(A_i,B_i)$, be the multilinear mapping from $\bigoplus_{j=1}^n\mathscr{G}(A_j,B_j)$ into $\mathscr{F}(A,B)$ defined by

$$f=\mathscr{L}(f_1,\dots,f_n),\qquad f(\xi)=M_0^{\xi-1}M_1^{-\xi}L[f_1(\xi),f_2(\xi),\dots,f_n(\xi)].$$

Then

$$\|\mathscr{L}(f_1,\dots,f_n)\|_{\mathscr{F}}\leqslant\prod_1^n\|f_j\|_{\mathscr{F}_j}$$

and consequently $\mathscr{L}$ can be extended to a multilinear mapping from $\bigoplus_1^n\mathscr{F}_j$ into $\mathscr{F}$ satisfying the preceding inequality. We will denote this extension also by $\mathscr{L}$. Let $0<s<1$ and set

$$\mathscr{L}_s(f_1,f_2,\dots,f_n)=f(s)\in C,$$

where $f=\mathscr{L}(f_1,\dots,f_n)$. This gives a multilinear mapping of $\bigoplus_1^n\mathscr{F}_j$ into $C=[A,B]_s$.

We will prove that $\mathscr{L}_s(f_1,f_2,\dots,f_n)$ depends only on the values of f_j at s. First let us observe that if $\varphi(\xi)$ is an analytic bounded continuous function in $0\leqslant s\leqslant 1$, then $\mathscr{L}(\varphi f_1,f_2,\dots,f_n)=\varphi\mathscr{L}(f_1,f_2,\dots,f_n)$. This is certainly valid for $f_j\in\mathscr{G}(A_j,B_j)$ and, by continuity, it is valid also for $f_j\in\mathscr{F}_j$. Suppose now that $f_1(s)=0$. Then

$$f_1(\xi)=\frac{e^{i\pi\xi}-e^{i\pi s}}{e^{i\pi\xi}-e^{-i\pi s}}\,g_1(\xi)\qquad\text{where}\qquad g_1\in\mathscr{F}.$$

Thus if $f=\mathscr{L}(f_1,f_2,\dots,f_n)$, we have

$$f=\mathscr{L}(\varphi g_1,\dots,f_2,\dots,f_n)=\varphi\mathscr{L}(g_1,\dots,f_2,\dots,f_n)$$

where $\varphi=(e^{i\pi\xi}-e^{i\pi s})/(e^{i\pi\xi}-e^{-i\pi s})$, and consequently $f(s)=0$, i. e. $\mathscr{L}_s(f_1,\dots,f_n)=f(s)=0$. More generally, we have that $\mathscr{L}_s(f_1,\dots,f_n)=0$ if one of the functions f_i vanishes at s. Let now $f_j,g_j\in\mathscr{F}_j$ be such that $f_j(s)=g_j(s)$. Then

$$\mathscr{L}_s(f_1,\dots,f_n)-\mathscr{L}_s(g_1,\dots,g_n)=\sum_{j=1}^n[\mathscr{L}_s(g_1,\dots,f_j,f_{j+1},\dots,f_n)-$$
$$-\mathscr{L}_s(g_1,\dots,g_j,f_{j+1},\dots,f_n)]=\sum_{j=1}^n\mathscr{L}_s(g_1,\dots,f_j-g_j,f_{j+1},\dots,f_n)=0$$

since $f_j(s) - g_j(s) = 0$. Consider now $\bigoplus_1^n C_j$ and the multilinear function $L_s(x_1, \ldots, x_n)$, $x_j \in C_j$, defined by

$$L_s(x_1, \ldots, x_n) = \mathscr{L}_s(f_1, \ldots, f_n) M_0^{1-s} M_1^s \in C,$$

where $f_j \in \mathscr{F}_j$ and $f_j(s) = x_j$. According to what we have just shown L_s is well defined. If $x_i \in A_i \cap B_i$ and we set $f_j = \text{constant} = x_j$ we obtain, as easily seen,

$$L_s(x_1, \ldots, x_n) = L(x_1, \ldots, x_n).$$

Thus L_s is an extension of L to $\bigoplus_1^n C_j$. As was seen above L_s maps $\bigoplus_1^n C_j$ into C. Let us estimate the norm of $L_s(x_1, \ldots, x_n)$. Given $\varepsilon > 0$ we can find $f_j \in \mathscr{F}_j$ such that $f_j(s) = x_j$, $\|f_j\|_{\mathscr{F}_j} \leqslant \|x_j\|_{C_j} + \varepsilon$.

According to one of the preceding inequalities setting $f = \mathscr{L}(f_1, \ldots, f_n)$, we have

$$\|f\|_{\mathscr{F}} \leqslant \prod_1^n \|f_j\|_{\mathscr{F}_j} \leqslant \prod_1^n (\|x_j\|_{C_j} + \varepsilon)$$

and thus

$$\|L_s(x_1, \ldots, x_n)\|_C = \|M_0^{1-s} M_1^s \mathscr{L}_s(f_1, \ldots, f_n)\|_c = M_0^{1-s} M_1^s \|f(s)\|_C$$

$$\leqslant M_0^{1-s} M_1^s \|f\|_{\mathscr{F}} \leqslant M_0^{1-s} M_1^s \prod_1^n (\|x_j\|_{C_j} + \varepsilon).$$

Since ε is arbitrary, we obtain

$$\|L_s(x_1, \ldots, x_n)\|_C \leqslant M_0^{1-s} M_1^s \prod_1^n \|x_j\|_{C_j}.$$

Now L_s is an extension of L and consequently the same inequality holds for L.

30.2. Suppose we first restrict $\mathscr{L}(L, x_1, \ldots, x_n)$ to $L \in \mathscr{M}_0 \cap \mathscr{M}_1$ instead of $L \in \mathscr{M}$. Then by the preceding 10.1 we have

$$\|L(x_1, \ldots, x_n)\|_C \leqslant \|L\|_{\mathscr{N}} \prod_1^n \|x_j\|_{C_j}.$$

Now if $L \in \mathscr{N} = [\mathscr{M}_0, \mathscr{M}_1]_s$, since $\mathscr{M}_0 \cap \mathscr{M}_1$ is dense in $\mathscr{N}$, there exists a sequence $L_n \in \mathscr{M}_0 \cap \mathscr{M}_1$ such that $\|L_n - L\|_{\mathscr{N}} \to 0$. Now $\|L_n - L\|_{\mathscr{M}_0 + \mathscr{M}_1} \leqslant \|L_n - L\|_{\mathscr{N}}$ and since $\mathscr{M}_0 + \mathscr{M}_1$ is continuously embedded in $\mathscr{M}$, it follows that $\|L_n(x_1, \ldots, x_n) - L(x_1, \ldots, x_n)\|_{A+B} \to 0$. On the other hand, $\|L_n - L_m\|_{\mathscr{N}} \to 0$ as n and $m \to \infty$, which implies that

$$\|L_n(x_1, \ldots, x_n) - L_m(x_1, \ldots, x_n)\|_C \leqslant \|L_n - L_m\|_{\mathscr{N}} \prod_1^n \|x_j\|_{C_j} \to 0.$$

Consequently $L_n(x_1, \ldots, x_n)$ converges in C and $A+B$, since limits are unique, we find that $L(x_1, \ldots, x_n) \in C$ and

$$\|L(x_1, \ldots, x_n)\|_C \leqslant \|L\|_{\mathscr{N}} \prod_1^n \|x_j\|_{C_j}.$$

30.3. Let $m_k(t) = \sup \|F_j(it) - F_l(it)\|_{\mathscr{M}_0}$, $j \geqslant k$, $l \geqslant k$, then if $F_k(it)$ converges in $\mathscr{M}_0$ as $k \to \infty$ for t on a set of positive measure E, it follows that $m_k(t)$ converges uniformly to zero as $k \to \infty$ for t in a subset E_1 of E, of positive measure. If $\sup_{t \in E_1} m_k(t) = \varepsilon_k$, applying 9.4, i), we obtain

$$\log\|F_j(s) - F_l(s)\|_{\mathscr{N}} = \log\|L_l - L_j\|_{\mathscr{N}} \leqslant \int_{-\infty}^{+\infty} \log\|F_l(it) - F_j(it)\|_{\mathscr{M}_0} \mu_0(s t)\, dt +$$

$$+ \int_{-\infty}^{+\infty} \log\|F_l(1+it) - F_j(1+it)\|_{\mathscr{M}_1} \mu_1(s, t)\, dt,$$

$$\mathscr{N} = [\mathscr{M}_0, \mathscr{M}_1]_s.$$

From the inequality $\log(a+b) \leqslant \log^+ a + \log^+ b + \log 2$ replacing $\log\|F_l(it) - F_j(it)\|_{\mathscr{M}}$ by $\log^+\|F_l\|_{\mathscr{M}} + \log^+\|F_j\|_{\mathscr{M}} + \log 2$, and $\|F_l - F_j\|_{\mathscr{M}_0}$ by ε_k for $t \in E_1$ above, we obtain

$$\log\|L_l - L_j\|_{\mathscr{N}} \leqslant 2c + \log 2 + \log \varepsilon_k \int_{E_1} \mu_0(s, t)\, dt$$

and since $\varepsilon_k \to 0$ as $k \to \infty$, it follows that $\|L_l - L_j\|_{\mathscr{N}} \to 0$ as j and $l \to \infty$. Consequently L_k converges in $\mathscr{N}$ to a limit L_0 and $L_k(x_1, \ldots, x_n)$ converges to $L_0(x_1, \ldots, x_n)$ in $C = [A, B]_s$.

On account of i) we have

$$l[L(x_1, \ldots, x_n)] = \lim_{k \to \infty} l[L_k(x_1, \ldots, x_n)] = l[L_0(x_1, \ldots, x_n)]$$

for every continuous linear functional on V. Therefore $L(x_1, \ldots, x_n) = L_0(x_1, \ldots, x_n) \in C = [A, B]_s$. Finally, from 9.4, i) and ii), it follows that $\|L_k\|_{\mathscr{N}} \leqslant e^c$ whence $\|L_0\|_{\mathscr{N}} \leqslant e^c$ and

$$\|L(x_1, \ldots, x_n)\|_C = \|L_0(x_1, \ldots, x_n\|_C \leqslant \|L_0\|_{\mathscr{N}} \prod_1^n \|x_j\|_{C_j} \leqslant e^c \prod_1^n \|x_j\|_{C_j}.$$

30.4. Let A be a Banach space and let S be a subset of A with the following property: for each $\varepsilon > 0$ there exists a totally bounded set T_ε such that every point of S is at distance less than ε from T_ε. Then X is totally bounded. In fact, given $\delta > 0$ let $\{x_j\}$ be a finite set of points such that each $x \in T_{\delta/2}$ is at distance less than $\delta/2$ from $\{x_j\}$; then since each point of S is at distance less than $\delta/2$ from $T_{\delta/2}$, it follows that each point of S is at distance less than δ from $\{x_j\}$.

Let now L_n be a sequence of completely continuous multilinear maps from $\bigoplus_1^n A_j$ to A converging in norm to L. Let S be the image of the unit sphere $\|x_j\|_{A_j} \leqslant 1$ of $\bigoplus_1^n A_j$ under L and S_n that under L_n. Clearly S_n is totally bounded, and given $\varepsilon > 0$ the distance between S_n and S is less than ε as soon as the norm of $L_n - L$ is less than ε. Hence S is totally bounded and L is completely continuous. Consider now the function $F(\xi)$. Let $f_j(\xi) \in \mathcal{G}(A_j, B_j)$ [2] be such that $\|f_j\|_{\mathcal{F}_j} \leqslant 1$. If for a given ξ we calculate the multilinear function $F(\xi)$ at the point $f_1(\xi), f_2(\xi), \ldots, f_n(\xi)$ of $\bigoplus_1^n (A_j \cap B_j)$ we obtain a function $f(\xi) \in \mathcal{F}(A, B)$ (this readily verified on account of the multilinear character of $F(\xi)$ and the fact that $F(\xi)(x_1, \ldots, x_n)$ is for fixed $x_j \in A_j \cap B_j$ a function in $\mathcal{F}(A, B)$. As we will see, for each s, $0 < s < 1$, $f(s)$ belongs to a totally bounded subset C_s of $[A, B]_s$. Let S_t be the image of the unit sphere of $\bigoplus_1^n A_j$ under $F(it)$ and consider the set $\bigcup_{t\in E} S_t$.

Given $\varepsilon > 0$ we can find a finite set $\{t_i\}$, $t_i \in E$, such that for each $t \in E$ there is a $t_j \in \{t_i\}$ with the property that $\|F(it) - F(it_j)\|_{\mathcal{M}_0} < \varepsilon$ which implies that the distance between S_t and S_{t_j} is less than ε, and this in turn implies that every point of $\bigcup_{t\in E} S_t$ is at distance less than ε from $\bigcup_i S_{t_i}$, but the sets S_{t_i} are totally bounded and consequently so is $\bigcup_i S_{t_i}$. Since this holds for every ε it follows that $\bigcup_{t\in E} S_t$ is totally bounded. Returning to the function $f(\xi) \in \mathcal{F}(A, B)$ we have $f(it) \in \bigcup_{t\in E} S_t$ for all $t \in E$, and $\|f(1+it)\|_B \leqslant \sup_t \|F(1+it)\|_{\mathcal{M}_1}$. Hence, by 9.6, it follows that for each s, $0 < s < 1$, there exists a compact subset K_s of $[A, B]_s$ such that $f(s) \in K_s$. Now the points $[f_1(s), \ldots, f_n(s)]$, $f_i \in \mathcal{G}(A_j, B_j)$, $\|f_i\|_{\mathcal{F}_j} \leqslant 1$, are dense in the unit sphere of $\bigoplus_1^n [A_j, B_j]_s$ and consequently the image of this sphere under $F(s)$ is also contained in K_s.

31.1. Let us begin observing that under the given assumptions we also have

$$\|L(x_1, \ldots, x_n)\|_A \leqslant M \|x_1\|_{A_1} \prod_2^n \|x_j\|_{A_j\cap B_j} \quad \text{if} \quad x_1 \in A_1,$$

$$\|L(x_1, \ldots, x_n)\|_B \leqslant M \|x_1\|_{B_1} \prod_2^n \|x_j\|_{A_j\cap B_j} \quad \text{if} \quad x_1 \in B_1,$$

[2] See 9.2.

where $M = \max(M_0, M_1)$. Suppose now that $x_1 \in A_1 + B_1$ and set $x_1 = y_1 + z_1$, where $y_1 \in A_1$, $z_1 \in B_1$, $\|y_1\|_{A_1} + \|z_1\|_{B_1} \leqslant \|x_1\|_{A_1+B_1} + \varepsilon$; then the preceding inequalities give

$$L(x_1, \ldots, x_n) = L(y_1, x_2, \ldots, x_n) + L(z_1, x_2, \ldots, x_n),$$

where

$$\|L(y_1, \ldots, x_n)\|_A \leqslant M \|y_1\|_{A_1} \prod_2^n \|x_j\|_{A_j\cap B_j},$$

$$\|L(z_1, \ldots, x_n)\|_B \leqslant M \|z_1\|_{B_1} \prod_2^n \|x_j\|_{A_j\cap B_j}.$$

Thus

$$\|L(x_1, \ldots, x_n)\|_{A+B} \leqslant M(\|x_1\|_{A_1+B_1} + \varepsilon) \prod_2^n \|x_j\|_{A_j\cap B_j}$$

and since ε is arbitrary

(1) $$\|L(x_1, \ldots, x_n)\|_{A+B} \leqslant M \|x_1\|_{A_1+B_1} \prod_2^n \|x_j\|_{A_j\cap B_j}.$$

Let now $f_1 \in \overline{\mathcal{F}}(A_1, B_1)$ and $f_j \in \mathcal{G}(A_j, B_j)$, $i = 2, \ldots, n$, and set

(2) $$f(\xi) = \int_\Gamma M_0^{\eta-1} M_1^{-\eta} L[f_1'(\eta), f_2(\eta), \ldots, f_n(\eta)]\, d\eta \quad (\xi = s + it,\ 0 < s < 1),$$

where Γ is a path joining the point $\frac{1}{2}$ with the point ξ and entirely contained in $0 < s < 1$. We will show that $f \in \overline{\mathcal{F}}(A, B)$ and that

$$\|f\|_{\overline{\mathcal{F}}} \leqslant \|f_1\|_{\overline{\mathcal{F}}_1} \prod_2^n \|f_j\|_{\mathcal{F}_j}$$

where $\mathcal{F}_j = \mathcal{F}(A_j, B_j)$, $\overline{\mathcal{F}}_1 = \overline{\mathcal{F}}(A_1, B_1)$.

From the multilinear character of L and (1) it is clear that the integrand of the integral above represents an $(A+B)$-valued, $(A+B)$-bounded, $(A+B)$-analytic function in $0 < s < 1$. Thus $f(\xi)$ is $(A+B)$-analytic and uniformly $(A+B)$-continuous in $0 < s < 1$ and it can therefore be extended to an $(A+B)$-continuous function in $0 \leqslant s \leqslant 1$. This extension, which we will also denote by $f(\xi)$, clearly satisfies the inequality $\|f(\xi)\|_{A+B} \leqslant c(1 + |\xi|)$.

Let now h be a positive real number and let $\delta(h)$ be the smallest real for which

$$\|f_j(\xi + it) - f_j(\xi)\|_{A_j\cap B_j} \leqslant \delta(h), \quad 0 \leqslant t \leqslant h,\ 0 < s < 1,\ j = 2, \ldots, n,$$

$$|M_0^{\xi+it-1} M_1^{-\xi-it} - M_0^{\xi-1} M_1^{-\xi}| \leqslant \delta(h)/\min(M_0, M_1), \quad 0 \leqslant t \leqslant h,\ 0 < s < 1,$$

and let $c \geqslant \|f_1'(\xi)\|_{A_1+B_1}+1$, $c \geqslant \|f_j(\xi)\|_{A_j \cap B_j}$; then from 30.1, (1), and 31.1, (1), we obtain

$$\|M_0^{\xi+it-1}M_1^{-\xi-it}L[f_1'(\xi+it), f_2+it), \ldots, f_n(\xi+it)]-$$
$$-M_0^{\xi-1}M_1^{-\xi}L[f_1'(\xi+it), f_2(\xi), \ldots, f_n(\xi)]\|_{A+B}$$
$$\leqslant nM\min(M_0, M_1)^{-1}c^n\delta(h).$$

Integrating this inequality with respect to t between 0 and h we obtain

$$\|f(\xi+ih)-f(\xi)-M_0^{\xi-1}M_1^{-\xi}L[f_1(\xi+ih)-f_1(\xi), f_2(\xi), \ldots, f_n(\xi)\|_{A+B}$$
$$\leqslant nM\min(M_0, M_1)^{-1}c^n\delta(h)h.$$

Setting

$$\xi_j = \xi+j\frac{ih}{m} \qquad (j = 0, 1, 2, \ldots, m)$$

by addition we obtain

$$\|f(\xi+ih)-f(\xi)-\sum_{j=0}^{m-1} M_0^{\xi_j-1}M_1^{-\xi_j}L[f_1(\xi_j+1)-$$
$$-f_1(\xi_j), f_2(\xi_j), \ldots, f_n(\xi_j)]\|_{A+B} \leqslant nh\delta\left(\frac{h}{m}\right)c^nM\min(M_0, M_1)^{-1}.$$

By continuity this inequality is valid for all ξ in the strip $0 \leqslant s \leqslant 1$, and since $\delta(h) \to 0$ as $h \to 0$ it follows that the sum

$$S_m(\xi, h) = \sum_{j=0}^{m-1} M_0^{\xi_j-1}M_1^{-\xi_j}L[f_1(\xi_{j+1})-f_1(\xi_j), f_2(\xi_j), \ldots, f_n(\xi_j)]$$

converges to $f(\xi+ih)-f(\xi)$ in $A+B$ as $m \to \infty$, $0 \leqslant s \leqslant 1$,

Now suppose that $\xi = it$; then $f_1(\xi_{j+1})-f_1(\xi_j) \in A_1$ and thus the value of the preceding sum belongs to A. Furthermore, if $m = 2^k$ and $k \to \infty$, $S_m(\xi, h)$ converges in A. In fact, let now $\delta(h)$ be the smallest positive real such that

$$\|f_j(it+i\tau)-f_j(it)\|_{A_j} \leqslant \delta(h), \qquad 0 \leqslant \tau \leqslant h,$$
$$\|M_0^{i(t+\tau)-1}M_1^{-i(t+\tau)}-M_0^{it-1}M_1^{-it}\| \leqslant \delta(h)M_0^{-1}, \qquad 0 \leqslant \tau \leqslant h.$$

Let

$$\xi_j = it+\frac{ijh}{m} \qquad (j = 0, \ldots, m);$$

then from 30.1, (1), and 31.1, (1), we obtain

$$\|M_0^{it-1}M_1^{-it}L[f_1(\xi_{j+1})-f_1(\xi_j), f_2(it), \ldots, f_n(it)]-$$
$$-M_0^{\xi_j-1}M_1^{\xi_j}L[f_1(\xi_{j+1})-f_1(\xi_j), f_2(\xi_j), \ldots, f_n(\xi_j)\|_A \leqslant n\delta(h)c^n\left(\frac{h}{m}\right)MM_0^{-1},$$

where

$$c \geqslant \left\|\frac{1}{\tau}[f_1(it+i\tau)-f_1(it)]\right\|_{A_1}+1 \quad \text{and} \quad c \geqslant \|f_j(it)\|_{A_i} \quad (j = 2, \ldots, n)$$

for all t and τ.

