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Absolutely p-summing operators in Hilbert space

by

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1. Introduction. A linear operator from one Hilbert space to another is absolutely p-summing ($1 \leqslant p < \infty$) if and only if it is a Hilbert-Schmidt operator. This result was proved for $p = 1$ by Grothendieck [1], for $1 \leqslant p \leqslant 2$ by Pietsch [4], and for $1 \leqslant p < \infty$ by Pełczyński [3]. The purpose of this paper is to give another proof of this result, using methods distinct from those of Pietsch and Pełczyński, but related to those of Grothendieck. These methods enable us to determine the best possible constants that occur in the fundamental norm inequalities.

Let us state the main results of this paper in detail. It will be convenient to state the essentially finite-dimensional case first. Let I_n denote the identity map from a real or complex n-dimensional inner product space into itself, and if $k \geqslant 0$ let

$$L_k = \int_0^\pi \sin^k \theta \, d\theta .$$

THEOREM 1. *Let E and F be real or complex Hilbert spaces, of which at least one is finite-dimensional, and let $n = \min(\dim E, \dim F)$. Every continuous linear operator from E into F is absolutely p-summing, for $1 \leqslant p < \infty$, and is a Hilbert-Schmidt operator. For each p ($1 \leqslant p < \infty$) the absolutely p-summing norm $a_p(T)$ is equivalent to the Hilbert-Schmidt norm $\sigma(T)$. Let $k_{G,p,n}$ and $K_{G,p,n}$ be respectively the largest and the smallest positive constants for which*

$$k_{G,p,n}\,\sigma(T) \leqslant a_p(T) \leqslant K_{G,p,n}\,\sigma(T)$$

for all T. Then

(i) *$k_{G,2,n} = K_{G,2,n} = 1$ (i.e., $\sigma(T) = a_2(T)$ for all T).*

(ii) *If $1 \leqslant p < 2$, $k_{G,p,n} = 1$, and $K_{G,p,n} = a_p(I_n)/\sqrt{n}$.*

(iii) *If $p > 2$, $k_{G,p,n} = a_p(I_n)/\sqrt{n}$, and $K_{G,p,n} = 1$.*

(iv) *If E and F are real,*

$$a_p(I_n) = \left(\frac{