Summing over j we get

$$\|M_0^{it-1}M_1^{-it}L[f_1(it+ih)-f_1(it), f_2(it), \ldots, f_n(it)] - S_m(it, h)\|_A$$
$$\leqslant n\delta(h)c^nhMM_0^{-1}$$

and from this inequality by addition we obtain

$$\|S_l(it, h)-S_{ml}(it, h)\|_A \leqslant nc^n\delta\left(\frac{h}{l}\right)hMM_0^{-1}.$$

Since $\delta(h) \to 0$ as $h \to 0$, from this we conclude that $S_{2^k}(it, h)$ converges in A as $k \to \infty$. Thus

$$f(it+ih)-f(it) = \lim_{k\to\infty} S_{2^k}(it, h) \in A.$$

To estimate the norm in A of $f(it+ih)-f(it)$ we observe that, on account of 11.1, (1),

$$\|L[f_1(it+ih)-f_1(it), f_2(it), \ldots, f_n(it)]\|_A$$
$$\leqslant M_0\|f_1(it+ih)-f_1(it)\|_{A_1}\prod_2^n \|f_j(it)\|_{A_j} \leqslant M_0h\|f_1\|_{\overline{\mathscr{F}}_1}\prod_2^n \|f_j\|_{\mathscr{F}_j},$$

where $\overline{\mathscr{F}}_1 = \overline{\mathscr{F}}(A_1, B_1)$, $\mathscr{F}_j = \mathscr{F}(A_j, B_j)$, whence by addition we get

$$\|S_m(it, h)\|_A \leqslant h\|f_1\|_{\overline{\mathscr{F}}_1}\prod_2^n \|f_j\|_{\mathscr{F}_j}.$$

From this it follows that

$$\frac{1}{h}\|f(it+ih)-f(it)\|_A \leqslant \|f_1\|_{\overline{\mathscr{F}}_1}\prod_2^n \|f_j\|_{\mathscr{F}_j}.$$

In a similar fashion we would obtain that $f(1+it+ih)-f(1+it) \in B$ and

$$\frac{1}{h}\|f(1+it+ih)-f(1+it)\|_B \leqslant \|f_1\|_{\overline{\mathscr{F}}_1}\prod_2^n \|f_j\|_{\mathscr{F}_j}.$$

Summarizing, we have proved that $f(\xi) \in \mathscr{F}(A, B)$ and that

$$\|f\|_{\mathscr{F}} \leqslant \|f_1\|_{\overline{\mathscr{F}}_1}\prod_2^n \|f_j\|_{\mathscr{F}_j}.$$

Consider now the multilinear map

$$\mathscr{L}(f_1, f_2, \dots, f_n), \quad f_1 \in \bar{\mathscr{F}}(A_1, B_1), \quad f_j \in \mathscr{G}(A_j, B_j), \quad j = 2, \dots, n,$$

with values in $\bar{\mathscr{F}}(A, B)$ defined by

$$\mathscr{L}(f_1, f_2, \dots, f_n) = f(\xi)$$

where $f(\xi)$ is the function defined in 31.1, (2). We have just shown that $\mathscr{L}$ is bounded with respect to the norms of $\bar{\mathscr{F}}(A_1, B_1)$ and $\mathscr{F}(A_j, B_j)$, $j = 2, \dots, n$. Furthermore $\mathscr{L}$ has the following obvious property: if g is a complex valued function which is continuous and bounded in $0 \leqslant s \leqslant 1$ and analytic in $0 \leqslant s \leqslant 1$, then for $j \geqslant 1$

$$\text{(3)} \quad \frac{d}{d\xi}\mathscr{L}(f_1, f_2, \dots, gf_j, \dots, f_n) = g(\xi)\frac{d}{d\xi}\mathscr{L}(f_1, f_2, \dots, f_n), \quad 0 < s < 1,$$

and if $f_1'(\xi) = g(\xi)\bar{f}_1'$, then

$$\text{(4)} \qquad \frac{d}{d\xi}\mathscr{L}(f_1, \dots, f_n) = g(\xi)\frac{d}{d\xi}\mathscr{L}(\bar{f}_1, f_2, \dots, f_n).$$

On account of its boundedness, as shown in 30.1, $\mathscr{L}$ can be extended to a bounded multilinear mapping $\bar{\mathscr{L}}$ of $\bar{\mathscr{F}}(A_1, B_1) \oplus \bigoplus_2^n \mathscr{F}(A_j, B_j)$ into $\bar{\mathscr{F}}(A, B)$. This extension $\bar{\mathscr{L}}$ also has properties (3) and (4). To see this let us consider the linear mappings $\mathscr{N}$ and $\mathscr{N}_1$ of $\bar{\mathscr{F}}(A, B)$ and $\bar{\mathscr{F}}(A_1, B_1)$ given by

$$\mathscr{N}(f) = \int_\Gamma g(\eta) f'(\eta)\, d\eta, \quad \mathscr{N}_1(f_1) = \int_\Gamma g(\eta) f_1'(\eta)\, d\eta,$$

where Γ is a path joining the point $\frac{1}{2}$ with the point ξ. As readily seen, these are special cases of the multilinear mapping $\mathscr{L}$ introduced above, and consequently, as we have already seen $\mathscr{N}$ and $\mathscr{N}_1$ are bounded linear mappings of $\bar{\mathscr{F}}(A, B)$ and $\bar{\mathscr{F}}(A_1, B_1)$ into themselves. Now we can reformulate (3) and (4) in terms of $\mathscr{N}$ and $\mathscr{N}_1$, as follows:

$$\mathscr{L}(f_1, f_2, \dots, gf_j, \dots, f_n) = \mathscr{N}\mathscr{L}(f_1, \dots, f_n), \quad 2 \leqslant j \leqslant n,$$
$$\mathscr{L}(\mathscr{N}_1 f_1, f_2, \dots, f_n) = \mathscr{N}\mathscr{L}(f_1, \dots, f_n).$$

Since the mappings $f_j \to gf_j$, $f_1 \to \mathscr{N}_1 f_1$ and $f \to \mathscr{N} f$ are continuous in the corresponding spaces, the identities above are preserved by passages to the limit, and thus they hold for the extended multilinear mapping $\bar{\mathscr{L}}$.

Let now $0 < s < 1$. If $\bar{\mathscr{L}}(f_1, \dots, f_n) = f$ set $\bar{\mathscr{L}}_s(f_1, \dots, f_n) = f'(s)$. Then clearly $\bar{\mathscr{L}}_s$ is a multilinear bounded mapping from $\bar{\mathscr{F}}(A_1, B_1) \oplus \bigoplus_2^n$

$\mathscr{F}(A_j, B_j)$ into $[A, B]^s$. We will show that the value of $\bar{\mathscr{L}}_s$ depends only on the values of $f_1'(s)$ and $f_j(s)$, $2 \leqslant j \leqslant n$. First let us show that if either $f_1(s) = 0$ or $f_j(s) = 0$ for some j, $2 \leqslant j \leqslant n$, then $\bar{\mathscr{L}}(f_1, \dots, f_n) = 0$. Let us assume first that $f_1'(s) = 0$. Since the value of $\bar{\mathscr{L}}$ only depends on f_1', by subtracting an appropriate constant from f_1 we may assume that also $f_1(s) = 0$. Consider now the function

$$\bar{f}_1(\xi) = f_1(\xi)\frac{\xi + s}{\xi - s} + \int_\Gamma f_1(\eta)\frac{2s}{(\eta - s)^2}\, d\eta,$$

where Γ is a path joining the point $\frac{1}{2}$ with the point ξ. It is easy to see that $\bar{f}_1 \in \bar{\mathscr{F}}(A_1, B_1)$. Furthermore

$$\bar{f}_1'(\xi) = \bar{f}_1'(\xi)\left(\frac{\xi - s}{\xi + s}\right).$$

Consequently by 31.1, (4), we have

$$\bar{\mathscr{L}}_s(f_1, \dots, f_n) = \left[\frac{d}{d\xi}\bar{\mathscr{L}}(f_1, \dots, f_n)\right]_{\xi = s} = \left[\frac{\xi - s}{\xi + s}\frac{d}{d\xi}\bar{\mathscr{L}}(\bar{f}_1, \dots, f_n)\right]_{\xi = s} = 0.$$

On the other hand, if $f_j(s) = 0$, $j \geqslant 2$, we set

$$\bar{f}_j = \frac{\xi + s}{\xi - s} f_j.$$

Then clearly $\bar{f}_j \in \mathscr{F}(A_j, B_j)$ and so, by 3), we have

$$\bar{\mathscr{L}}_s(f_1, \dots, f_n) = \left[\frac{d}{d\xi}\bar{\mathscr{L}}(f_1, \dots, f_n)\right]_{\xi = s}$$
$$= \left[\frac{\xi - s}{\xi + s}\frac{d}{d\xi}\bar{\mathscr{L}}(f_1, \dots, \bar{f}_j, \dots, f_n)\right]_{\xi = s} = 0.$$

Let us set $C = [A, B]^s$, $C_1 = [A_1, B_1]^s$, $C_j = [A_j, B_j]_s$ and consider a multilinear mapping $\bar{L}_s$ of $\bigoplus_1^n C_j$ to C defined as follows: if $x_j \in C_j$, $j = 1, \dots, n$, and $f_1 \in \bar{\mathscr{F}}(A_1, B_1)$, $f_j \in \mathscr{F}(A_j, B_j)$, are such that $f_1'(s) = x_1$, $f_j(s) = x_j$, $j = 2, \dots, n$, then

$$\bar{L}_s(x_1, \dots, x_n) = M_0^{1-s} M_1^s \bar{\mathscr{L}}_s(f_1, \dots, f_n).$$

First of all let us verify that $\bar{L}_s$ is well defined. Suppose that for $\bar{f}_1 \in \mathscr{F}(A_1, B_1)$, $\bar{f}_j = \mathscr{F}(A_j, B_j)$, we also have $\bar{f}_1'(s) = x_1$, $\bar{f}_j(s) = x_j$, $j = 2, \dots, n$.

Then

$$\bar{\mathscr{L}}_s(f_1, \ldots, f_n) - \bar{\mathscr{L}}_s(\bar{f}_1, \ldots, \bar{f}_n)$$

$$= \sum_{j=1}^{n} [-\bar{\mathscr{L}}_s(\bar{f}_1, \ldots, \bar{f}_j, f_{j+1}, \ldots, f_n) + \bar{\mathscr{L}}_s(\bar{f}_1, \ldots, \bar{f}_{j-1}, f_j, \ldots, f_n)]$$

$$= \sum_{j=1}^{n} \bar{\mathscr{L}}_s(\bar{f}_1, \ldots, \bar{f}_{j-1}, f_j - \bar{f}_j, \ldots, f_n).$$

Since $f_j(s) - \bar{f}_j(s) = 0$ for $2 \leqslant f \leqslant n$ and $f_1'(s) - \bar{f}_1'(s) = 0$, as we saw above, all terms in the last sum vanish. Consequently

$$\bar{\mathscr{L}}_s(f_1, \ldots, f_n) = \bar{\mathscr{L}}_s(\bar{f}_1, \ldots, \bar{f}_n)$$

and $\bar{L}_s(x_1, \ldots, x_n)$ is well defined. If we choose $f_1, \ldots, f_n$ in such a way that $\|f_1\|_{\bar{\mathscr{F}}_1} \leqslant \|x_1\|_{C_1} + \varepsilon$, $\|f_j\|_{\mathscr{F}_j} \leqslant \|x_j\|_{C_j} + \varepsilon$, $j = 2, \ldots, n$, we find that

$$\|\bar{L}_s(x_1, \ldots, x_n)\|_C = M_0^{1-s} M_1^s \|\bar{\mathscr{L}}_s(f_1, \ldots, f_n)\|_C$$

$$\leqslant M_0^{1-s} M_1^s \|\bar{\mathscr{L}}(f_1, \ldots, f_n)\|_{\bar{\mathscr{F}}} \leqslant M_0^{1-s} M_1^s \|f_1\|_{\bar{\mathscr{F}}_1} \prod_1^n \|f_j\|_{\mathscr{F}_j}$$

$$\leqslant M_0^{1-s} M_1^s \prod_1^n (\|x_j\|_{C_j} + \varepsilon).$$

Now it is readily verified that if $x_1 \epsilon C_1$ and $x_j \epsilon A_j \cap B_j$, then $\bar{L}_s(x_1, \ldots, x_n)$ coincides with $L(x_1, \ldots, x_n)$. For this purpose it is enough to set $f_1(\xi) = x_1 \xi$, $f_j(\xi) = \text{const} = x_j$, $j = 2, \ldots, n$. This concludes the proof of 31.1.

31.2. Our statement is a special case of 11.1. In fact, we consider the multilinear functional $\mathscr{L}(L, x_1, \ldots, x_n)$ with values in $A+B$ defined for $L \epsilon \mathscr{M}_0 + \mathscr{M}_1$, $x_j \epsilon (A_j \cap B_j)$ by

$$\mathscr{L}(L, x_1, \ldots, x_n) = L(x_1, \ldots, x_n)$$

and apply the result in 11.1.

32.1. We begin discussing the duals of certain Banach space valued functions. Let A be a Banach space, A^* its dual. Consider the space $A'(A^*)$ of functions with values in some space containing A^* and such that $g(t_2) - g(t_1) \epsilon A^*$ for any t_1 and t_2 and that

$$\left\| \frac{1}{t_2 - t_1} [g(t_2) - g(t_1)] \right\|_{A^*} \leqslant M, \quad t_1 \neq t_2.$$

We reduce this space modulo constant functions and in the quotient space $A(A^*)$ we introduce the norm

$$\|g\|_A = \sup_{t_1, t_2} \left\| \frac{g(t_1) - g(t_2)}{t_1 - t_2} \right\|_{A^*}.$$

We consider on the other hand the space $C_0(A)$ of continuous A-valued functions $f(t)$, $-\infty < t < \infty$, with compact support. In $C_0(A)$ we introduce the norm

$$\|f\| = \int_{-\infty}^{+\infty} \|f(t)\|_A \, dt.$$

The completion of C_0 with respect to this norm can be identified with the space $L^1(A)$ of strongly measurable A-valued functions $f(t)$ such that

$$\|f\| = \int_{-\infty}^{+\infty} \|f(t)\|_A \, dt < \infty$$

reduced modulo functions vanishing almost everywhere.

For $f(t) \epsilon C_0(A)$ and $g \epsilon A'(A^*)$ we define

$$\int_{-\infty}^{+\infty} \langle f(t), dg(t) \rangle \tag{1}$$

as the limit of the Riemann sums

$$\sum \langle f(\tau_j), g(t_{j+1}) - g(t_j) \rangle, \quad t_j \leqslant \tau_j \leqslant t_{j+1},$$

as $\sup_j (t_{j+1} - t_j) \to 0$. On account of the uniform continuity of $f(t)$ this limit is easily shown to exist. Furthermore, since

$$\left| \sum \langle f(\tau_j), g(t_{j+1}) - g(t_j) \rangle \right| \leqslant \sum \|f(\tau_j)\|_A \|g(t_{j+1}) - g(t_j)\|_{A^*}$$

$$\leqslant \sum \|f(\tau_j)\|_A \|g\|_A (t_{j+1} - t_j),$$

we have

$$\left| \int_{-\infty}^{+\infty} \langle f(t), dg(t) \rangle \right| \leqslant \|g\|_A \int_{-\infty}^{+\infty} \|f(t)\|_A \, dt = \|g\|_A \|f\|, \tag{2}$$

which shows that, for given g, the integral (1) represents a linear functional of f which is bounded with respect to the norm $\|f\|$, and the norm of this linear functional does not exceed $\|g\|_A$.

More generally, if $f(t)$ is a continuous A-valued function such that $\int_{-\infty}^{+\infty} \|f(t)\|_A \, dt < \infty$, there exists a sequence $f_n(t) \epsilon C_0(A)$ such that $\|f_n - f\| \to 0$ and we define

$$\int_{-\infty}^{+\infty} \langle f(t), dg(t) \rangle = \lim_{n \to \infty} \int_{-\infty}^{+\infty} \langle f_n(t), dg(t) \rangle.$$

Evidently for given g this generalized integral still represents a linear functional of f which is bounded with respect to the norm $\|f\|$ of f, the norm of the functional not exceeding $\|g\|_A$. Suppose now that $f(t)$ has the form $\varphi(t)x$ where $x \in A$ and $\varphi(t)$ is a continuous integrable numerical function. Then from the definition of (1) it follows that

$$\int_{-\infty}^{+\infty} \langle f(t), dg(t)\rangle = \int_{-\infty}^{+\infty} \varphi(t) d\langle x, g(t)\rangle,$$

where the integral on the right is a Stieltjes integral. Since $\langle x, g(t)\rangle$ is a Lipschitz function of t we can write

$$\int_{-\infty}^{+\infty} \langle f(t), dg(t)\rangle = \int_{-\infty}^{+\infty} \varphi(t)\left[\frac{d}{dt}\langle x, g(t)\rangle\right] dt. \tag{3}$$

Conversely, every linear functional l on C_0, which is bounded with respect to the norm $\|f\|$, has the form (1), where $\|g\|_A$ does not exceed the norm of l. To show this we first extend l to $L^1(A)$ and consider $l[x^\chi(t)]$ where $x \in A$ and $\chi(\tau, t)$ is the characteristic function of the interval $[0, \tau)$ if $\tau \geqslant 0$, or minus the characteristic function of $[\tau, 0)$ if $\tau < 0$. Clearly $l[x^\chi(\tau, t)]$ is a continuous linear functional of x for each given τ. Consequently

$$l[x^\chi(\tau, t)] = \langle x, g(\tau)\rangle,$$

where $g(\tau) \in A^*$. On the other hand,

$$\|g(\tau_1) - g(\tau_2)\|_{A^*} = \sup_x l\{x[\chi(\tau_1, t) - \chi(\tau_2, t)]\}, \qquad \|x\|_A \leqslant 1,$$

$$\|g(\tau_1) - g(\tau_2)\|_{A^*} \leqslant \sup_x \|l\| \int_{-\infty}^{+\infty} \|x[\chi(\tau_1, t) - \chi(\tau_2, t)]\|_A dt$$

$$\leqslant \|l\| \int_{-\infty}^{+\infty} |\chi(\tau_1, t) - \chi(\tau_2, t)| dt = \|l\| |\tau_2 - \tau_1|.$$

Consequently $g(t) \in A'$ and $\|g\|_A \leqslant \|l\|$.

Let now $f(t) \in C_0(A)$ and $S_n(t) = f(k/n)$ if $k/n \leqslant t < (k+1)/n$. Then

$$l[f(t)] = l[S_n(t)] + l[f(t) - S_n(t)]$$

$$= l\left\{\sum_k f\left(\frac{k}{n}\right)\left[\chi\left(\frac{k+1}{n}, t\right) - \chi\left(\frac{k}{n}, t\right)\right]\right\} + l[f(t) - S_n(t)]$$

$$= \sum_k \left\{l\left[f\left(\frac{k}{n}\right)\chi\left(\frac{k+1}{n}, t\right)\right] - l\left[f\left(\frac{k}{n}\right)\chi\left(\frac{k}{n}, t\right)\right]\right\} + l[f(t) - S_n(t)]$$

$$= \sum_k \left\langle f\left(\frac{k}{n}\right), g\left(\frac{k+1}{n}\right) - g\left(\frac{k}{n}\right)\right\rangle + l[f(t) - S_n(t)].$$

Now, as $n \to \infty$, $\|f(t) - S_n(t)\| \to 0$, and the sum in the last expression converges to

$$\int_{-\infty}^{+\infty} \langle f(t), dg(t)\rangle.$$

Consequently

$$l(f) = \int_{-\infty}^{+\infty} \langle f(t), dg(t)\rangle, \tag{4}$$

where $\|g\|_A \leqslant \|f\|$. But as we saw above, we also have $\|l\| \leqslant \|g\|_A$ whence $\|l\| = \|g\|_A$. By continuity this representation of $l(f)$ is also valid for continuous A-valued functions $f(t)$ such that

$$\int_{-\infty}^{+\infty} \|f(t)\|_A dt < \infty.$$

Consider now the interpolation pairs (A, B) and (A^*, B^*). Since both A^* and B^* are continuously embedded in $(A \cap B)^*$ we have a bilinear functional on $(A \cap B) \oplus (A^* + B^*)$, which we will denote by $\langle x, y\rangle$, which is the value of the continuous linear functional $y \in (A \cap B)^*$ at $x \in (A \cap B)$. Clearly, if $y \in A^*$ or B^* we have

$$|\langle x, y\rangle| \leqslant \|x\|_A \|y\|_{A^*} \quad \text{and} \quad |\langle x, y\rangle| \leqslant \|x\|_B \|y\|_{B^*}$$

respectively, and thus by 11.1 we also have

$$|\langle x, y\rangle| \leqslant \|x\|_C \|y\|_{C'},$$

whenever $y \in C' = [A^*, B^*]^s$, where $C = [A, B]_s$. Thus with each $y \in C'$ there is associated a linear functional on $A \cap B$ which is continuous with respect to the norm of C. Since $A \cap B$ is dense in C this linear functional can be extended uniquely to C, with norm not exceeding $\|y\|_{C'}$.

Let now $f \in \mathcal{G}(A, B)$ and $g \in \mathcal{F}(A^*, B^*)$. Let $0 < s < 1$, and consider

$$I = -i\left[\int_{-\infty}^{+\infty} \langle f(it)\mu_0(s, t), dg(it)\rangle + \int_{-\infty}^{+\infty} \langle f(1+it)\mu_1(s, t), dg(1+it)\rangle\right],$$

where μ_0 and μ_1 are the Poisson kernels for the strip introduced in 9.4. Since $f(\xi) = \sum x_j f_j(\xi)$ where $x_j \in A \cap B$ and the $f_j(\xi)$ are complex valued continuous functions in $0 \leqslant s \leqslant 1$ analytic in $0 < s < 1$ and tending to zero at infinity. By (3) we can write I as

$$I = -i\sum_j \left[\int_{-\infty}^{+\infty} f_j(it)\frac{d}{dt}\langle x_j, g(it)\rangle \mu_0(s, t) dt + \right.$$

$$\left. + \int_{-\infty}^{+\infty} f_j(1+it)\frac{d}{dt}\langle x_j, g(1+it)\rangle \mu_1(s, t) dt\right].$$

Now consider the functions

$$h_j(\xi) = \left\langle x_j, \frac{d}{d\xi} g(\xi) \right\rangle.$$

They are bounded analytic functions in the strip $0 < s < 1$ and have non-tangential limits almost everywhere on the boundary of $0 < s < 1$, equal to

$$-i\frac{d}{dt}\langle x_j, g(it)\rangle \quad \text{and} \quad -i\frac{d}{dt}\langle x_j, g(1+it)\rangle,$$

respectively. Consequently

$$I = \sum_j f_j(s)\langle x_j, g'(s)\rangle = \left\langle \sum x_j f_j(s), g'(s)\right\rangle = \langle f(s), g'(s)\rangle.$$

Thus if $x = f(s)$ and $y = g'(s)$ we have $\langle x, y\rangle = I$. Now suppose that $x \in C = [A, B]_s$ and that $f \in \mathscr{F}(A, B)$ is such that $f(s) = x$. Then if $f_n \in \mathscr{G}(A, B)$ are such that $\|f_n - f\|_{\mathscr{F}} \to 0$, that is, such that $\|f_n(it) - f(it)\|_A \to 0$ and $\|f_n(1+it) - f(1+it)\|_B \to 0$ uniformly in t, we have

$$\|f_n(s) - x\|_C = \|f_n(s) - f(s)\|_C \leqslant \|f_n - f\|_{\mathscr{F}} \to 0$$

and consequently

$$\langle x, y\rangle = \lim_{n\to\infty}\langle f_n(s), g'(s)\rangle$$

$$= \lim_{n\to\infty}\Big[-i\int_{-\infty}^{+\infty}\langle f_n(it)\mu_0(s,t), dg(t)\rangle - i\int_{-\infty}^{+\infty}\langle f_n(1+it)\mu_1(s,t), dg(1+it)\rangle -$$

$$-i\int_{-\infty}^{+\infty}\langle f(it)\mu_0(s,t), dg(t)\rangle - i\int_{-\infty}^{+\infty}\langle f(1+it)\mu_1(st), dg(1+it)\rangle\Big]$$

which is the desired representation of the linear functional on C associated with $y \in C'$.

Now suppose that l is a bounded linear functional on $C = [A, B]_s$. Since C is a factor space of $\mathscr{F}(A, B)$, l induces a linear functional l on $\mathscr{F}(A, B)$ with the same norm.

Consider now the mapping $\varphi\colon \mathscr{F}(A, B) \to L^1(A)\oplus L^1(B)$ given by

$$\varphi(f) = [f(it)\mu_0(s,t), f(1+it)\mu_1(s,t)].$$

This mapping is linear and one-one. On the image of $\mathscr{F}(A, B)$ under φ define the linear functional λ by

$$\lambda[\varphi(f)] = \bar{l}(f).$$

Then by 9.4, iii), we have

$$|\lambda[\varphi(f)]| = |l(f)| = |l[f(s)]| \leqslant \|l\|\,\|f(s)\|_s$$

$$\leqslant \|l\|\Big[\int_{-\infty}^{+\infty}\|f(it)\|_A\mu_0(s,t)\,dt + \int_{-\infty}^{+\infty}\|f(1+it)\|_B\mu_1(s,t)\,dt\Big] = \|l\|\,\|\varphi(f)\|,$$

where $\|\varphi(f)\|$ stands for the norm of $\varphi(f)$ as an element of $L^1(A)\oplus L^1(B)$, which shows that λ is a bounded linear functional with norm not exceeding $\|l\|$. Now we extend λ to all of $L^1(A)\oplus L^1(B)$ with preservation of norm and represent it as in (4) obtaining

(5) $\lambda[\varphi(f)] = l(f)$

$$= \int_{-\infty}^{+\infty}\langle f(it)\mu_0(s,t), dg_0(t)\rangle + \int_{-\infty}^{+\infty}\langle f(1+it)\mu_1(s,t), dg_1(t)\rangle,$$

where $g_0(t) \in A'(A^*) = A_0'$, $g_1(t) = A'(B^*) = A_1'$ and $\max(\|g_0\|_{A_0}, \|g_1\|_{A_1}) = \|\lambda\| \leqslant \|l\|$. The functions $g_0(t)$ and $g_1(t)$ are determined up to an additive constant. Suppose now that $x \in A \frown B$ and that $h(\xi)$ is a complex valued function which is continuous in $0 \leqslant s \leqslant 1$, analytic in $0 < s < 1$ and tends to zero at infinity. Setting $f(\xi) = xh(\xi)$ we obtain

$$l[f] = l[f(s)] = h(s)l(x)$$

$$= \int_{-\infty}^{+\infty} h(it)\mu_0(s,t)\frac{d}{dt}\langle x, g_0(t)\rangle dt + \int_{-\infty}^{+\infty} h(1+it)\mu_1(s,t)\frac{d}{dt}\langle x, g_1(t)\rangle dt.$$

Evidently, if $h(s) = 0$, then the right-hand side of the expression above vanishes, and this, as we will prove, implies that the functions

$$\frac{d}{dt}\langle x, g_0(t)\rangle \quad \text{and} \quad \frac{d}{dt}\langle x, g_1(t)\rangle$$

are the boundary values on $s = 0$ and $s = 1$ respectively, of a function $k(x, \xi)$ of ξ which is analytic and bounded in $0 < s < 1$. Let us accept this fact for the moment and draw conclusions from it. First of all $k(x, \xi)$ clearly depends linearly on x. Furthermore

$$|k(x, \xi)| \leqslant \max\Big[\sup_t\Big|\frac{d}{dt}\langle x, g_0(t)\rangle\Big|, \sup_t\Big|\frac{d}{dt}\langle x, g_1(t)\rangle\Big|\Big]$$

$$\leqslant \max[\|x\|_A\|g_0\|_{A_0}, \|x\|_B\|g_1\|_{A_1}] \leqslant \|x\|_{A\frown B}\max[\|g_0\|_{A_0}, \|g_1\|_{A_1}]$$

which means that $k(x, \xi)$ is, for each ξ in $0 < s < 1$, a bounded linear functional on $A \frown B$. Define now the function $k(\xi)$ with values in $(A \frown B)^*$ by

$$\langle x, k(\xi)\rangle = k(x, \xi).$$

Then, since the elements of $(A \frown B)$ form determining space of linear functionals on $(A \frown B)^*$, and since $\langle x, k(\xi)\rangle$ is analytic in $0 < s < 1$, for each $x \in A \cap B$, it follows that $k(\xi)$ is an $(A \cap B)^*$-valued bounded analytic function. Consider now the function

$$g(\xi) = \int_{\Gamma} k(\eta)\,d\eta$$

where Γ is a path entirely contained in $0 < s < 1$ and joining the point $1/2$ with the point ξ. This function $g(\xi)$ is uniformly $(A \cap B)^*$-continuous (since its derivative is $(A \cap B)^*$-bounded) and therefore it can be extended continuously to $0 \leqslant s \leqslant 1$. Furthermore, if ξ is in $0 < s < 1$ and $x \in A \cap B$ we have

$$\langle x, g(\xi+ih)\rangle - \langle x, g(\xi)\rangle = i\int_0^h \langle x, k(\xi+i\tau)\rangle\,d\tau$$

and by letting $s \to 0$ in $\xi = s+it$, since

$$\langle x, k(s+it+i\tau)\rangle \to \frac{d}{dt}\langle x, g_0(t+\tau)\rangle$$

for almost all τ we obtain

$$\langle x, g(it+ih) - g(it)\rangle = i\int_0^h \frac{d}{dt}\langle x, g_0(t+\tau)\rangle\,d\tau = i\langle x, g_0(t+h) - g_0(t)\rangle$$

and since this holds for all x, we obtain

$$g(it+ih) - g(it) = i[g_0(t+h) - g_0(t)] \in A^* \tag{6}$$

and

$$\left\|\frac{1}{h}[g(it+ih) - g(it)]\right\|_{A^*} = \left\|\frac{1}{h}[g_0(t+h) - g_0(t)]\right\|_{A^*} \leqslant \|g_0\|_{A_0}.$$

Similarly we obtain

$$\begin{gathered} g(1+it+ih) - g(1+it) = i[g_1(t+h) - g_1(t)] \in B^*, \\ \left\|\frac{1}{h}[g(1+it+ih) - g(1+it)]\right\|_{B^*} \leqslant \|g_1\|_{A_1}, \end{gathered} \tag{7}$$

and by 9.1, ii), we conclude that $g(\xi) \in \mathscr{F}(A^*, B^*)$ and $\|g\|_{\mathscr{F}} \leqslant \max(\|g_0\|_{A_0}, \|g_1\|_{A_1}) \leqslant \|l\|$.

Now, from (5), (6), and (7) we obtain

$$\begin{aligned} l[f(s)] &= l(f) \\ &= -i\int_{-\infty}^{+\infty} \langle f(it)\mu_0(s,t), dg(it)\rangle - i\int_{-\infty}^{+\infty} \langle f(1+it)\mu_1(s,t), dg(1+it)\rangle \end{aligned}$$

and as we saw above the value of this integral is precisely $\langle f(s), g'(s)\rangle$. Thus, we have $l[f(s)] = \langle f(s), g'(s)\rangle$ or, setting $g'(s) = y$ and $f(s) = x$,

$l(x) = \langle x, y\rangle$ where $y \in C' = [A^*, B^*]^s$ and $\|y\|_{C'} \leqslant \|g\|_{\mathscr{F}} \leqslant \|l\|$. But as we showed already, for linear functionals of this form we have $\|l\| \leqslant \|y\|_{C'}$, whence $\|y\|_{C'} = \|l\|$. In other words, with the bilinear functional $\langle x, y\rangle$, $x \in A \cap B$, $y \in [A^*, B^*]^s$, extended to $[A, B]_s \oplus [A^*, B^*]^s$, $[A^*, B^*]^s$ becomes the dual of $[A, B]_s$.

There remains to prove our assertion about the functions $\langle x, g_1(t)\rangle$ and $\langle x, g_2(t)\rangle$ being the boundary values of an analytic function. Let $0 < \sigma < 1$, and let $\xi(\eta)$ be a function mapping conformally the circle $|\eta| < 1$ onto the strip $0 < s < 1$, such that $\xi(0) = \sigma$.

We may take for example

$$\xi(\eta) = \frac{1}{\pi i}\log\left[\frac{\eta e^{-i\pi\sigma} - e^{i\pi\sigma}}{\eta - 1}\right], \tag{8}$$

where log stands for the principal branch of the logarithm. Evidently the mapping $\xi(\eta)$ can be extended to a continuous map from $|\eta| \leqslant 1$ with the points $\eta = 1$ and $\eta = e^{2\pi i\sigma}$ removed onto the closed strip $0 \leqslant s \leqslant 1$. Let now $\mu(\xi)$ be a complex valued bounded continuous function in $0 \leqslant s \leqslant 1$ which is harmonic in $0 < s < 1$. Then

$$\begin{aligned} \int_{-\infty}^{+\infty} \mu(it)\mu_0(\sigma,t)\,dt + \int_{-\infty}^{+\infty} \mu(1+it)\mu_1(\sigma,t)\,dt &= \mu(s) = \mu[\xi(0)] \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mu[\xi(e^{i\theta})]\,d\theta. \end{aligned}$$

Since the boundary values of $\mu(\xi)$ are continuous and bounded, but otherwise arbitrary, if $g_0(t)$ and $g_1(t)$ are two bounded continuous functions in $-\infty < t < \infty$, and $h(\theta) = g_0[-i\xi(e^{i\theta})]$ if the real part of $\xi(e^{i\theta})$ vanishes, and $h(\theta) = g_1[i - i\xi(e^{i\theta})]$ if the real part of $\xi(e^{i\theta})$ is 1, then

$$\int_{-\infty}^{+\infty} \mu_0(\sigma,t)g_0(t)\,dt + \int_{-\infty}^{+\infty} \mu_1(\sigma,t)g_1(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta. \tag{9}$$

This identity evidently holds also for any two bounded measurable functions g_0 and g_1, provided $h(\theta)$ is defined accordingly. Suppose now that g_0 and g_1 have the property that

$$\int_{-\infty}^{+\infty} f(it)g_0(t)\mu_0(\sigma,t)\,dt + \int_{-\infty}^{+\infty} f(1+it)g_1(t)\mu_1(\sigma,t)\,dt = 0 \tag{10}$$

for every function $f(\xi)$ which is continuous and bounded in $0 \leqslant s \leqslant 1$, tends to zero at infinity, is analytic in $0 < s < 1$, and vanishes at $\xi = \sigma$. Then setting

$$f(\xi)=\left(\frac{e^{i\pi\xi}-e^{i\pi s}}{e^{i\pi\xi}-e^{-i\pi s}}\right)^{n}e^{\varepsilon\xi^2},\qquad \varepsilon>0,$$

and letting ε tend to zero, we conclude that (10) also holds for

$$f(\xi)=\left(\frac{e^{i\pi\xi}-e^{i\pi s}}{e^{i\pi\xi}-e^{-i\pi s}}\right)^{n},\qquad n=1,2,\ldots$$

Applying (9) to the left-hand side of (10) with

$$f(\xi)=\left(\frac{e^{i\pi\xi}-e^{i\pi s}}{e^{i\pi\xi}-e^{-i\pi\xi}}\right)^{n}$$

we find that if $h(\theta)$ is the function associated with the pair $g_0(t)$, $g_1(t)$ then

$$\frac{1}{2\pi}\int_0^{2\pi}h(\theta)e^{in\theta}d\theta=\int_{-\infty}^{+\infty}f(it)g_0(t)\mu_0(\sigma,t)\,dt+\int_{-\infty}^{+\infty}f(1+it)g_1(t)\mu_1(\sigma,t)\,dt=0.$$

Therefore the Fourier series of $h(\theta)$ contains only terms with non-negative index, that is

$$h(\theta)=\sum_0^\infty a_n e^{in\theta}.$$

Since $h(\theta)$ is bounded, the function

$$h(\eta)=\sum a_n\eta^n$$

is analytic and bounded in $|\eta|<1$ and has non-tangential limit $h(\theta)$ at $\eta=e^{i\theta}$ for almost all θ. If we set $g(\xi)=h(\eta)$ where $\xi(\eta)$ is given by (8) we obtain a bounded analytic function in the strip whose non-tangential limits at $s=0$ and $s=1$ coincide almost everywhere with $g_0(t)$ and $g_1(t)$, as we wished to show. This completes the proof of 12.1.

32.2. To prove our assertion we will use the theorem of Eberlein according to which the unit sphere of a Banach space is weakly compact if and only if every sequence of elements in the sphere has a subsequence converging weakly to a limit, and the fact that a Banach space is reflexive if and only if its unit sphere is weakly compact.

Let $C=[A,B]_s$, $0<s<1$, and let $x_n\in C$, $\|x_n\|_C\leqslant 1$. We will show first that it is possible to extract a subsequence $\{x_{n_j}\}$ from $\{x_n\}$ so that $l(x_{n_j})$ converges for every continuous linear functional l on C. Once this is established we will prove the existence of an element x to which $\{x_{n_j}\}$ converges weakly.

To show that we can extract from $\{x_n\}$ a sequence $\{x_{n_j}\}$ such that $l(x_{n_j})$ converges for every l it will be sufficient to prove the following slightly weaker statement; given a sequence $\{x_n\}$, $x_n\in C$, $\|x_n\|_C\leqslant 1$ and a positive number ε, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\overline{\lim\limits_{j,i\to\infty}}|l(x_{n_j}-x_{n_i})|\leqslant\|l\|\varepsilon$ for every l. If this holds, then it is possible to extract from $\{x_n\}$ a weakly convergent sequence by an obvious diagonal process.

So let $x_n\in C$, $\|x_n\|_C\leqslant 1$ and $\varepsilon>0$ be given. Let $f_n\in\mathscr{F}(A,B)$, $\|f_n\|_{\mathscr{F}}\leqslant 2$ be such that $f_n(s)=x_n$. Assume that A is reflexive, and let a be real and such that $e^{(1-s)a}<\varepsilon/8$. Then for the functions $h_n(\xi)=f_n(\xi)e^{(\xi-s)a}$ we have $h_n(s)=x_n$, $\|h_n(it)\|_A\leqslant 2e^{-as}$, $\|h_n(1+it)\|_B\leqslant 2e^{(1-s)a}<\varepsilon/4$. Consider now the A-valued functions $h_n(it)\mu_0(s,t)^{1/2}$. The norms of these functions are uniformly square integrable, that is, these functions belong to a bounded subset of the space $L^2(A)$ of the strongly measurable A-valued functions of t, $-\infty<t<\infty$, of square integrable norm (see [8]). By a theorem of Phillips, if A is reflexive so is $L^2(A)$ and consequently any closed sphere in $L^2(A)$ is sequentially weakly compact, and the sequence $h_n(it)\mu_0(s,t)^{1/2}$ has a subsequence $h_{n_j}(it)\mu_0(s,t)^{1/2}$ which converges weakly in $L^2(A)$. Let now l be a continuous linear functional on C. By 12.1 we have

(1)
$$l(x_n)=-i\int_{-\infty}^{+\infty}\langle h_n(it)\mu_0(s,t),dg(it)\rangle-i\int_{-\infty}^{+\infty}\langle h_n(1+it)\mu_1(s,t)dg(1+it)\rangle,$$

where $g\in\overline{\mathscr{F}}(A^*,B^*)$ and $\|g\|_{\overline{\mathscr{F}}}\leqslant 2\|l\|$.

Consider the space $C_0(A)$ (see 32.1) and the linear functional λ on $C_0(A)$ defined by

$$\lambda(f)=\int_{-\infty}^{+\infty}\langle f(t)\mu_0(s,t)^{1/2},dg(it)\rangle.$$

Then as we showed in 32.1, (1),

$$|\lambda(f)|=\left|\int_{-\infty}^{+\infty}\langle f(t)\mu_0(s,t)^{1/2},dg(it)\rangle\right|\leqslant\|g(it)\|_A\int_{-\infty}^{+\infty}\|f(t)\mu_0(s,t)^{1/2}\|_A\,dt$$
$$\leqslant\|g\|_{\overline{\mathscr{F}}}\Big[\int\|f(t)\|_A^2\,dt\Big]^{1/2}\Big[\int_{-\infty}^{+\infty}\mu_0(s,t)\,dt\Big]^{1/2},$$

that is, λ is continuous with respect to the norm of $L^2(A)$. Since $C_0(A)$ is dense in $L^2(A)$, λ can be extended to a bounded linear functional on $L^2(A)$, which we will also denote by λ. Returning to (1) we can write

$$l(x_n)=-i\lambda[h_n(it)\mu_0(s,t)^{1/2}]-i\int_{-\infty}^{+\infty}\langle h_n(1+it)\mu_1(s,t),dg(1+it)\rangle$$

setting $n = n_j$ above, since $h_{n_j}(it)\mu_0(s,t)^{1/2}$ converges weakly in $L^2(A)$, $\lambda[h_{n_j}(it)\mu_0(s,t)^{1/2}]$ converges and consequently

$$\overline{\lim_{j,l\to\infty}}|l(x_{n_j}-x_{n_k})| = \overline{\lim_{j,k\to\infty}}\Big|\int_{-\infty}^{+\infty}\langle[h_{n_j}(1+it)-h_{n_k}(1+it)]\mu_1(s,t),\, dg(1+it)\rangle\Big|$$

and applying 32.1, (2), we find that

$$\Big|\int_{-\infty}^{+\infty}\langle h_{n_j}(1+it)-h_{n_k}(1+it)]\mu_1(s,t),\, dg(1+it)\rangle\Big|$$

$$\leqslant \|g\|_{\overline{\mathscr{F}}}\int_{-\infty}^{+\infty}[\|h_{n_j}(1+it)\|_B+\|h_{n_k}(1+it)\|_B]\mu_1(s,t)\,dt$$

$$\leqslant 2\|l\|(\varepsilon/2)\int_{-\infty}^{+\infty}\mu_1(s,t)\,dt \leqslant \|l\|\varepsilon.$$

Thus

$$\overline{\lim_{j,k\to\infty}}|l(x_{n_j}-x_{n_k})| \leqslant \|l\|\varepsilon$$

as we wished to show. Consequently if $x_n \epsilon C$, $\|x_n\|_C \leqslant 1$, there is a subsequence $\{x_{n_j}\}$ which converges weakly.

Now we will show that the sequence $\{x_{n_j}\}$ has a weak limit. Let as before $f_n \epsilon \mathscr{F}(A,B)$, $\|f_n\|_{\overline{\mathscr{F}}} \leqslant 2$, be such that $f_n(s) = x_n$. Consider the A-valued functions $f_n(it)\mu_0(s,t)^{1/2}$. As we pointed out above, these functions have bounded norms in $L^2(A)$, and since $L^2(A)$ is reflexive, we can select a subsequence from $f_{n_j}(it)\mu_0(s,t)^{1/2}$ which converges weakly to a limit in $L^2(A)$. By a theorem of Banach there exists a sequence of finite convex combinations of the f_{n_j}, say

$$h_m(t) = \sum_j a_{m_j} f_{n_j}(it),$$

$$a_{m_j} = 0 \text{ if } j \leqslant m, \qquad a_{m_i} \geqslant 0, \qquad \sum a_{m_j} = 1,$$

such that h_m converges strongly to the same limit in $L^2(A)$. By restricting ourselves to a subsequence of the $\{h_m\}$ we may assume that $h_m(t)$ converges in A for almost all t. Let $h_m(\xi) = \sum_j a_{m_j} f_{n_j}(\xi)$. Then by 9.4 we see that

$$h_m(s) = \sum_j a_{m_j} f_{n_j}(s) = \sum_j a_{m_j} x_{n_j}$$

converges in $C = [A,B]_s$ to a limit x. Then

$$\lim_{j\to\infty} l(x_{n_j}) = \lim_{m\to\infty}\sum a_{m_j} l(x_{n_j}) = \lim_{m\to\infty} l\Big(\sum a_{m_j} x_{n_j}\Big) = l(x),$$

which shows that x is the weak limit of the sequence x_{n_j}. This completes the proof of 12.2.

32.3. Let $C = [A,B]_s$ and $C_1 = [A_1,B_1]_\sigma$. We will show first that $C \subset C_1$ and $\|x\|_C \geqslant \|x\|_{C_1}$ for every $x \epsilon C$.

For let $f \epsilon \mathscr{F}(A,B)$ and $g(\xi) = f[\alpha(1-\xi)+\beta\xi]$. Then $g(it) = f[(\beta-\alpha)it+\xi]_{\xi=\alpha} = f[\xi+it(\beta-\alpha)]e^{(\xi-\alpha)^2}]_{\xi=\alpha}$ and consequently

$$\|g(it)\|_{A_1} \leqslant \|f[\xi+it(\beta-\alpha)]\|_{\mathscr{F}} = \|f\|_{\mathscr{F}},$$

$$\|g(it)\|_{A_1} \leqslant \|f[\xi+it(\beta-\alpha)]e^{(\xi-\alpha)^2}\|_{\mathscr{F}}$$

and the expression on the right tends to zero as $|t|\to\infty$. Furthermore,

$$\|g(it+ih)-g(it)\|_{A_1} \leqslant \|f[\xi+ih(\beta-\alpha)]-f(\xi)\|_{A_1}$$

and the right-hand side of this inequality tends to zero with h. Consequently $g(it)$ is an A_1-valued continuous function of t, tending to zero at infinity and $\|g(it)\|_{A_1} \leqslant \|f\|_{\mathscr{F}}$.

Similarly one shows that $g(1+it)$ is a B_1-valued continuous function tending to zero at infinity and $\|g(1+it)\|_{B_1} \leqslant \|f\|_{\mathscr{F}}$. On the other hand, $g(\xi)$ is a bounded $(A+B)$-valued analytic function and so, by 9.1, $g \epsilon \mathscr{F}(A_1,B_1)$ and its norm, as an element of this space, does not exceed $\|f\|_{\mathscr{F}}$. Now given $x \epsilon C = [A,B]_s$, we let $f \epsilon \mathscr{F}(A,B)$, $f(s) = x$, $\|f\|_{\mathscr{F}} \leqslant \|x\|_C + \varepsilon$, and $g(\xi) = f[\alpha(1-\xi)+\beta\xi]$. Then $g(\sigma) = f(s) = x$, and therefore $x \epsilon C_1 = [A_1,B_1]_\sigma$ and $\|x\|_{C_1} \leqslant \|g\|_{\mathscr{F}} \leqslant \|f\|_{\mathscr{F}} \leqslant \|x\|_C+\varepsilon$. Since ε is arbitrary the desired conclusion follows.

Similarly one shows that if $C = [A,B]^s$ and $C_1 = [A_1,B_1]^\sigma$, then $C \subset C_1$ and $\|x\|_C \geqslant \|x\|_{C_1}$ for every $x \epsilon C$. For let $f \epsilon \overline{\mathscr{F}}(A,B)$ and

$$g(\xi) = \frac{1}{\beta-\alpha}f[\alpha(1-\xi)+\beta\xi],$$

then $g \epsilon \overline{\mathscr{F}}(A_1,B_1)$ and the norm of g as an element of this space does not exceed $\|f\|_{\overline{\mathscr{F}}}$. Given $x \epsilon C$ we find $f \epsilon \overline{\mathscr{F}}(A,B)$ such that $f'(s) = \|x\|_C+\varepsilon$. Setting

$$g(\xi) = \frac{1}{\beta-\alpha}f[\alpha(1-\xi)+\beta\xi]$$

we get $g'(\sigma) = f'(s) = x$, whence $x \epsilon C_1$ and

$$\|x\|_{C_1} \leqslant \|g\|_{\overline{\mathscr{F}}} \leqslant \|f\|_{\overline{\mathscr{F}}} \leqslant \|x\|_C+\varepsilon.$$

Consider now the spaces $(A \cap B)$ and $(A_1 \cap B_1)$. Since $A \cap B$ is continuously embedded in both A_1 and B_1 the inclusion map I of $A \cap B$ into $A_1 \cap B_1$ is continuous, and so is the adjoint I^* of I which maps $(A_1 \cap B_1)^*$ into $(A \cap B)^*$. Furthermore, since $A \cap B$ is assumed to be dense in $A_1 \cap B_1$, I^* is one-to-one. Let A_{I}^* and A_{II}^* be respectively the subspaces of $(A \cap B)^*$ and $(A_1 \cap B_1)^*$ of linear functionals which are continuous with respect to the norm of A_1 and introduce in A_{I}^* and A_{II}^*

the norm of linear functionals on A. Let B_{I}^* and B_{II}^* be similarly defined. Then clearly I^* maps A_{II}^*, B_{II}^* and $A_{\mathrm{II}}^*+B_{\mathrm{II}}^*$ isometrically onto A_{I}^*, B_{I}^* and $A_{\mathrm{I}}^*+B_{\mathrm{I}}^*$ respectively, and thus also $[A_{\mathrm{II}}^*, B_{\mathrm{II}}^*]_\sigma$ onto $[A_{\mathrm{I}}^*, B_{\mathrm{I}}^*]_\sigma$. Let now l be a continuous linear functional on $[A, B]_s$. Then l is of the form $l(x) = \langle x, y\rangle$ where $y\in[A^*, B^*]^s$. Now as we saw above $[A^*, B^*]^s \subset [A_{\mathrm{I}}^*, B_{\mathrm{I}}^*]^\sigma$ the inclusion being norm decreasing; consequently there is a $z\in[A_{\mathrm{II}}^*, B_{\mathrm{II}}^*]^\sigma$ such that $I^*(z) = y$ and $\|z\| \leqslant \|y\| = \|l\|$. Furthermore, for $x\in A \cap B$ we have

$$l(x) = \langle x, y\rangle = \langle x, z\rangle,$$

whence $\|x\|_C = \sup_{\|l\|\leqslant 1}|l(x)| \leqslant \sup_{\|z\|\leqslant 1}|\langle x, z\rangle| \leqslant \|x\|_{C_1}$, since $\langle x, z\rangle$ is a continuous linear functional on C_1 (see 12.1) of norm equal to the norm of z in $[A_{\mathrm{II}}^*, B_{\mathrm{II}}^*]^\sigma$ (3). Since $A \cap B$ is dense in $[A, B]_s$ and the norms of $[A_1, B_1]_\sigma$ and $[A, B]_s$ coincide in $A \cap B$, it follows that $[A, B]_s$ is isometrically embedded in $[A_1, B_1]_\sigma$. Now by assumption $A \cap B$ is dense in $A_1 \cap B_1$ with respect to the norm of $A_1 \cap B_1$ and therefore it will also be dense in $A_1 \cap B_1$ with respect to the (smaller) norm of $[A_1, B_1]_\sigma$. Consequently, since $[A, B]_s$ is a closed subspace of $[A_1, B_1]_\sigma$, from $[A, B]_s \supset A \cap B$ follows $[A, B]_s \supset A_1 \cap B_1$. But $A_1 \cap B_1$ is dense in $[A_1, B_1]_\sigma$ and therefore $[A \cap B]_s \supset [A_1, B_1]_\sigma$.

Finally assume that $A \subset B$. Then of course we have $A = A \cap B$ but the norms of these spaces need not be equal. Nevertheless they are equivalent. In fact, both A and B are continuously embedded in a topological vector space V and this implies that if $\{x_n\}$ is a sequence of elements in A such $\|x_n - x_1\|_A \to 0$ and $\|x_n - x_2\|_B \to 0$ then $x_n \to x_1$ and $x_n \to x_2$ in V whence $x_2 = x_1\in A$. Therefore the inclusion map of A into B is closed, and since it is everywhere defined on A, it is continuous, and $\|x\|_B \leqslant c\|x\|_A$ for $x\in A$. Consequently

$$\|x\|_A \leqslant \|x\|_{A\cap B} = \max(\|x\|_A, \|x\|_B) \leqslant \|x\|_A \max(1, c).$$

On the other hand, $A+B = B$ and the norms of these two spaces are again equivalent, since the identity mapping $B \to A+B$ is norm decreasing, and therefore continuous. Furthermore, the mapping is onto and therefore has a continuous inverse. Now let us show that $A_1 \subset B_1$. Let $x\in A_1$ and let $f\in\mathscr{F}(A, B)$ be such that $f(\alpha) = x$. Consider the function $g(\xi) = e^{(\xi-\beta)^2}f\left(\xi\dfrac{\alpha}{\beta}\right)$. Then $g(\xi)$ is an $(A+B)$-continuous function of ξ in $0 \leqslant s \leqslant 1$ tending to zero at infinity, $g(it)$ is A-continuous and tends to zero at infinity, and $g(1+it)$ is $(A+B)$-continuous and therefore also B-continuous and tends to zero at infinity. In other words, $g(\xi)\in\mathscr{F}(A, B)$. Thus $x = f(\alpha) = g(\beta)\in[A, B]_\beta = B_1$ and $A_1 \subset B_1$. From this it follows

(3) Thus the norms of C and C_1 coincide on $A \cap B$.

that the spaces A_1 and $A_1 \cap B_1$ coincide and that their norms are equivalent. Since $A \cap B$ is dense in A_1, it is also dense in $A_1 \cap B_1$.

32.2. That V is a complete metric space is well known. It is also well known that if the sequence f_n converges to f in measure on every set of finite measure, then $d(f_n, f) \to 0$. Consequently, to prove that X is continuously embedded in V it will be enough to show that $\|f_n\|_X \to 0$ implies that f_n converges in measure to zero on every set of finite measure. For suppose that for some positive number ε, and a subsequence f_{n_j} of f_n we had $|f_n| > \varepsilon$ on a subset E_{n_j} of measure larger than δ, $\delta > 0$, of a set E of finite measure. Then if χ_j denotes the characteristic function of the set E_{n_j} we would have $\varepsilon\chi_j \leqslant |f_{n_j}|$ and consequently $\|\chi_j\|_X \to 0$. We now select a subsequence of χ_j, which we will also denote by χ_j, such that $\sum_1^\infty\|\chi_j\|_X < \infty$. Let $S_N = \sum_1^N \chi_j$ and suppose that $\lim_{N\to\infty} S_N$ is finite almost everywhere; then outside a subset D of E of measure less than $\delta/2$ we would have $\lim_{N\to\infty} S_N < M$ for some $M < \infty$, and $\int_{E-D} S_N dx \leqslant M|E-D|$. On the other hand, we also have

$$\int_{E-D} S_N dx = \sum_1^N \int_{E-D} \chi_j dx = \sum_1^N |(E-D) \cap E_{n_j}| \geqslant N\delta/2$$

and this would be impossible for sufficient large N. Consequently $S_N \to \infty$ on a set of positive measure and $S_N \to \infty$ uniformly on a set D of positive measure. Let χ be the characteristic function of D; then given any integer m we would have $m \leqslant S_N$ for sufficiently large N and

$$\|\chi\|_X \leqslant \frac{1}{m}\|S_N\|_X \leqslant \frac{1}{m}\sum_1^\infty \|\chi_j\|_X.$$

But since $\sum_1^\infty\|\chi_j\|_X$ is finite and m is arbitrary this would imply that $\|\chi\|_X = 0$ and consequently $\chi = 0$ almost everywhere, and the set D would have measure zero, which is a contradiction.

The proof our second assertion is immediate. If f_n is such that $\sum_1^\infty\|f_n\|_X < \infty$, then $S_N = \sum|f_n|$ converges to a limit in X. Consequently the series of functions $\sum|f_n|$ converges, in measure on every subset of finite measure of $\mathscr{M}$. Since the series has positive terms this implies that the series converges almost everywhere. Consequently the series $\sum f_n(x)$ converges absolutely almost everywhere on $\mathscr{M}$, and its limit must be $f(x)$ almost everywhere since $\sum_1^N f_n$ converges in measure to f.

33.3. Let $\varphi(t)$ be a concave non-negative function on $0 \leqslant t < \infty$, $\varphi(0) = 0$. Let λ_n and a_n be two sequences of positive numbers and suppose that $\sum \lambda_n < \infty$. Then we have the inequality

$$\frac{\sum_1^\infty \lambda_n \varphi(a_n)}{\sum_1^\infty \lambda_n} \leqslant \varphi\left[\frac{\sum_1^\infty \lambda_n a_n}{\sum_1^\infty \lambda_n}\right]$$

which is the analogue of Jensen's inequality for convex functions. The analogue of Jensen's integral inequality is also valid for concave functions and is proved in a similar way.

Let now $\varphi(x, t)$, $x \in \mathcal{M}$, $0 \leqslant t < \infty$, be a concave non-negative function of t for each x vanishing at $t = 0$. Consider the space $\varphi(X)$. This class of functions is evidently closed under multiplication by scalars. So to show that it is a linear space we merely have to prove that $g_1, g_2 \in \varphi(X)$ implies that $g_1 + g_2 \in \varphi(X)$. Let $|g_j(x)| \leqslant \lambda_j \varphi[x, f_j(x)]$ almost everywhere, $j = 1, 2$, with $f_j \in X$, $f_j \geqslant 0$ and $\|f_j\|_X = 1$, then

$$|g_1 + g_2| \leqslant \lambda_1 \varphi(x, f_1) + \lambda_2 \varphi(x, f_2) \leqslant (\lambda_1 + \lambda_2)\varphi\left[x, \frac{\lambda_1 f_1 + \lambda_2 f_2}{\lambda_1 + \lambda_2}\right]$$

and consequently $g_1 + g_2 \in \varphi(X)$.

Let now g_n be a finite or infinite sequence of elements in $\varphi(X)$ such that $\sum \|g_n\|_{\varphi(X)} < \infty$. Then we can find numbers λ_n, $\lambda_n \leqslant \|g_n\|_X + \varepsilon/2^n$, and functions $f_n \geqslant 0$ in X with $\|f_n\|_X \leqslant 1$ such that $|g_n| \leqslant \lambda_n \varphi(x, f_n)$. From the inequality for concave functions stated above we obtain

$$\sum |g_n| \leqslant \sum \lambda_n \varphi(x, f_n) \leqslant \varphi\left[x, \frac{\sum \lambda_n f_n}{\sum \lambda_n}\right]\left(\sum \lambda_n\right).$$

Now according to 13.2, since $\sum \|\lambda_n f_n\|_X \leqslant \sum \lambda_n \leqslant \varepsilon + \sum \|g_n\|_X$, it follows that $\sum \lambda_n f_n(x)$ converges almost everywhere to a function $f(x)$ in X of norm not exceeding $\sum_1^\infty \lambda_n$. Consequently $\sum |g_n(x)|$ is finite almost everywhere and belongs to $\varphi(X)$ and since ε is arbitrary we find that $\|\sum |g_n|\,\|_{\varphi(X)} \leqslant \sum \|g_n\|_{\varphi(X)}$. Thus the norm introduced in $\varphi(X)$ is subadditive. The homogeneity of the norm is clear so that the only property of the norm that remains to be shown is that $\|g\|_{\varphi(X)} = 0$ implies $g = 0$.

If $\|g\|_{\varphi(X)} = 0$ for each n, $n \geqslant 1$, there exists $f_n \in X$, $f_n \geqslant 0$, $\|f_n\|_X \leqslant 1$ such that $|g| \leqslant \frac{1}{n^3}\varphi(x, f_n)$ almost everywhere. Clearly this inequality will also hold almost everywhere simultaneously for all n. Now since $\varphi(x, t)$ is concave in t and $\varphi(x, 0) = 0$ it follows that for $0 \leqslant \lambda \leqslant 1$ we have $\varphi(x, \lambda t) \geqslant \lambda \varphi(x, t)$. Consequently

$$|g| \leqslant \frac{1}{n^3}\varphi(x, f_n) \leqslant \frac{1}{n}\varphi\left(x, \frac{1}{n^2}f_n\right) \quad \text{for all } n, \text{ almost everywhere.}$$

Now the series $\sum \left\|\frac{1}{n^2}f_n\right\|_X$ is convergent, and thus, by 13.2, the series $\sum \frac{1}{n^2}f_n$ converges almost everywhere, and, in particular, $\frac{1}{n^2}f_n$ converges to zero almost everywhere. Let now x be a point where $|g| \leqslant \frac{1}{n}\varphi\left(x, \frac{1}{n^2}f_n\right)$ holds for all n, and where $\frac{1}{n^2}f_n \to 0$ as $n \to \infty$; at such a point g must obviously vanish. Consequently $g(x) = 0$ almost everywhere.

Finally let us show that $\varphi(X)$ is complete. For this purpose it is enough to show that if g_n is such that $\sum_1^\infty \|g_n\|_{\varphi(X)} < \infty$ then the partial sums $\sum_1^N g_n$ converge in X. We have shown above that under these assumptions the series $\sum |g_n|$ converges almost everywhere to a function in $\varphi(X)$; and this implies that the series $\sum g_n$ also converges almost everywhere to a function g in $\varphi(X)$. But then we have $g - \sum_1^N g_n = \sum_{N+1}^\infty g_n$ and, as we saw above, the pointwise sum of the series $\sum_{N+1}^\infty g_n$ has norm not exceeding $\sum_{N+1}^\infty \|g_n\|_{\varphi(X)}$.

Thus $\|g - \sum_1^N g_n\|_{\varphi(X)}$ tends to zero as $N \to \infty$.

33.4. Properties i) and ii) of the function f^{**} are obvious. Properties iii) and iv) are immediate consequences of the inequalities

$$\int_E f^s g^{1-s}\,dx \leqslant \Big[\int_E f\,dx\Big]^s \Big[\int_E g\,dx\Big]^{1-s},$$

$$\frac{1}{t}\int_E \varphi(f)\,dx \leqslant \frac{|E|}{t}\varphi\left[\frac{1}{|E|}\int_E f\,dx\right] \leqslant \varphi\left[\frac{1}{t}\int_E f\,dx\right], \qquad |E| \leqslant t,$$

valid for non-negative functions. The first inequality is nothing but Hölder's inequality, and the second follows from the analogue of Jensen's integral inequality and the fact that $\varphi(\lambda t) \geqslant \lambda\varphi(t)$ for $\lambda \leqslant 1$. From properties i) and ii) it follows at once that the norm introduced in X^* is actually subadditive and homogeneous. Furthermore, if $\|f\|_{X^*} = 0$ implies that $\|f^{**}\|_X = 0$ whence $f^{**} = 0$ almost everywhere, and consequently $f = 0$ almost everywhere.

To show that X^* is complete consider a sequence of functions f_n in X^* such that $\sum\|f_n\|_{X^*} < \infty$. Then $\sum\|f_n^{**}\|_X < \infty$ and consequently, by 13.2, the series $\sum_1^\infty f_n^{**}$ converges for almost all t; but if t is value for which $\sum_1^\infty f_n^{**}(t)$ converges, we have

$$\Big(\sum_1^N |f_n|\Big)^{**}(t) \leqslant \sum_1^N f_n^{**}(t) \leqslant \sum_1^\infty f_n^{**}(t) < \infty$$

which shows that the integral of $\sum_1^N |f_n|$ on any set of finite measure has a bound independent of N, and by the monotone convergence theorem it follows that the series $\sum_1^\infty |f_n|$ converges almost everywhere and that $(\sum_1^\infty |f_n|)^{**} \leqslant \sum_1^\infty f^{**}$ for almost all t. Let now g be the sum of the series $\sum_1^\infty f_n$. Then $g^{**} \leqslant \sum_1^\infty |f_n|^{**} = \sum_1^\infty f_n^{**}$ and this last function belongs to X on account of the fact that $\sum\|f_n^{**}\|_X < \infty$ (see 13.2). Consequently $g \in X^*$. Furthermore,

$$\Big\|g - \sum_1^N f_n\Big\|_{X^*} = \Big\|\sum_{N+1}^\infty f_n\Big\|_X \leqslant \Big\|\Big(\sum_{N+1}^\infty |f_n|\Big)^{**}\Big\|_X$$

$$\leqslant \Big\|\sum_{N+1}^\infty f_n^{**}\Big\|_X \leqslant \sum_{N+1}^\infty \|f_n^{**}\|_X = \sum_{N+1}^\infty \|f_n\|_{X^*}$$

and the last expression tends to zero as $N \to \infty$. Consequently the partial sums of $\sum_1^\infty f_n$ converge to a limit in X^*, which proves the completeness of X^*.

33.5. That $X_1 + X_2$ and $X_1 \cap X_2$ are Banach lattices is clear, except perhaps for the validity of the inequality $\|g\|_{X_1+X_2} \leqslant \|f\|_{X_1+X_2}$ whenever $|g| \leqslant |f|$ almost everywhere. Let $f = f_1 + f_2$ with $\|f_1\|_{X_1} + \|f_2\|_{X_2} \leqslant \|f\|_{X_1+X_2} + \varepsilon$; then $g = f_1 g/f + f_2 g/f$, where g/f is defined to be zero wherever $f = 0$. Since $|f_i g/f| \leqslant |f_i|$, we have $\|f_i g/f\|_{X_j} \leqslant \|f_i\|_{X_j}$ and consequently $\|g\|_{X_1+X_2} \leqslant \|f_1\|_{X_1} + \|f_2\|_{X_2} \leqslant \|f\|_{X_1+X_2} + \varepsilon$. Since ε is arbitrary, the desired inequality follows. Concerning the space $X_1^{1-s}X_2^s$ let f_n be a sequence of functions in $X = X_1^{1-s}X_2^s$ such that $\sum_1^\infty \|f_n\|_X < \infty$. Then given $\varepsilon > 0$ there exist positive numbers λ_n and functions g_n and h_n in X_1 and X_2 respectively such that

$$\lambda_n \leqslant \|f_n\|_X + \varepsilon/2^n,$$

$$\|g_n\|_{X_1} \leqslant 1, \quad \|h_n\|_{X_2} \leqslant 1, \quad |f_n| \leqslant \lambda_n |g_n|^{1-s} |h_n|^s.$$

Then from Hölder's inequality we obtain

$$\sum_1^\infty |f_n| \leqslant \sum_1^\infty \lambda_n |g_n|^{1-s} |h_n|^s \leqslant \Big(\sum_1^\infty \lambda_n |g_n|\Big)^{1-s} \Big(\sum_1^\infty \lambda_n |h_n|\Big)^s$$

$$\leqslant \Big(\sum_1^\infty \lambda_n\Big) \Big[\Big(\sum_1^\infty \lambda_n |g_n|\Big)\Big(\sum \lambda_n\Big)^{-1}\Big]^{1-s} \Big[\Big(\sum \lambda_n |h_n|\Big)\Big(\sum \lambda_n\Big)^{-1}\Big]^s.$$

Now the expression within the first pair of square brackets on the right represents a function in X_1 of norm not exceeding 1, and the expression in the second pair of square brackets represents a function in X_2 of norm less that or equal to 1. Consequently $\sum_1^\infty |f_n|$ is a function in X of norm not exceeding $\sum_1^\infty \lambda_n = \sum \|f_n\|_X + \varepsilon$, and since ε is arbitrary, it follows that $\|\sum_1^\infty |f_n|\|_X \leqslant \sum \|f_n\|_X$. This gives, in particular, the subadditivity of the norm introduced in X. The homogeneity of the norm is clear so that the only remaining property of the norm we have to prove is that $\|f\|_X = 0$ implies $f = 0$ almost everywhere. Suppose that $\|f\|_X = 0$; then for each integer n, $n > 0$, there exist functions $g_n \in X_1$ and $h_n \in X_2$ such that $\|g_n\|_{X_1} \leqslant 1$, $\|h_n\|_{X_2} \leqslant 1$ and $|f| \leqslant n^{-2} |g_n|^{1-s} |h_n|^s$. But then we have

$$\sum \|n^{-1(1-s)} g_n\|_{X_1} < \infty, \qquad \sum \|n^{-1/s} h_n\|_{X_2} < \infty$$

and by 13.2 we find that $n^{-1/(1-s)} g_n$ and $n^{-1/s} h_n$ tend almost everywhere to zero as $n \to \infty$. Consequently, since $|f| \leqslant |n^{-1/(1-s)} g_n|^{1-s} |n^{-1/s} h_n|^s$ almost everywhere for all n, it follows that $f = 0$ almost everywhere.

To show that X is complete, let $f_n \in X$ be such that $\sum \|f_n\|_X < \infty$. Then, as we saw above, $\sum |f_n|$ is finite almost everywhere. Let f be the pointwise sum of the series $\sum_1^\infty f_n$. Then $|f| \leqslant \sum |f_n|$ and, as we saw above, the right-hand side of this inequality is a function in X. Consequently $f \in X$. Furthermore,

$$\Big\|f - \sum_1^N f_n\Big\|_X = \Big\|\sum_{N+1}^\infty f_n\Big\|_X \leqslant \Big\|\sum_{N+1} |f_n|\Big\|_X \leqslant \sum_{N+1}^\infty \|f_n\|_X,$$

which tends to zero as $N \to \infty$. Consequently the partial sums of the series $\sum f_n$ converge to f in X.

Next let us consider the space $\varphi_1(X)^{1-s}\varphi_2(X)^s$. First let us show that if the functions $\varphi_1(x, t)$ and $\varphi_2(x, t)$ are non-negative and concave in t, so is $\varphi_1^{1-s}(x, t)\varphi_2(x, t)^s$. In fact, we have

$$\tfrac{1}{2}[\varphi_1^{1-s}(x, t_1)\varphi_2^s(x, t_1) + \varphi_1^{1-s}(x, t_2)\varphi_2^s(x, t_2)]$$

$$\leqslant \Big[\frac{\varphi_1(x, t_1) + \varphi_1(x, t_2)}{2}\Big]^{1-s} \Big[\frac{\varphi_2(x, t_1) + \varphi_2(x, t_2)}{2}\Big]^s \leqslant \varphi_1\Big(x, \frac{t_1 + t_2}{2}\Big)^{1-s} \varphi_2\Big(x, \frac{t_1 + t_2}{2}\Big)^s.$$

On the other hand, since $\varphi_1(x, 0) = \varphi_2(x, 0) = 0$, we have $\lambda\varphi_1(x, t) \leqslant \varphi_i(x, \lambda t)$ whenever $0 \leqslant \lambda \leqslant 1$; furthermore, since $\varphi_i(x, t)$ is a non-decreasing function of t, if $t_1 \leqslant t_2$ we have

$$\varphi_i(x, t_2) \geqslant \varphi_i\left(x, \frac{t_1+t_2}{2}\right) \geqslant \frac{t_1+t_2}{2t_2}\varphi_i(x, t_2) \geqslant \frac{1}{2}\varphi_i(x, t_2).$$

Consequently

$$\varphi_1^{1-s}(x, t_1)\varphi_2^s(x, t_2) \leqslant 2^s\varphi_1^{1-s}\left(x, \frac{t_1+t_2}{2}\right)\varphi_2^s\left(x, \frac{t_1+t_2}{2}\right)$$
$$\leqslant 2\varphi_1^{1-s}\left(x, \frac{t_1+t_2}{2}\right)\varphi_2^s\left(x, \frac{t_1+t_2}{2}\right).$$

From these inequalities we find that if

$$|f| \leqslant \lambda\varphi_1^{1-s}(x, |g|)\varphi_2^s(x, |h|)$$

almost everywhere, then

$$|f| \leqslant 2\lambda\varphi_1^{1-s}\left(x, \frac{|g|+|h|}{2}\right)\varphi_2\left(x, \frac{|g|+|h|}{2}\right)^s = 2\lambda\varphi\left(x, \frac{|g|+|h|}{2}\right),$$

where $\varphi(x, t) = \varphi_1^{1-s}(x, t)\varphi_2^s(x, t)$, which shows that $\varphi_1(X)^{1-s}\varphi_2(X)^s$ is contained in $\varphi(X)$ and the norm in the second space does not exceed twice the norm in the first. The reverse inclusion and the fact that the norm in the first space does not exceed that of the second, are obvious.

Next let us consider the lattices X^*. Let X_1 and X_2 be two Banach lattices on $(0, \infty)$. Let $f \in (X_1^*)^{1-s}(X_2^*)^s$. Then given $\varepsilon > 0$ there exist $g \in X_1^*$ and $h \in X_2^*$, $g \geqslant 0$, $h \geqslant 0$, $\|g\|_{X_1^*} \leqslant 1$, $\|h\|_{X_2^*} \leqslant 1$, such that $|f| \leqslant \lambda g^{1-s}h^s$ with $\lambda \leqslant \|f\| + \varepsilon$ where $\|f\|$ denotes the norm of f in $(X_1^*)^{1-s}(X_2^*)^s$. Then from inequality iii) in 13.4 it follows that $f^{**} \leqslant \lambda(g^{1-s}h^s)^{**} \leqslant \lambda(g^{**})^{1-s}(h^{**})^s$ and since $\|g^{**}\|_{X_1} = \|g\|_{X_1^*}$, $\|h^{**}\|_{X_2} = \|h\|_{X_2^*}$, it follows that $f^{**} \in X_1^{1-s}X_2^s$ and that the norm of f^{**} as an element of this space does not exceed λ, and this in turn implies that $f \in (X_1^{1-s}X_2^s)^*$ and that the corresponding norm of f is dominated by λ.

The proof that $(X_1^{1-s}X_2^s)^* \subset (X_1^*)^{1-s}(X_2^*)^s$ under the additional conditions postulated is more complicated. We begin with some remarks.

If the measure space $\mathscr{M}$ is non-atomic, given a positive t less than the measure of the total space $\mathscr{M}$ and a subset E_1 of $\mathscr{M}$ of measure less than t, there exists a second subset E_2 such that $E_2 \supset E_1$ and $|E_2| = t$. Consequently the definition of f^{**} can be modified as follows:

$$f^{**}(t) = \frac{1}{t}\sup\int_E |f|\,dx,$$

where the supremum is taken over all sets of measure *equal* to t, if t is less than the measure of $\mathscr{M}$, or $E = \mathscr{M}$ in the remaining cases. Given a measurable function $f(x)$ on $\mathscr{M}$, we denote by $f_*(t)$ the distribution function of $|f(x)|$, i. e. the function in $(0, \infty)$ whose value for any given t, $0 < t \leqslant \infty$, is the measure of the set where $|f(x)| \geqslant t$.

Of course, we allow $+\infty$ as a value for $f_*(t)$ and we complete the definition by setting $f_*(0) = +\infty$, even if $\mathscr{M}$ has finite total measure. Furthermore, if $f_*(t) = +\infty$ for $t < t_0$, we modify the value of $f_*(t_0)$, if necessary, and set $f_*(t_0) = +\infty$.

The distribution function is non-increasing and continuous on the left. On the other hand, we denote by $f^*(t)$ the left-continuous non-increasing rearrangement of $|f(x)|$ in $[0, \infty)$ for which $f^*(0) = +\infty$ and $f^*(+\infty) = \lim_{t\to\infty} f^*(t)$. The function $f^*(t)$ is uniquely defined and is related to $f_*(t)$ as follows:

$$(1) \qquad f_*[f^*(t)] \geqslant t, \quad f^*[f_*(t)] \geqslant t;$$

an analogous relation between f^*, f_* and $|f(x)|$ is given by the inequality

$$(2) \qquad f^*\{f_*[|f(x)|]\} \geqslant |f(x)|$$

which follows from the second inequality in (1).

An equivalent way of defining $f^*(t)$ is this: $f^*(t)$ is the non-increasing left continuous function in $(0, \infty)$ which is equimeasurable with $f(x)$, that is, such that the sets $\{x|\ |f(x)| > \lambda\}$ and $\{t\ |\ f^*(t) > \lambda\}$ have the same measure for $\lambda > 0$, and for which $f^*(0) = +\infty$ and $f^*(\infty) = \lim_{t\to\infty} f^*(t)$.

It is not difficult to see, and well known, that for all t less than the measure of $\mathscr{M}$ we have

$$\sup_{|E|=t}\int_E |f(x)|\,dx = \int_0^t f^*(s)\,ds,$$

and, if the measure of $\mathscr{M}$ is finite,

$$\int |f(x)|\,dx = \int_0^t f^*(s)\,ds$$

whenever $t > |\mathscr{M}|$. Consequently we have

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$$

or in terms of our operator S_1, $f^{**} = S_1 f^*$. Evidently we have $f^{**}(t) \geqslant f^*(t)$. Consider now the operators S_1 and S_2. If $g(t) \geqslant 0$ we have

$$(3)\qquad S_2S_1g = \int_t^\infty \frac{ds}{s^2}\int_0^s g(v)\,dv = \int_0^t dv\int_t^\infty \frac{g(v)}{s^2}\,ds + \int_t^\infty dv\int_v^\infty \frac{g(v)}{s^2}\,ds$$

$$= \frac{1}{t}\int_0^t g(v)\,dv + \int_t^\infty \frac{g(v)}{v}\,dv = S_1g+S_2g.$$

On the other hand, if $g_1(t)$ and $g_2(t)$ are non-negative functions, by Hölder's inequality we have

$$(4)\qquad S_2(g_1^{1-s}g_2^s) = \int_t^\infty g_1(v)^{1-s}g_2(v)^s\frac{dv}{v} \leqslant \left[\int_t^\infty g_1(v)\frac{dv}{v}\right]^{1-s}\left[\int_t^\infty g_2(v)\frac{dv}{v}\right]^s$$

$$= (S_2g_1)^{1-s}(S_2g_2)^s.$$

Now we are ready to show that condition i) implies the desired results. Let c be a bound for the norms of the operators S_1 and S_2 in X_1 and X_2. Suppose that $f\epsilon(X_1^{1-s}X_2^s)^*$ and let us denote by $\|f\|$ its norm in this space. Then, if $\lambda > \|f\|$, there exist two functions $g_1(t)\geqslant 0$ and $g_2(t)\geqslant 0$ in X_1 and X_2 respectively such that $\|g_1\|_{X_1}\leqslant 1$, $\|g_2\|_{X_2}\leqslant 1$ and $f^{**}(t)\leqslant \lambda g_1(t)^{1-s}g_2(t)^s$. Let

$$h_1 = \frac{1}{c^2}S_2g_1,\qquad h_2 = \frac{1}{c^2}S_2g_2,\qquad h_i(0)=\infty,\qquad h_i(+\infty) = \lim_{t\to\infty}h_i(t).$$

Then from the preceding inequality and (4) we obtain $S_2f^{**}\leqslant$ $\leqslant \lambda S_2(g_1^{1-s}g_2^s)\leqslant \lambda(S_2g_1)^{1-s}(S_2g_2)^s = c^2\lambda h_1^{1-s}h_2^s$.

On the other hand, we have $f^{**} = S_1f^*$ whence by (3) we find that

$$(5)\qquad S_2f^{**} = S_2S_1f^* = S_1f^*+S_2f^*\geqslant f^*$$

which combined with the preceding inequality gives

$$(6)\qquad f^*\leqslant c^2\lambda h_1^{1-s}h_2^s.$$

Define now $f_1(x) = h_1\{f_*[|f(x)|]\}$ and $f_2(x) = h_2\{f_*[|f(x)|]\}$. Since $|f(x)|$ and $f^*(t)$ are equimeasurable, $f_i(x) = h_i\{f_*[|f(x)|]\}$ is equimeasurable with $h_i\{f_*[f^*(t)]\}$, which, since h_i is non-increasing, is a non-increasing function of t. Consequently $f_i^*(t) = h_if_*[f^*(t)]$ except perhaps at the points of discontinuity of $f_i^*(t)$. Now the first inequality in (1) and the non-increasing character of $h_i(t)$ imply that $h_i\{f_*[f^*(t)]\}\leqslant h_i(t)$, and this combined with the preceding result, implies that $f_i^*(t)\leqslant h_i(t)$ except perhaps at the points of discontinuity of $f_i^*(t)$. Hence we obtain

$$f_i^{**} = S_1f_i^*\leqslant S_1h_i = \frac{1}{c^2}S_1S_2g_i.$$

But the operators S_1 and S_2 are bounded in X_i and their norm does not exceed c. Consequently $f_i^{**}\epsilon X_i$ and $\|f_i^{**}\|_{X_i}\leqslant 1$ which implies that $f_i\epsilon X_i^*$ and $\|f_i\|_{X_i^*}\leqslant 1$. Now from (6) and (2) it follows that

$$|f(x)|\leqslant f^*\{f_*[|f(x)|]\}\leqslant c^2\lambda h_1\{f_*[|f(x)|]\}^{1-s}h_2\{f_*[|f(x)|]\}^s$$

$$= c^2\lambda f_1(x)^{1-s}f_2(x)^s.$$

Since $f_i\epsilon X_i^*$ and $\|f_i\|_{X_i^*}\leqslant 1$, it follows that $f\epsilon(X_1^*)^{1-s}(X_2^*)^s$ and that its norm as an element of this space does not exceed $c^2\lambda = c^2(\|f\|+\varepsilon)$, where $\|f\|$ denotes the norm of f as an element of $(X_1^{1-s}X_2^s)^*$. Since ε is arbitrary the desired conclusion follows.

Now we will show that the assumed properties of the operators H^s imply that i) is satisfied. We will limit ourselves to show that the integral defining S_1 is absolutely convergent and that it represents a bounded operator in X_i, an almost identical argument being applicable to S_2.

First let us consider any positive number a, and the integral

$$\int_{-a}^0 e^{s/2}H^sf(t)\,ds = \int_{-a}^0 f(te^s)e^s\,ds$$

where $f(t)$ is a non-negative function belonging to, say, X_1. This integral can be interpreted in two different ways. Either as a possibly divergent Lebesgue integral depending on a parameter t, or as the Riemann integral of the X_1-valued function $H^sf(t)$ of s. In the second sense the integral is meaningful because of the assumed continuity of the X_1-valued function $H^sf(t)$ of s. In other words, the Riemann sums of the integral are functions in X_1 which converge to a limit with respect to the norm of X_1. But convergence with respect to the norm implies convergence in measure on every set of finite measure and consequently the Riemann sums of the integral converge in measure with respect to t on every set of finite measure, and the limit is finite almost everywhere. Assume now that $f(t)$ is integrable on every interval (b,c), $0<b<c$. Let ε and δ be two positive numbers and let $f = f_1+f_2$ where f_1 is continuous and

$$\int_{b\exp(-a)}^c |f_2(t)|\,dt < \frac{1}{2a}\varepsilon\delta.$$

Since f_1 is continuous, the Riemann sums of $\int_{-a}^0 f_1(te^s)e^s\,ds$ will converge to their limit uniformly in $b\leqslant t\leqslant c$. On the other hand, if $-a = s_1 < s_2 < \ldots < s_n = 0$ denotes a subdivision of $(-a,0)$ and $s_j\leqslant\sigma_j\leqslant s_{j+1}$ we have

$$\int_b^c\left|\sum_1^{n-1}f_2(te^{-\sigma_j})e^{-\sigma_j}(s_{j+1}-s_j)\right|dt\leqslant\sum_1^{n-1}(s_{j+1}-s_j)\int_{b\exp(-a)}^c|f_2(t)|\,dt < \frac{1}{2}\varepsilon\delta$$

so that, if the subdivision is so fine that the Riemann sum of $f_1(te^s)e^s$ differs by less than $\delta/2$ from the corresponding integral for all t in (b, c), the set of values of t in (b, c) for which the Riemann sum of $f(te^s)e^s$ differs from the corresponding integral by more than δ has measure less than ε. Thus the Riemann sums of $f(te^s)e^s$ converge in measure to the Lebesgue integral

$$\int_{-a}^{0} f(te^s)e^s ds$$

on every interval (b, c).

Now these Riemann sums also converge in measure to the integral in the vectorial sense, whence it follows that the two definitions of the integral coincide almost everywhere provided that $f(t)$ is assumed to be integrable on every interval (b, c). Now we shall remove the assumption of integrability of $f(t)$. Suppose that $f(t)$ is non-integrable in some interval $(be^{-a/2}, b)$ and let $f_n(t)$ be the function $f(t)$ truncated at height n. Then

$$\int_{-a}^{0} e^{s/2}H^s f(t)\,ds \geqslant \int_{-a}^{0} e^{s/2}H^s f_n(t)\,ds = \int_{-a}^{0} f_n(te^s)e^s ds = \frac{1}{t}\int_{t\exp(-a)}^{t} f_n(s)\,ds$$

where the two first integrals are taken in the vectorial sense and the remaining ones in the sense of Lebesgue, and the first integral represents a function which is finite almost everywhere and the last tends to infinity with n for all t in the interval $(b, be^{a/2})$, which is impossible. Hence $f(t)$ must be integrable on every closed interval contained in $0 < t < \infty$.

Consider now the integral $\int_{-\infty}^{0} e^{s/2}H^s f(t)\,dt$. Since the norm of H^s does not exceed $e^{a|s|}$ where $a < 1/2$, this integral, taken as a vectorial integral, converges absolutely and represents a bounded operator on f. If $f \geqslant 0$ we have in addition

$$\int_{-\infty}^{0} e^{s/2}H^s f(t)\,dt \geqslant \int_{-a}^{0} e^{s/2}H^s f(t)\,dt = \int_{-a}^{0} f(te^s)e^s ds$$

where the last integral is taken in the sense of Lebesgue. This shows that the Lebesgue integral $\int_{-\infty}^{0} f(te^s)e^s ds$ is finite for almost all t and is majorized by $\int_{-\infty}^{0} e^{s/2}H^s f(t)\,dt$. Hence the Lebesgue integrals

$$\int_{-\infty}^{0} f(te^s)e^s ds = \frac{1}{t}\int_{0}^{t} f(s)\,ds = S_1 f$$

represent a bounded operator in X_1 as we wished to show.

To obtain the same conclusion about S_2 we argue as above with the integrals

$$\int_{t}^{\infty} \frac{f(s)}{s}\,ds = \int_{0}^{\infty} f(te^s)\,ds = \int_{0}^{\infty} e^{-s/2}H^s f(t)\,ds.$$

33.6. We begin by showing that $X(B)$ is complete with respect to its norm. For this it is enough to show that if $f_n \in X(B)$ is a sequence such that $\sum_1^\infty \|f_n\|_{X(B)} < \infty$, then the partial sums of the series $\sum f_n$ converge to a limit in $X(B)$. In fact, let $g_n(x) = \|f_n(x)\|_B$; then on account of the definition of the norm of $X(B)$ our assumption is that $\sum_1^\infty \|g_n\|_X < \infty$.

Since X is complete, the partial sums of the series $\sum_1^\infty g_n$ converge to a limit in X and thus they converge also to the same limit function in measure on every set of finite measure. Since the limit function is finite almost everywhere and $g_n(x) \geqslant 0$, convergence in measure of the partial sums implies convergence almost everywhere to a finite limit, that is

$$\sum_1^\infty g_n(x) = \sum_1^\infty \|f_n(x)\|_B < \infty$$

for almost all x, and the series $\sum_1^\infty f_n(x)$ converges in B for almost all x. Let now $h(x) = \sum_1^\infty f_n(x)$; then

$$\Big\|h(x) - \sum_1^N f_n(x)\Big\|_B \leqslant \sum_{N+1}^\infty \|f_n(x)\|_B,$$

$$\Big\|h - \sum_1^N f_n\Big\|_{X(B)} \leqslant \Big\|\sum_{N+1}^\infty \|f_n(x)\|_B\Big\|_X \leqslant \sum_{N+1}^\infty \|f_n\|_{X(B)}$$

and the last expression tends to zero as $N \to \infty$, that is $h \in X(B)$ and the partial sums of $\sum f_n$ converge to h in $X(B)$.

Before proceeding to the proof of i) and ii) we will establish some facts about vector valued measurable functions. First of all let us observe that if B_1 is continuously embedded in B_2, then a B_1-measurable function is also B_2-measurable. Thus if X_1 is continuously embedded in X_2, then $X_1(B_1)$ is contained in $X_2(B_2)$ and the inclusion map is continuous.

Assume now that (B_0, B_1) is an interpolation pair. Let $f(x)$ be a function with values in $B_0 \cap B_1$ which is both B_0 and B_1-measurable; then $f(x)$ is also $(B_0 \cap B_1)$-measurable. To show this let $g_n(x)$ and $h_n(x)$ be two sequences of simple functions with values in B_0 and B_1 respectively such that $\|g_n(x)-f(x)\|_{B_0}\to 0$ and $\|h_n(x)-f(x)\|_{B_1}\to 0$ almost everywhere. Let $E_1\subset E_2\subset\ldots\subset E_m\subset\ldots$ be sets such that $\mathscr{M}-\bigcup E_m$ has measure zero and that $\|g_n(x)-f(x)\|_{B_0}+\|h_n(x)-f(x)\|_{B_1}\to 0$ uniformly on each of the sets E_m. For each m let $n=n(m)$ be such that $\|g_n-f\|_{B_0}+\|h_n-f\|_{B_1}<1/m$ on E_m, and split E_m into a union of finitely many disjoint sets E_{mk} in such a way that g_n and h_n be constant on each of the sets E_{mk}. Now select a point x_k on each set E_{mk} and define $s_m(x)=f(x_k)$ for $x\in E_{mk}$ and $s_m(x)=0$ for $x\notin E_m$. Then for $x\in E_{mk}$ we have $g_n(x)=g_n(x_k)$, $h_n(x)=h_n(x_k)$ and consequently

$$\begin{aligned}\|s_m(x)-f(x)\|_{B_0\cap B_1}&=\|f(x_k)-f(x)\|_{B_0\cap B_1}\leqslant\|f(x_k)-f(x)\|_{B_0}+\|f(x_k)-f(x)\|_{B_1}\\&\leqslant\|f(x_k)-g_n(x_k)\|_{B_0}+\|f(x)-g_n(x)\|_{B_0}+\|f(x_k)-h_n(x_k)\|_{B_1}\\&\quad+\|f(x)-h_n(x)\|_{B_1}\leqslant\frac{2}{m}.\end{aligned}$$

Consequently, as $m\to\infty$, s_m converges uniformly to f with respect to the norm of $B_0\cap B_1$ on each of the sets $E_1, E_2, \ldots$, etc. Thus $f(x)$ is $(B_1\cap B_2)$-measurable.

Next consider the space $X(B)$. We will show that functions in $X(B)$ with countably many values form a dense subspace of $X(B)$. In fact, let $f(x)\in X(B)$ and let $g_n(x)$ be a sequence of simple functions and E_m a sequence of disjoint sets with union $\mathscr{M}$ such that $\|g_n-f\|_B$ tends uniformly to zero on each of the sets E_m. Further, let D_1 be the set where $\|f(x)\|_B\geqslant 1$, and D_k, $k>1$, the set where $1/(k-1)>\|f(x)\|_B\geqslant 1/k$. Given $\varepsilon>0$, for each pair (m,k) let $n=n(m,k)$ be so large that $\|g_n(x)-f(x)\|_B<\varepsilon k^{-1}\|f\|_{X(B)}^{-1}$ in $E_m\cap D_k$ and define $h(x)=g_n(x)$, $n=n(m,k)$ for $x\in E_m\cap D_k$, and $h(x)=0$ if $\|f(x)\|_B=0$. Then clearly $h(x)$ has countably many values and $\|h(x)-f(x)\|_B\leqslant\varepsilon\|f(x)\|_B\|f\|_{X(B)}^{-1}$, whence it follows that $h\in X(B)$ and that $\|h-f\|_{X(B)}\leqslant\varepsilon$.

We are now ready to prove i). Let $F(x,\xi)$ be a function in $\mathscr{G}[X_0(B_0),X_1(B_1)]$ of the form

$$F(x,\xi)=e^{\delta\xi^2}\sum f_n(x)e^{\lambda_n\xi},$$

where $\delta>0$, the λ_n are real and $f_n\in X_0(B_0)\cap X_1(B_1)$. We know now that f_n is measurable as a function with values in $B_0\cap B_1$ and consequently it is also measurable as a function with values in B. By 9.4, ii), for any given x we have

$$\text{(1)}\qquad \|F(x,s)\|_B\leqslant\left[\frac{1}{1-s}\int_{-\infty}^{+\infty}\|F(x,it)\|_{B_0}\mu_0(s,t)\,dt\right]^{1-s}\left[\frac{1}{s}\int_{-\infty}^{+\infty}\|F(x,1+it)\|_{B_1}\mu_1(s,t)\,dt\right]^{s}.$$

Setting

$$g(x)=\frac{1}{1-s}\int_{-\infty}^{+\infty}\|F(x,it)\|_{B_0}\mu_1(s,t)\,dt$$

we find that

$$\text{(2)}\qquad \|g\|_{X_0}=\left\|\frac{1}{1-s}\int_{-\infty}^{+\infty}\|F(x,it)\|_{B_0}\mu_0(s,t)\,dt\right\|_{X_0}\leqslant\frac{1}{1-s}\int_{-\infty}^{+\infty}\|(\|F(x,it)\|_{B_0})\|_{X_0}\mu_0(s,t)\,dt.$$

We postpone the detailed justification of this last inequality and proceed with the proof. Denoting by $|||F|||$ the norm of F as an element of $\mathscr{F}[X_0(B_0),X_1(B_1)]$ we have

$$\|(\|F(x,it)\|_{B_0})\|_{X_0}=\|F(x,it)\|_{X_0(B_0)}\leqslant|||F|||.$$

Thus from (2) we obtain

$$\|g\|_{X_0}\leqslant|||F|||\frac{1}{1-s}\int_{-\infty}^{+\infty}\mu_0(s,t)\,dt=|||F|||.$$

Similarly we find that $\|h\|_{X_1}\leqslant|||F|||$, where

$$h(x)=\frac{1}{s}\int_{-\infty}^{+\infty}\|F(x,1+it)\|_{B_1}\mu_1(s,t)\,dt.$$

Thus (1) can be expressed as

$$\|F(x,s)\|_B\leqslant g(x)^{1-s}h(x)^s$$

where $g\in X_0$, $h\in X_1$, $\|g\|_{X_0}\leqslant|||F|||$ and $\|h\|_{X_1}\leqslant|||F|||$ and this implies that $\|F(x,s)\|_B\in X$. Consequently $F(x,s)\in X(B)$ and $\|F(x,s)\|_{X(B)}=\|(\|F(x,s)\|_B)\|_X\leqslant|||F|||$. Let now f be an element of $[X_0(B_0),X_1(B_1)]_s$ and $F(x,z)$ a function in $\mathscr{F}[X_0(B_0),X_1(B_1)]$ such that $F(x,s)=f(x)$ and $|||F|||\leqslant\|f\|+\varepsilon$, where $\|f\|$ denotes the norm of f as an element of $[X_0(B_0),X_1(B_1)]_s$. Let F_n be a sequence of functions in $\mathscr{G}[X_0(B_0),X_1(B_1)]$ such that $|||F_n-F|||\to 0$. Then we have

$$\|F_n(x,s)-f(x)\|=\|F_n(x,s)-F(x,s)\|\leqslant|||F_n-F|||\to 0$$

where the first two expressions denote norms in $[X_0(B_0),X_1(B_1)]_s$. Consequently $F_n(x,s)$ converges to $f(x)$ in $[X_0(B_0),X_1(B_1)]_s$. On the other hand, we have

$$\|F_n(x,s)-F_m(x,s)\|_{X(B)}\leqslant|||F_n-F_m|||\to 0,$$

$$\|F_n(x,s)\|_{X(B)}\leqslant|||F_n|||\to|||F|||\leqslant\|f\|+\varepsilon.$$

Consequently $F_n(x, s)$ also converges to a limit in $X(B)$ of norm not exceeding $\|f\|+\varepsilon$. Now, $X(B)$ and $[X_0(B_0), X_1(B_1)]$ are both continuously embedded in $(X_0+X_1)(B_0+B_1)$, whence it follows that the limits of $F_n(x, s)$ coincide. Consequently $f \in X(B)$ and $\|f\|_{X(B)} \leqslant \|f\|+\varepsilon$; now ε is arbitrary and therefore we have $\|f\|_{X(B)} \leqslant \|f\|$.

To prove the second half of property i) let us consider the class S_s of simple functions in $X(B)$ defined as follows: $f \in S_s$ if and only if there exist $g(x) \in X_0, h(x) \in X_1, \|g\|_{X_0} \leqslant 1, \|h\|_{X_1} \leqslant 1$ such that $\|f(x)\|_B = (1+\varepsilon)\|f\|_{X(B)}\, g(x)^{1-s}h(x)^s$ and the non-zero values of g and h have positive upper and lower bounds.

Given $f(x) \in S_s$ we can write

$$f(x) = \sum_1^m \chi_j(x) u_j$$

where the χ_j are characteristic functions of disjoint measurable sets and $u_j \in B$. Let now $\varphi_j(\xi)$ be functions in $\mathscr{F}(B_0, B_1)$ such that $\varphi_j(s) = u_j\|u_j\|_B^{-1}$ and $|||\varphi_j||| \leqslant 1+\varepsilon$, where $|||\varphi_j|||$ denotes the norm of φ_j as an element of $\mathscr{F}(B_0, B_1)$, and set

$$F(x, \xi) = (1+\varepsilon)\|f\|_{X(B)}\, g(x)^{1-\xi}h(x)^{\xi}\sum_1^m \chi_j(x)\varphi_j(\xi),$$

if $\|f(x)\|_B \neq 0$ and $F(x, \xi) = 0$ otherwise. Then for each x, $F(x, \xi)$ is a function in $\mathscr{F}(B_0, B_1)$, which is continuous, uniformly with respect to x. If $\chi(x)$ is the characteristic function of the support of $\|f(x)\|_B$, then $\|F(x, \xi)\|_{B_0+B_1} \leqslant c\chi(x)$. But $\|f(x)\|_B$ is simple and therefore we have $\chi(x) \leqslant c\|f(x)\|_B$ which implies that $\chi(x) \in X \subset X_0+X_1$. Consequently $\|F(x, \xi)\|_{B_0+B_1} \in X_0+X_1$ or $F(x, \xi) \in (X_0+X_1)(B_0+B_1)$ for each ξ. Now for each ξ in $0 < \sigma < 1$ we have

$$\left\|\frac{1}{\eta}[F(x, \xi+\eta)-F(x, \xi)]-\frac{d}{d\xi}F(x, \xi)\right\|_{B_0+B_1} \leqslant c(\eta)\chi(x)$$

with $c(\eta)$ tending to zero with η, which implies that the increment quotient has a limit in $(X_0+X_1)(B_0+B_1)$ when η tends to zero. In other words, $F(x, \xi)$ is a function of ξ with values in $(X_0+X_1)(B_0+B_1)$ which is analytic in $0 < \sigma < 1$. Furthermore, since

$$\|F(x, \xi_1)-F(x, \xi_2)\|_{B_0+B_1} \leqslant c(\xi_1, \xi_2)\chi(x),$$

where $c(\xi_1, \xi_2)$ tends to zero with $|\xi_1-\xi_2| \to 0$ uniformly for ξ_1 and ξ_2 in the interval $0 \leqslant \sigma \leqslant 1$, we infer that $F(x, \xi)$, as a function with values in $(X_0+X_1)(B_0+B_1)$, is continuous in $0 \leqslant \sigma \leqslant 1$.

Now let us consider the values of $F(x, \xi)$ on $\xi = it$. Evidently, for each x we have $F(x, it) \in B_0$ and

$$\|F(x, it)\|_{B_0} \leqslant c(t)\chi(x),$$

$$\|F[x, i(t+\tau)]-F(x, it)\|_{B_0} \leqslant c(t, \tau)\chi(x),$$

where $c(t)$ and $c(t, \tau)$ tend to zero as $|t| \to \infty$ and $\tau \to 0$ respectively. But since both $g(x)$ and $h(x)$ are bounded below on their supports which contain that of $\|f(x)\|_B$, it follows that $\chi(x) \leqslant cg(x)$ and therefore $\chi(x) \in X_0$. Thus the preceding inequalities imply that, for each t, $F(x, it) \in X_0(B_0)$, and that as function of t with values in $X_0(B_0)$, $F(x, it)$ is continuous and tends to zero as $|t| \to \infty$.

Similarly we conclude that $F(x, 1+it)$ is a continuous $X_1(B_1)$-valued function of t which tends to zero at infinity.

Now we apply 9.1, i), and conclude that $F(x, \xi)$, as a function of ξ, belongs to $\mathscr{F}[X_0(B_0), X_1(B_1)]$. Now we estimate the norm of F as an element of this space. Since $|||\varphi_j||| \leqslant 1+\varepsilon$, we have $\|\varphi_j(it)\|_{B_0} \leqslant 1+\varepsilon$ and

$$\|F(x, it)\|_{B_0} \leqslant (1+\varepsilon)^2\|f\|_{X(B)}\, g(x),$$

whence it follows that

$$\|F(x, it)\|_{X_0(B_0)} \leqslant (1+\varepsilon)^2\|f\|_{X(B)};$$

similarly we obtain

$$\|F(x, it)\|_{X_1(B_1)} \leqslant (1+\varepsilon)^2\|f\|_{X(B)}$$

which implies that the norm of F as an element of $\mathscr{F}[X_0(B_0), X_1(B_1)]$ does not exceed $(1+\varepsilon)^2\|f\|_{X(B)}$. But $F(x, s) = f(x)$ which shows that $f \in [X_0(B_0), X_1(B_1)]_s$ and $\|f\| \leqslant (1+\varepsilon)^2\|f\|_{X(B)}$, where $\|f\|$ denotes the norm of f as an element of $[X_0(B_0), X_1(B_1)]_s$.

Now we will show that S_s is dense in $X(B)$. Given $f \in X(B)$ and $\delta > 0$, let $k \in X(B)$ have countably many values and be such that $\|f-k\|_{X(B)} < \delta/2$. Since $\|k(x)\|_B \in X$, there exist two functions g and h, $g \in X_0$, $h \in X_1$, $\|g\|_{X_0} \leqslant 1$, $\|h\|_{X_1} \leqslant 1$ such that $\|k(x)\|_B \leqslant (1+\varepsilon/2)\|k(x)\|_{X(B)}\, g(x)^{1-s}h(x)^s$. Let $u_1, u_2, \ldots, u_m, \ldots$ be the non-zero values of $k(x)$ and let $\chi_m(x) = 1$ if $k(x) = u_j$, $j \leqslant m$, and $1/m \leqslant g(x) \leqslant m$, $1/m \leqslant h(x) \leqslant m$ and $\chi_m(x) = 0$ otherwise. Evidently we have $\|\chi_m(x)k(x)-k(x)\|_B \leqslant \|k(x)\|_B$ and $\|\chi_m(x)k(x)-k(x)\|_B \to 0$ as $m \to \infty$ for every x. Consequently $\|(\|\chi_m(x)k(x)-k(x)\|_B)\|_X = \|\chi_m k-k\|_{X(B)} \to 0$ as $m \to \infty$ and $\|\chi_m k\|_{X(B)} \to \|k\|_{X(B)}$ as $m \to \infty$; furthermore $\|\chi_m(x)k(x)\|_B \leqslant (1+\varepsilon/2)\|k\|_{X(B)}[\chi_m(x)g(x)]^{1-s}[\chi_m(x)h(x)]^s$ and taking m so large that $(1+\varepsilon/2)\|k\|_{X(B)} \leqslant (1+\varepsilon)\|\chi_m k\|_{X(B)}$ and $\|\chi_m k-k\|_{X(B)} < \delta/2$ we will have

$$\|\chi_m(x)k(x)\|_B \leqslant (1+\varepsilon)\|\chi_m k\|_{X(B)}[\chi_m g(x)]^{1-s}[\chi_m h(x)]^s.$$

If we had equality sign here, since $\|\chi_m g\|_{X_0} \leqslant \|g\|_{X_0}$ and $\|\chi_m h\|_{X_1} \leqslant \|h\|_{X_1}$ and the positive values of $\chi_m g$ and $\chi_m h$ are between $1/m$ and m it would follow that $\chi_m k \in S_\varepsilon$. Actually, the equality sign in the relation above can be obtained by replacing h, if necessary, by a smaller function. This new function, as readily seen, also has its positive values bounded away from zero. Thus we can conclude that $\chi_m k \in S_\varepsilon$. Since $\|\chi_m k - f\|_{X(B)} \leqslant \|\chi_m k - k\|_{X(B)} + \|k - f\|_{X(B)} < \delta$, we have proved that S_ε is dense in $X(B)$.

Let now f be any function in $X(B)$. Given $\varepsilon > 0$, we construct inductively a sequence f_n of functions in S_ε as follows: we select first f_1 in such a way that $\|f - f_1\| \leqslant \frac{1}{2}\|f\|_{X(B)}$ and $\|f_1\|_{X(B)} \leqslant \frac{1}{2}(1+\varepsilon)\|f\|_{X(B)}$. Having chosen $f_1, f_2, \ldots, f_m$ in such a way that

$$\Big\|f - \sum_1^m f_j\Big\|_{X(B)} \leqslant \frac{1}{2^m}\|f\|_{X(B)}, \qquad \|f_m\|_{X(B)} \leqslant \frac{1}{2^m}(1+\varepsilon)\|f\|_{X(B)},$$

we select f_{m+1} in such a way that the above inequalities be valid with m replaced by $m+1$. Due to the density of S_ε in $X(B)$ this is always possible. Consider now the series $\sum_1^\infty f_m$; its partial sums obviously converge to f in $X(B)$. On the other hand, since $f_m \in S_\varepsilon$, we have $f_m \in [X_0(B_0), X_1(B_1)]_s$ and

$$\|f_m\| \leqslant (1+\varepsilon)^2\|f_m\|_{X(B)} \leqslant \frac{1}{2^m}(1+\varepsilon)^3\|f\|_{X(B)},$$

where $\|f_m\|$ denotes the norm of f_m in $[X_0(B_0), X_1(B_1)]_s$. Consequently the series $\sum_1^\infty f_m$ also converges in $[X_0(B_0), X_1(B_1)]_s$ and its sum has norm not exceeding $(1+\varepsilon)^3\|f\|_{X(B)}$. But the two sums of the series coincide. Consequently we infer that $f \in [X_0(B_0), X_1(B_1)]_s$ and $\|f\| \leqslant (1+\varepsilon)^3\|f\|_{X(B)}$, where $\|f\|$ denotes the norm of f as an element of $[X_0(B_0), X_1(B_1)]_s$. Since ε is arbitrary, it follows that $\|f\| \leqslant \|f\|_{X(B)}$. Thus i) is established, except for the justification of (2).

Referring to 2) let us observe that $\|F(x, it)\|_{B_0}$ is a continuous function of t for all x; consequently for any given $a > 0$, and all x we have

$$\int_{-a}^{a} \|F(x, it)\|_{B_0}\mu_0(s, t)\,dt = \lim_{n\to\infty} R_n(x), \tag{3}$$

where $R_n(x)$ is a Riemann sum of the integral corresponding to the subdivision of $(-a, a)$ into intervals of length a/n. On the other hand, we have

$$\big\|\big(\|F(x, it+ih)\|_{B_0} - \|F(x, it)\|_{B_0}\big)\big\|_{X_0} \leqslant \big\|\big(\|F(x, it+ih) - F(x, it)\|_{B_0}\big)\big\|_{X_0}$$
$$= \|F(x, it+ih) - F(x, it)\|_{X_0(B_0)},$$

and since $F(x, it)$ as a function of t with values in $X_0(B_0)$ is continuous, the last expression above tends to zero with h for each t. But then the first expression also does, which shows that $\|F(x, it)\|_{B_0}$, as a function of t with values in X_0, also is continuous. Thus the $R_n(x)$, which as elements of X_0 are also Riemann sums of the integral in (3) interpreted as a vectorial Riemann integral, converge, in X_0 as $n \to \infty$. But if $I(x)$ is the limit of R_n in X_0, then $R_n(x)$ also converges to $I(x)$ in measure, and since $R_n(x)$ tends to the integral in (3) for all x, it follows that this integral coincides with $I(x)$ and

$$\Big\|\int_{-a}^{+a} \|F(x, it)\|_{B_0}\mu_0(s, t)\,dt\Big\|_{X_0} = \lim_{n\to\infty}\|R_n\|_{X_0}.$$

Let now ϱ_n be the Riemann sum of the integral

$$\int_{-a}^{+a} \big\|\big(\|F(x, it)\|_{B_0}\big)\big\|_{X_0}\mu_0(s, t)\,dt$$

constructed with the same points of the interval $(-a, a)$ as R_n; then evidently $\|R_n\|_{X_0} \leqslant \varrho_n$. Since ϱ_n tends to the preceding integral as $n \to \infty$, it follows that

$$\Big\|\int_{-a}^{+a} \|F(x, it)\|_{B_0}\mu_0(s, t)\,dt\Big\|_{X_0} = \lim_{n\to\infty}\|R_n\|_{X_0} \leqslant \lim_{n\to\infty}\varrho_n \tag{4}$$
$$= \int_{-a}^{+a} \big\|\big(\|F(x, it)\|_{B_0}\big)\big\|_{X_0}\mu_0(s, t)\,dt \leqslant \int_{-\infty}^{+\infty} \big\|\big(\|F(x, it)\|_{B_0}\big)\big\|_{X_0}\mu_0(s,t)\,dt.$$

Finally, since $\|F(x, it)\|_{B_0} \leqslant \sum\|f_n(x)\|_{B_0}$ where the f_n are the functions that enter in the definition of F, we have

$$\Big\|\int_a^b \|F(x, it)\|_{B_0}\mu_0(s, t)\,dt\Big\|_{X_0} \leqslant \Big\|\int_a^b \sum\|f_n(x)\|_{B_0}\mu_0(s, t)\,dt\Big\|_{X_0}$$
$$= \Big\|\sum\|f_n(x)\|_{B_0}\Big\|_{X_0}\int_a^b \mu_0(s, t)\,dt, \qquad a < b,$$

and the last expression tends to zero as $a \to +\infty$ or $b \to -\infty$. Consequently

$$\int_{-a}^{a} \|F(x, it)\|_{B_0}\mu_0(s, t)\,dt$$

converges in X_0 as well as pointwise everywhere as a tends to infinity. Letting a tend to infinity in (4) we obtain (2).

We pass now to the proof of ii). We will show first that $[X_0(B_0), X_1(B_1)]^s \subset X(B)$, the inclusion being norm-decreasing, provided that $X(B)$ is closed in $X_0(B_0) + X_1(B_1)$.

Let $f(x)\in[X_0(B_0), X_1(B_1)]^s$ be given and let $F(x,\xi)$ be a function in $\mathscr{F}[X_0(B_0), X_1(B_1)]$ such that $\frac{d}{d\xi}F(x,\xi)=f(x)$ and the norm $|||F|||$ of F as an element of this space is less than, say, $\|f\|+\varepsilon$, where $\|f\|$ is the norm of f as an element of $[X_0(B_0), X_1(B_1)]^s$. Then

$$F_h(x,\xi)=\frac{1}{hi}[F(x,\xi+hi)-F(x,\xi)]e^{h\xi^2},\qquad h>0,$$

belongs to $\mathscr{F}[X_0(B_0), X_1(B_1)]$; now in the proof of i) we showed that $[X_0(B_0), X_1(B_1)]_s$ is contained in $X(B)$ and that the norm of the former majorizes that of the latter, without using the hypothesis on X made in i). Consequently we can assert that $F_h(x,s)\in X(B)$ and that $\|F_h(x,s)\|_{X(B)}$ does not exceed the norm of F_h in $\mathscr{F}[X_0(B_0), X_1(B_1)]$, which is readily seen to be majorized by $e^h|||F|||$. Thus we have $\|F_h(x,s)\|_{X(B)}\leqslant e^h|||F|||\leqslant\leqslant e^h(\|f\|+\varepsilon)$. But as $h\to 0$, $F_h(x,s)$ converges to $f(x)$ in $X_0(B_0)+X_1(B_1)$ and belongs eventually to the sphere of $X(B)$ with radius $\|f\|+2\varepsilon$ and center at zero. Since this sphere is closed in $X_0(B_0)+X_1(B_1)$, it follows that $f\in X(B)$ and $\|f\|_{X(B)}\leqslant\|f\|+2\varepsilon$ which, since ε is arbitrary implies that $\|f\|_{X(B)}\leqslant\|f\|$.

Let f be a function in $X(B)$ with countably many values and let $g\in X_0$, $h\in X_1$ be such that $\|g\|_{X_0}\leqslant 1$, $\|h\|_{X_1}\leqslant 1$, $\|f(x)\|_B=\|f\|_{X(B)}(1+\varepsilon)g(x)^{1-s}h(x)^s$. Denote by $u_1, u_2, \ldots, u_m, \ldots$ the non-zero values of f and by $\chi_j(x)$ the characteristic function of the set where $f(x)=u_j$. Let $\varphi_j(\xi)$ be functions in $\mathscr{F}(B_0, B_1)$ such that $\varphi_j(s)=u_j\|u_j\|_B^{-1}$ and with norm in $\mathscr{F}(B_0, B_1)$ not exceeding $1+\varepsilon$. Define now

$$F(x,\xi)=\|f\|_{X(B)}(1+\varepsilon)g(x)^{1-\xi}h(x)^{\xi}\sum_{1}^{\infty}\chi_j(x)\varphi_j(\xi)\tag{5}$$

if $h(x)g(x)\neq 0$ and $F(x,\xi)=0$ otherwise. Let Γ be a path in the strip $0\leqslant\sigma\leqslant 1$ joining the points ξ and $1/2$ and set

$$F_1(x,\xi)=\int_\Gamma F(x,\eta)\,d\eta,\qquad F_2(x,\xi)=\int_\Gamma F_1(x,\eta)\,d\eta,$$

where the integrals here are understood as integrals of (B_0+B_1)-valued functions depending on the parameter x. Now, for each ξ, $F(x,\xi)$ is a (B_0+B_1)-valued measurable function of x, and consequently so are the Riemann sums for the first integral above, assuming that the same points of Γ are used to construct a given Riemann sum for all x; thus $F_1(x,\xi)$ which is the pointwise limit of such sums, is also (B_0+B_1)-measurable. Similarly we conclude that $F_2(x,\xi)$ is (B_0+B_1)-measurable. Since $\|\varphi_j(\xi)\|_{B_0+B_1}\leqslant 1+\varepsilon$, we have

$$\begin{aligned}\|F(x,\xi)\|_{B_0+B_1}&\leqslant(1+\varepsilon)^2\|f\|_{X(B)}|g(x)^{1-\xi}h(x)^{\xi}|\\&\leqslant(1+\varepsilon)^2\|f\|_{X(B)}[g(x)+h(x)]\end{aligned}\tag{6}$$

and from this we obtain

$$\left\{\begin{aligned}&\|F_1(x,\xi_2)-F_1(x,\xi_1)\|_{B_0+B_1}\leqslant(1+\varepsilon)^2\|f\|_{X(B)}[g(x)+h(x)]|\xi_2-\xi_1|,\\&\|F_2(x,\xi_2)-F_2(x,\xi_1)-(\xi_2-\xi_1)F_1(x,\xi_1)\|_{B_0+B_1}\\&=\Big\|\int_{\xi_1}^{\xi_2}[F_1(x,\eta)-F_1(x,\xi_1)]\,d\eta\Big\|_{B_0+B_1}\\&\qquad\leqslant(1+\varepsilon)^2\|f\|_{X(B)}[g(x)+h(x)]|\xi_2-\xi_1|^2.\end{aligned}\right.\tag{7}$$

Since $(g+h)\in X_0+X_1$, these inequalities show that F, F_1 and F_2 have for each ξ values in $(X_0+X_1)(B_0+B_1)$; that F_1 as a function of ξ with values in this space is continuous in $0\leqslant\sigma\leqslant 1$ and that F_1 is the derivative of F_2.

Now the preceding argument can be repeated with ξ replaced by it and B_0+B_1 replaced by B_0, and the inequality

$$\|F_1(x,it_2)-F_1(x,it_1)\|_{B_0}\leqslant(1+\varepsilon)^2\|f\|_{X(B)}g(x)|t_2-t_1|$$

would follow, showing that $F_1(x,it)$, as a function of t, has values in $X_0(B_0)$ and that

$$\|F_1(x,it_2)-F(x,it_1)\|_{X_0(B_0)}\leqslant(1+\varepsilon)^2\|f\|_{X(B)}|t_2-t_1|.$$

Similarly we would obtain

$$\|F_2(x_1,1+it_2)-F(x,1+it_1)\|_{X_1(B_1)}\leqslant(1+\varepsilon)^2\|f\|_{X(B)}|t_2-t_1|.$$

From these two inequalities and (6) we conclude that F_1 belongs to $\overline{\mathscr{F}}[X_0(B_0), X_1(B_1)]$ and that its norm in this space does not exceed $(1+\varepsilon)^2\|f\|_{X(B)}$. But $\frac{d}{d\xi}F_1(s)=f(x)$; in fact, the increment quotient of F_1 converges to its derivative in $X_0(B_0)+X_1(B_1)$, and therefore also in $(X_0+X_1)(B_0+B_1)$, and it converges pointwise to $F(x,s)$ in B_0+B_1; since the limits coincide, it follows that the vectorial derivative $\frac{d}{d\xi}F_1$ at s equals $F(x,s)$, and, as readily seen, $F(x,s)=f(x)$. Thus we have proved that f belongs to $[X_0(B_0), X_1(B_1)]^s$ and its norm in this space does not exceed $(1+\varepsilon)^2\|f\|_{X(B)}$. Since ε is arbitrary, denoting with $\|f\|$ the norm of f in $[X_0(B_0), X_1(B_1)]^s$, we have $\|f\|\leqslant\|f\|_{X(B)}$.

Finally let f be any given element of $X(B)$ and f_n a sequence of functions in $X(B)$ with countably many values such that $\|f_n-f\|_{X(B)}\to 0$. Denoting as above by $\|k\|$ the norm of an element k of $[X_0(B_0), X_1(B_1)]^s$, we have $(f_n-f_m)\in[X_0(B_0), X_1(B_1)]^s$ and $\|f_n-f_m\|\leqslant\|f_n-f_m\|_{X(B)}\to 0$, which shows that f_n converges in $[X_0(B_0), X_1(B_1)]^s$ to an element k. Since $\|f_n\|\leqslant\|f_n\|_{X(B)}$, we have $\|k\|=\lim\|f_n\|\leqslant\lim\|f_n\|_{X(B)}=\|f\|_{X(B)}$. Now, both $X(B)$ and $[X_0(B_0), X_1(B_1)]^s$ are continuously embedded in

$(X_0+X_1)(B_0+B_1)$; therefore the two limits of the sequence f_n coincide, that is, $f = k$. Consequently $f \in [X_0(B_0), X_1(B_1)]^s$ and $\|f\| \leqslant \|f\|_{X(B)}$. This concludes the proof of ii).

34.1. and **34.2.** We begin by showing that under the assumptions of 14.1, iii), the function $\min(t^k, t^r)$ belongs to X. For this purpose let $g(t) \geqslant 0$ be a non-vanishing element of X and let

$$h(t) = t^r \int_0^t g(s)\frac{ds}{s^{1+r}} + t^k \int_t^\infty g(s)\frac{ds}{s^{k+1}}.$$

Then, according to ii) and iii), $h(t)$ belongs to X. Furthermore, $h(t)$ is continuous, positive, and

$$\varliminf_{t\to\infty} \frac{h(t)}{t^r} \geqslant \int_0^\infty g(s)\frac{ds}{s^{r+1}} > 0, \qquad \varliminf_{t\to 0} \frac{h(t)}{t^k} \geqslant \int_0^\infty g(s)\frac{ds}{s^{k+1}} > 0$$

consequently, for sufficiently large c, we will have $ch(t) \geqslant \min(t^k, t^r)$ and this implies the desired conclusion.

Next let us prove that ii) implies that

$$\int_0^1 g(s)\frac{ds}{s^{1+j}}$$

is a bounded linear functional of g for $0 \leqslant j \leqslant r$. Let $\chi(t)$ be the characteristic function of the interval $(1, 2)$. Then, if $g \geqslant 0$, we have

$$\chi(t)\int_0^1 g(s)\frac{ds}{s^{1+j}} \leqslant t^r \int_0^t g(s)\frac{ds}{s^{1+r}}.$$

But according to iii) the integral on the right represents an element of X of norm not exceeding $c\|g\|_X$. Consequently, we have

$$\|\chi\|_X \int_0^1 g(s)\frac{ds}{s^{1+j}} \leqslant c\|g\|_X$$

and since $\|\chi\|_X > 0$, the desired conclusion follows for $g \geqslant 0$. The general case is reduced to this by replacing g by $|g|$.

Let us now turn to the operator $\mathscr{S}$ in 14.2. Let us assume that X satisfies the condition postulated in 14.1, iii), and show that if $\psi(y)$ is any infinitely differentiable function with compact support in R^n, then the integral

$$w = \int_1^\infty t^{n-1+j}\left\{\int \tau_y F\left(\frac{1}{t}\right)\psi(ty)\,dy\right\}dt$$

is absolutely convergent whenever $F(t) \in X(B)$ and $0 \leqslant j \leqslant r$, and represents an element w of B such that $\tau_y w$ is a continuous bounded B-valued function. In fact

$$\int_1^\infty t^{n-1+j}\left\{\int \left\|\tau_y F\left(\frac{1}{t}\right)\right\|_B |\psi(ty)|\,dy\right\}dt = \int_1^\infty t^{n-1+j}\left\|F\left(\frac{1}{t}\right)\right\|_B\left\{\int |\psi(ty)|\,dy\right\}dt$$

$$\leqslant c\int_1^\infty t^{-1+j}\left\|F\left(\frac{1}{t}\right)\right\|_B dt = c\int_0^1 \|F(s)\|_B\frac{ds}{s^{j+1}} \leqslant c\|F\|_{X(B)}.$$

On the other hand,

$$\|\tau_y w - w\|_B = \left\|\int_1^\infty t^{n-1+j}dt\left\{\int \tau_z F\left(\frac{1}{t}\right)[\psi(tz-ty)-\psi(tz)]\,dz\right\}\right\|_B$$

$$\leqslant \int_1^\infty t^{n-1+j}\left\|F\left(\frac{1}{t}\right)\right\|_B\left\{\int |\psi(tz-ty)-\psi(tz)|\,dz\right\}dt$$

$$\leqslant \int_1^\infty t^{-1+j}\left\|F\left(\frac{1}{t}\right)\right\|_B\left\{\int |\psi(z-ty)-\psi(z)|\,dz\right\}dt.$$

But since ψ is infinitely differentiable, we have

$$\int |\psi(z-ty)-\psi(z)|\,dz \leqslant c\min(t|y|, 1)$$

so that if we assume that $|y| \leqslant 1$ we have

$$\|\tau_y w - w\|_B \leqslant c|y|\int_1^{1/|y|} t^j\left\|F\left(\frac{1}{t}\right)\right\|_B dt + c\int_{1/|y|}^\infty t^{-1+j}\left\|F\left(\frac{1}{t}\right)\right\|_B dt$$

$$= c|y|\int_{|y|}^1 \|F(s)\|_B\frac{ds}{s^{j+2}} + c\int_0^{|y|}\|F(s)\|_B\frac{ds}{s^{j+1}};$$

now as $|y| \to 0$, the second term in the last expression tends to zero, and so does the first. For if δ is a positive number and $|y| < \delta$ we have

$$|y|\int_{|y|}^1 \|F(s)\|_B\frac{ds}{s^{j+2}} = |y|\int_{|y|}^\delta \|F(s)\|_B\frac{ds}{s^{j+2}} + |y|\int_\delta^1 \|F(s)\|_B\frac{ds}{s^{j+2}}$$

$$\leqslant \int_{|y|}^\delta \|F(s)\|_B\frac{ds}{s^{j+1}} + \frac{|y|}{\delta}\int_\delta^1 \|F(s)\|_B\frac{ds}{s^{j+1}}$$

and the second term in the last expression tends to zero with $|y|$ and the first is arbitrarily small if δ is sufficiently small.

This shows that if we differentiate j times, $j \leqslant r$, under the integral sign τ_y applied to the second integral in the definition of $\mathscr{S}$ we obtain an absolutely convergent integral representing an element w of B such that $\tau_y w$ is a B-valued continuous function of y. The same thing is evidently true for the first term in the definition of $\mathscr{S}$. Thus every element w in the range of $\mathscr{S}$ has the property that $\tau_y w$ as a B-valued function of y has r continuous strong derivatives. Furthermore, if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index and we write

$$|\alpha| = \alpha_1+\alpha_2+\ldots+\alpha_n, \qquad \left(\frac{\partial}{\partial y}\right)^\alpha = \left(\frac{\partial}{\partial y_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial y_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial y_n}\right)^{\alpha_n}$$

where $y_1, y_2, \ldots, y_n$ are the coordinates of E^n, we have

$$\left\|\left(\frac{\partial}{\partial y}\right)^\alpha \tau_y[\mathscr{S}(F, u)]\right\|_B \leqslant c\|u\|_B + c\|F\|_{X(B)}, \qquad 0 \leqslant |\alpha| \leqslant r. \tag{1}$$

Let now w denote a derivative of order r of $\tau_y[\mathscr{S}(F, u)]$ at $y = 0$, let $\mu(y)$ be a measure in E^n with compact support and with moments of all orders less than $k-r$ equal to zero and consider the function

$$G(t) = t^r \int (\tau_{ty} w)\, d\mu(y). \tag{2}$$

We will show that G belongs to $X(B)$ and that

$$\|G\|_{X(B)} \leqslant c\|u\|_B + c\|F\|_{X(B)}. \tag{3}$$

By differentiating $\tau_y \mathscr{S}$ r times under the integral sign, from 14.2, i), we obtain

$$w = \int (\tau_z u)\psi_2^1(z)\,dz + \int_1^\infty s^{n-1+r}\left\{\int \tau_z F\left(\frac{1}{s}\right)\psi_1^1(sz)\,dz\right\}ds$$

where ψ_1^1 and ψ_2^1 are infinitely differentiable functions with compact support. Substituting in (2) and inverting the order of integration we obtain

$$t^{-r}G(t) = \int \tau_z u\left[\int \psi_2^1(z-ty)\,d\mu(y)\right]dz + \tag{4}$$
$$+ \int_1^\infty s^{n-1+r}\left\{\int \tau_z F\left(\frac{1}{s}\right)\left[\int \psi_1^1(sz-tsy)\,d\mu(y)\right]dz\right\}ds.$$

Expanding ψ_j^1 by Taylor's formula at the point z we obtain

$$\psi_j^1(z-ty) = P_j(z, ty) + R_j(z, ty)$$

where P_j is a polynomial of degree $k-r-1$ in the coordinates of ty whose coefficients are bounded functions of z, and the remainder R_j is dominated

by $c|ty|^{k-r}$. Since all moments of $\mu(y)$ of orders less than $k-r$ are zero, we find that

$$\left|\int \psi_2^1(z-ty)\,d\mu(y)\right| = \left|\int R_2\,d\mu(y)\right| \leqslant ct^{k-r}.$$

Since, on the other hand, we also have

$$\left|\int \psi_2^1(z-ty)\,d\mu(y)\right| \leqslant c,$$

it follows that

$$\left|\int \psi_2^1(z-ty)\,d\mu(y)\right| \leqslant c\min(t^{k-r}, 1).$$

Consider now the expression

$$\int\left|\int \psi_1^1(sz-sty)\,d\mu(y)\right|dz = s^{-n}\int\left|\int \psi_1^1(z-sty)\,d\mu(y)\right|dz.$$

If the supports of ψ_1^1 and μ are contained in a sphere of radius ϱ, we have

$$\psi_1^1(z-sty) = 0$$

for $|z| > 2\varrho$, $|y| \leqslant \varrho$ and $st \leqslant 1$. Consequently for $st \leqslant 1$ we have

$$\int\left|\int \psi_1^1(z-sty)\,d\mu(y)\right|dz = \int_{|z|<2\varrho}\left|\int \psi_1^1(z-sty)\,d\mu(y)\right|dz$$
$$\leqslant \int_{|z|<2\varrho}\left|\int R_1(z, tsy)\,d\mu(y)\right|dz \leqslant c(st)^{k-r}.$$

On the other hand,

$$\int\left|\int \psi_1^1(z-sty)\,d\mu(y)\right|dz \leqslant \int\left[\int |\psi_1^1(z-sty)|\,dz\right]d|\mu(y)| \leqslant c,$$

whence

$$\int\left|\int \psi_1^1(sz-sty)\,d\mu(y)\right|dz \leqslant cs^{-n}\min[(st)^{k-r}, 1].$$

Substituting in (4) we obtain

$$\|G(t)\|_B \leqslant c\|u\|_B\min(t^k, t^r) + ct^r\int_0^\infty s^{r-1}\left\|F\left(\frac{1}{s}\right)\right\|_B \min[(st)^{k-r}, 1]\,ds$$
$$= c\|u\|_B\min(t^k, t^r) + ct^k\int_t^\infty \|F(s)\|\frac{ds}{s^{k+1}} + ct^r\int_0^t \|F(s)\|\frac{ds}{s^{r+1}}.$$

Since $\min(t^k, t^r)$ belongs to X, inequality (3) follows on account of iii).

Suppose now that in the preceding situation we have $r = 0$ and $\mu(y) = \varphi(y)\,dt$. Then

$$G(t) = \int (\tau_{ty}w)\varphi(y)\,dy = t^{-n}\int (\tau_y w)\varphi\left(\frac{1}{t}y\right)dy = Tw$$

and (1) and (3) show that $w = \mathscr{S}(F, u)$ belongs to $\Lambda(x, y)$ and

$$\|w\|_\Lambda \leqslant c\|u\|_B + c\|F\|_{X(B)}.$$

Thus $\mathscr{S}$ maps $X(B)+B$ continuously into $\Lambda(B, X)$.

Let us turn to the functions ψ_1 and ψ_2 in the definition of $\mathscr{S}$. As we will show below, for $\mathscr{S}$ to be the right inverse of $\mathscr{F}$ it is enough that

$$\int \psi_1(y)\varphi(z)\log|z-y|\,dz\,dy = -1, \tag{5}$$

$$\psi_2(y) = \int_0^1 t^{n-1}\Big[\int \varphi(ty-z)\psi_1(z)\,dz\Big]\,dt. \tag{6}$$

First let us show that condition (5) can actually be fulfilled. Consider the function

$$\int \log|y-z|\varphi(\lambda z)\,dz = \lambda^{-n}\int \log|y-z|\varphi(z)\,dz - \lambda^{-n}\log\lambda\int\varphi(z)\,dz, \qquad \lambda > 0.$$

Since the integral of φ vanishes, the last term can be dropped. This shows that if

$$\int \log|y-z|\varphi(\lambda z)\,dz \tag{7}$$

vanishes identically in y for $\lambda = 1$, then it vanishes identically for all λ. Assuming this to be the case and denoting by μ and $\hat\varphi(y)$ the distribution Fourier transforms of $\log|y|$ and $\varphi(y)$ respectively, the transforms of $\varphi(\lambda y)$ and (7) will be given by $\lambda^{-n}\hat\varphi(\lambda^{-1}y)$ and $\mu\lambda^{-n}\hat\varphi(\lambda^{-1}y)$, and we will have $\mu\lambda^{-n}\hat\varphi(\lambda^{-1}y) = 0$ for all λ. Since $\hat\varphi(0) = 0$, this implies that the support of μ is the origin and that μ is the Fourier transform of a polynomial, which it is not. Consequently

$$\int \log|y-z|\varphi(z)\,dz$$

is not identically zero. But since $\varphi(z)$ is spherically symmetric, the above integral represents a spherically symmetric function of y and thus there exists a spherically symmetric function $\psi_1(y)$ such that

$$\int \psi_1(y)\int\log|y-z|\varphi(z)\,dz = -1$$

which is (5).

Let us turn now to $\psi_2(y)$. Let

$$G(y) = \int\varphi(y-z)\psi_1(z)\,dz.$$

Then, since φ and ψ_1 are spherically symmetric, so is G, and, since φ has zero integral,

$$\int G(y)\,dy = \int\psi_1(z)\int\varphi(y-z)\,dy\,dz = 0.$$

If we calculate the integral of G in spherical coordinates we find that

$$\int_\Sigma d\sigma\int_0^\infty G(t\nu)t^{n-1}dt = 0$$

where $|\nu| = 1$. But the inner integral is independent of ν. Therefore

$$\int_0^\infty G(t\nu)t^{n-1}dt = 0.$$

But G has compact support, thus $G(t\nu) = 0$ for $t \geqslant a$ and

$$\int_0^a G(t\nu)t^{n-1}dt = 0.$$

Now by definition we have

$$\psi_2(y) = \int_0^1 t^{n-1}\Big[\int\varphi(ty-z)\psi_1(z)\,dz\Big]dt = \int_0^1 t^{n-1}G(ty)\,dt \tag{8}$$

which shows that $\psi_2(y)$ is infinitely differentiable. Furthermore, if $y = \varrho\nu$, where $|y| = \varrho$, then

$$\psi_2(y) = \int_0^1 t^{n-1}G(t\varrho\nu)\,dt = \varrho^{-n}\int_0^\varrho t^{n-1}G(t\nu)\,dt.$$

Thus $\psi_2(y) = 0$ for $|y| \geqslant a$.

Now $\psi_2(y)$ is spherically symmetric, consequently

$$\begin{aligned}\int\psi_2(y)\,dy &= \int_{|y|<a} dy\int_0^1 t^{n-1}G(ty)\,dy = \int_0^1 t^{n-1}\Big[\int_{|y|<a} G(ty)\,dy\Big]dt \\ &= \int_0^1\frac{dt}{t}\int_{|y|<ta} G(y)\,dy = \int_0^a\frac{dt}{t}\int_{|y|<t} G(y)\,dy \\ &= -\int_0^a \log\frac{t}{a}\,d\Big[\int_{|y|<t} G(y)\,dy\Big] = -\int_{|y|<a} G(y)\log\Big(\frac{|y|}{a}\Big)dy\end{aligned}$$

and since the integral of G is zero, we finally obtain

$$\int\psi_2(y)\,dy = -\int G(y)\log|y|\,dy.$$

But the expression on the right is precisely the integral in (5). Thus

$$\int\psi_2(y)\,dy = 1.$$

Now let us calculate $\mathcal{S}\mathcal{I}u$. We have

$$(9)\qquad \mathcal{S}\mathcal{I}u = \lim_{\lambda\to\infty}\Big\{\int(\tau_y u)\psi_2(y)\,dy + \int_1^\lambda t^{n-1}\int\tau_z\Big[t^n\int(\tau_y u)\varphi(ty)\,dy\Big]\psi_1(tz)\,dz\,dt\Big\}.$$

Now the second term on the right can be also written as

$$\int(\tau_y u)\Big\{\int_1^\lambda t^{2n-1}\Big[\int\varphi(ty-tz)\psi_1(tz)\,dz\Big]dt\Big\}dy$$

$$= \int(\tau_y u)\Big\{\int_1^\lambda t^{n-1}\Big[\int\varphi(ty-z)\psi_1(z)\,dz\Big]dt\Big\}dy$$

$$= \int(\tau_y u)\Big[\int_1^\lambda t^{n-1}G(ty)\,dt\Big]dy.$$

Substituting in (9) and using the expression for ψ_2 given in (8), we find that

$$\mathcal{S}\mathcal{I}u = \lim_{\lambda\to\infty}\Big[\int(\tau_y u)\int_0^\lambda t^{n-1}G(ty)\,dt\Big],$$

but

$$\int_0^\lambda t^{n-1}G(ty)\,dt = \lambda^n\int_0^1 t^{n-1}G(t\lambda y)\,dy = \lambda^n\psi_2(\lambda y)$$

and consequently

$$\mathcal{S}\mathcal{I}u = \lim_{\lambda\to\infty}\lambda^n\int(\tau_y u)\psi_2(\lambda y)\,dy.$$

Since the integral of ψ_2 is equal to one and $(\tau_y u)$ is a continuous function of y in the appropriate topology, it follows that the limit above must be equal to $\tau_0 u = u$. Thus, we have shown that $\mathcal{S}\mathcal{I}u = u$.

Now let us show that, up to equivalence of norms, the space $\Lambda(B, X)$ is independent of the choice of the function φ used in its definition. For suppose we have two functions φ and denote by $\Lambda_1(B, X)$, $\Lambda_2(B, X)$ the corresponding spaces, $\mathcal{I}_1$ and $\mathcal{I}_2$ the corresponding operators $\mathcal{I}$ and $\mathcal{S}_1$ and $\mathcal{S}_2$ their left inverses. As we saw, the operator $\mathcal{S}$ of 14.2 maps $X(B)\oplus B$ into $\Lambda(B, X)$, regardless of the choice of ψ_1 and ψ_2. Consequently we may assert that $\mathcal{S}_1$ maps $X(B)\oplus B$ into $\Lambda_2(B, X)$, and since it also maps $X(B)\oplus B$ onto $\Lambda_1(B, X)$, it follows that $\Lambda_1(B, X) \subset \Lambda_2(B, X)$. Similarly we conclude that $\Lambda_2(B, X) \subset \Lambda_1(B, X)$. Consequently $\Lambda_1(B, X) = \Lambda_2(B, X)$. To show that the norms of these spaces are equivalent we use the fact that they are both continuously embedded in B; thus we may consider them as an interpolation pair and form the space $\Lambda_1+\Lambda_2$, which coincides with Λ_1 and Λ_2 but has a smaller norm.

Since the inclusion mapping $\Lambda_1 \to \Lambda_1+\Lambda_2$ is continuous and onto, the open mapping theorem implies that the norms of Λ_1 and $\Lambda_1+\Lambda_2$ are equivalent. The same argument shows the equivalence of the norms of Λ_2 and $\Lambda_1+\Lambda_2$, whence the desired conclusion follows.

We pass now to the proof of 14.1, iii). Given $z \in R^n$ we define

$$\mu_z(y) = \sum_{j=0}^m \binom{m}{j}(-1)^j\delta(y-jz), \qquad m \geqslant k-r,$$

where δ is Dirac's δ-function with support at the origin. Evidently all moments of μ_z of orders less than m are zero. Returning to 1) and 2), given $u\in\Lambda(B, X)$ we set $F = Tu$. Then $\mathcal{S}(F, u) = \mathcal{S}(Tu, u) = u$ and 1) becomes

$$\Big\|\Big(\frac{\partial}{\partial x}\Big)^a\tau_y u\Big\|_B \leqslant c\|u\|_B + c\|Tu\|_{X(B)} = c\|u\|_\Lambda, \qquad 0 \leqslant |a| \leqslant r.$$

On the other hand, setting $\mu = \mu_z$, (2) becomes

$$G(t) = t^r\int(\tau_{ty}w)\,d\mu_z(y) = t^r\sum_{j=0}^m\binom{m}{j}(-1)^j\tau_{jtz}w = t^r\Delta_{tz}w$$

and (3) gives

$$\|t^r\Delta_{tz}w\|_{X(B)} \leqslant c\|u\|_\Lambda.$$

As readily verified from the derivation of (3), the constant c here can be taken to be independent of z, provided that $|z| = 1$. This shows that the elements u of $\Lambda(B, X)$ have the properties described in 14.1, iii). Thus half of iii) is established.

To prove the second half, let u be an element of B with the properties postulated. We assume first that r is even and choose any function $\eta(z)$ infinitely differentiable, spherically symmetric, supported in $|z| \leqslant 1$ and with moments of orders less than m equal to zero. Setting

$$\Delta = \sum_{j=1}^n\frac{\partial^2}{\partial y_j^2}\tau_y\Big]_{y=0}$$

we define

$$g(t) = t^r\sup_{|z|=1}\|\Delta_{tz}(\Delta^{r/2}u)\|_B.$$

Then $g\in X$ and, for any $z\in R^n$, we have

$$\|t^r\Delta_{tz}(\Delta^{r/2}u)\|_B \leqslant |z|^{-r}g(t|z|).$$

From this we obtain

(10) $$\Big\| \int t^r \Delta_{tz}(\Delta^{r/2}u)\eta(z)\,dz \Big\|_B$$

$$\leqslant \int |z|^{-r} g(t|z|)\,|\eta(z)|\,dz \leqslant c \int\limits_{|z|<1} |z|^{-r} g(t|z|)\,dz$$

$$= c\int_0^1 s^{n-1-r} g(ts)\,ds \leqslant c\int_0^1 g(ts)\frac{ds}{s^{r+1}} \leqslant ct^r \int_0^t g(s)\frac{ds}{s^{r+1}}.$$

On the other hand, since the integral of η is zero, we have

$$t^r \int \Delta_{tz}(\Delta^{r/2}u)\eta(z)\,dz = t^{r-n}\int \tau_z(\Delta^{r/2}u)\sum_{j=1}^{m}\binom{m}{j}(-1)^j\frac{1}{j^n}\eta\left(\frac{z}{jt}\right)dz.$$

Now it is readily seen that

$$\tau_z(\Delta^{r/2}u) = \left(\sum_{j=1}^{n}\frac{\partial^2}{\partial z_j^2}\right)^{r/2}\tau_z u$$

whence integrating by parts in the last integral we obtain

(11) $$t^r \int \Delta_{tz}(\Delta^{r/2}u)\eta(z)dz = t^{-n}\int (\tau_z u)\varphi\left(\frac{1}{t}z\right)dz$$

where

$$\varphi(z) = \frac{1}{j^r}\sum_{j=1}^{m}\binom{m}{j}(-1)^j\frac{1}{j^n}\zeta\left(\frac{z}{j}\right), \qquad \zeta(z) = \left(\sum_{j=1}^{n}\frac{\partial^2}{\partial z_i^2}\right)^{r/2}\eta(z).$$

Since $\eta(z)$ has compact support, $\zeta(z)$ does not vanish identically unless $\eta(z)$ does, which of course we assume not to be the case. But then neither $\varphi(z)$ vanishes identically, for, since $\zeta(z)$ has compact support, there exists z such that $\zeta(z/m)\neq 0$ and $\zeta(z/j) = 0$ for $1\leqslant j < m$. In addition, φ is clearly spherically symmetric and its moments of order less than $m+r$ are zero. In fact, since the moments of $\eta(z)$ of order less than m are zero, the moments of $\zeta(z)$ of order less than $m+r$ are zero and consequently the same is true for $\varphi(z)$.

Combining (10) with (11) we find that $Tu\,\epsilon\, X(B)$ and that $\|Tu\|_{X(B)} \leqslant c\|g\|_X$. This concludes the proof of iii) when γ is even.

When γ is odd we replace the left-hand side of (11) by

$$t^r\sum_{j=1}^{n}\int \Delta_{tz}\left(\frac{\partial}{\partial y_j}\tau_y\Delta^{(r-1)/2}u\right)\frac{\partial}{\partial z_j}\eta(z)\,dz$$

and use the same argument.

14.3. We begin proving that τ_y restricted to B is a strongly continuous group of isometries.

That τ_y is an isometry when restricted to B is an immediate consequence of 4. To show that τ_y is strongly continuous given $u\,\epsilon\, B$ we let $F(\xi)\,\epsilon\,\mathscr{F}(B_0, B_1)$ be such that $F(s) = u$. Then $\tau_y u - u = \tau_y F(s) - F(s)$; but $\tau_y F(\xi) - F(\xi)$ is a function in $\mathscr{F}(B_0, B_1)$ and the function

$$\tau_y F(it) - F(it)$$

is a B_0-valued function which is uniformly continuous in t and tends to zero as $t\to\infty$, uniformly in y. Since for each t this function tends to zero with y we conclude that

$$\sup_t \|\tau_y F(it) - F(it)\|_{B_0} \to 0$$

as $y\to 0$. Similarly we obtain

$$\sup_t \|\tau_y F(1+it) - F(1+it)\|_B \to 0$$

whence it follows that $\tau_y F - F$ tends to zero in $\mathscr{F}(B_0, B_1)$ as $y\to 0$. Now this implies that $\|\tau_y u - u\|_B \to 0$ as we wished to show.

Next let us show that X satisfies conditions i) and ii) of 14.1. Let $g(t)\,\epsilon\, X$; then $|g(t)| \leqslant \lambda h(t)^{1-s}k(t)^s$ where $\|h\|_{X_0}\leqslant 1$, $\|k\|_{X_1}\leqslant 1$ and $\lambda \leqslant 2\|g\|_X$. Then

$$t^k\int_0^t |g(\sigma)|\frac{d\sigma}{\sigma^{k+1}} \leqslant \lambda t^k \int_0^t h(\sigma)^{1-s}k(\sigma)^s\frac{d\sigma}{\sigma^{k+1}} \leqslant \lambda\left[t^k\int_0^t h(\sigma)\frac{d\sigma}{\sigma^{k+1}}\right]^{1-s}\left[t^k\int_0^t k(\sigma)\frac{d\sigma}{\sigma^{k+1}}\right]^s;$$

but the expressions in square brackets represent functions in X_0 and X_1 of norms not exceeding a fixed constant and this implies that the first of the preceding integrals represents a function in X with norm not exceeding $c\|g\|_X$. The other integral in 14.1 can be treated in a similar way. Evidently $X_0 + X_1$ also satisfies conditions i) and ii).

Let us now write $\bar{B} = B_0 + B_1$, $\bar{X} = X_0 + X_1$ and consider the operators $\mathscr{I}$ and $\mathscr{S}$ introduced in 14.2 mapping $\Lambda(\bar{B}, \bar{X})$ into $\bar{X}(\bar{B})\oplus\bar{B}$ and conversely, and let us assume that $\mathscr{S}$ has been chosen in such a way that $\mathscr{S}$ is a left inverse of $\mathscr{I}$. Evidently $X_i(B_i)\oplus B_i$ $(i = 0, 1)$ and $X(B)\oplus B$ are continuously embedded in $\bar{X}(\bar{B})\oplus\bar{B}$, and $\Lambda(B_i, X_i)$ $(i = 0, 1)$ and $\Lambda(B, X)$ are continuously embedded in $\Lambda(\bar{B}, \bar{X})$. Assume now that $X(B) = [X_0(B_0), X_1(B_1)]_s$. Then since $\mathscr{I}$ maps $\Lambda(B_i, X_i)$ continuously into $X_i(B_i)\oplus B_i$ by 4, it also maps $[\Lambda(B_0, X_0); \Lambda(B_1, X_1)]_s$ into

$$[X_0(B_0)\oplus B_0,\ X_1(B_1)\oplus B_1]_s = [X_0(B_0)\oplus X_1(B_1)]_s \oplus [B_0, B_1]_s = X(B)\oplus B,$$

as the reader will have no difficulty in verifying.

On the other hand, $\mathcal{S}$ maps $X(B)\oplus B$ onto $\Lambda(B, X)$ and consequently $\mathcal{S}\mathcal{I}$, which is the identity, maps $[\Lambda(B_0, X_0), \Lambda(B_1, X_1)]_s$ continuously into $\Lambda(B, X)$. Thus $[\Lambda(B_0, X_0), \Lambda(B_1, X_1)]_s$ is continuously embedded in $\Lambda(B, X)$.

Now, $\mathcal{S}$ maps $X_i(B_i)\oplus B_i$ continuously into $\Lambda(B_i, X_i)$ $(i = 0, 1)$ and therefore it maps $[X_0(B_0)\oplus B_0, X_1(B_1)\oplus B_1]_s = X(B)\oplus B$ into $[\Lambda(B_0), X_0, \Lambda(B_1, X_1)]_s$. But the image of $X(B)\oplus B$ under $\mathcal{S}$ is $\Lambda(B, X)$. Consequently

$$\Lambda(B, X) \subset [\Lambda(B_0, X_0), \Lambda(B_1, X_1)]_s.$$

We already proved the reverse inclusion and its continuity, and thus the open mapping theorem yields the desired conclusion.

In the case where $X(B) = [X_0(B_0), X_1(B_1)]^s$ the result sought is obtained by using 7 instead of 4 in the preceding argument.

Bibliography

[1] A. P. Calderón, *Intermediate spaces and interpolation*, Studia Math., Special Series 1 (1963), p. 31-34.

[2] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Coll. Pub. 31 (1957).

[3] S. G. Krein, *On an interpolation theorem in the theory of operators*, Doklady Akad. Nauk 130 (1960), p. 491-494.

[4] — *On the notion of normal scale of spaces*, ibidem 132 (1960), p. 510-513.

[5] J. L. Lions, *Une construction d'espaces d'interpolation*, Comptes Rendus Acad. Sc. Paris 83 (1960), p. 1853-1855.

[6] G. G. Lorentz, *Some new functional spaces*, Annals of Math. 51 (1950), p. 37-55.

[7] E. M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. 83 (1956), p. 482-492.

[8] — and G. Weiss, *Interpolation of operators with change of measures*, ibidem 87 (1958), p. 159-172.

[9] M. H. Taibleson, Doctoral Dissertation, University of Chicago, 1962.

[10] A. Zygmund, *Trigonometrical series*, (*I*), (*II*), London 1959.

THE UNIVERSITY OF CHICAGO, CHICAGO, ILL.

Reçu par la Rédaction le 15. 8. 1963

STUDIA MATHEMATICA, T. XXIV. (1964)

A ring of analytic functions

by

R. M. BROOKS (Baton Rouge, La)*

This paper is devoted to an investigation of a topological ring of analytic functions. Specifically, this ring, denoted by R, is the set of functions analytic on the unit disc with the usual addition and scalar multiplication, the Hadamard product for its ring multiplication, and the compact-open topology. The ring R is identified algebraically with a subring $\hat{R}$ of the ring of continuous functions on the non-negative integers X. The operations in $\hat{R}$ are the usual pointwise operations, and the structure of $\hat{R}$ is determined by considering its isomorph $\hat{R}$.

In Section 2 we are concerned with the problems of identifiying the maximal ideal space of R and describing the maximal ideals intrinsically. We first show, using theorems on general rings of continuous functions, that the maximal ideals are in one-to-one correspondence with the points of the Stone-Čech compactification βX of X. We next give an intrinsic description of the maximal ideals, using the properties of the power series expansions of analytic functions. Using this description we strengthen the previous theorem appreciably and show that the maximal ideal space with the hull-kernel topology is homeomorphic to βX. Finally, the Hadamard product is used to give a simple characterization of the dual space of the topological linear space of analytic functions on the unit disc. This dual space is isomorphic to the set of functions in R whose radius of convergence exceeds one, which is exactly the intersection of the maximal ideals corresponding to points of $\beta X - X$ (the dense maximal ideals of R).

In Section 3 we continue the investigation of the maximal ideals by studying the structure of their associated residue class rings. The complex number field C is isomorphically embedded in R/M, where M is a maximal ideal of R. If M corresponds to a point of X, then R/M and the isomorph C^* of C are identical; whereas, if M corresponds to a point of $\beta X - X$, then R/M is a transcendental extension of C^* having transcendence degree $\mathfrak{c}$, the cardinality of the continuum. Moreover, we show,

* This research has been supported in part by NSF and the AFOSR.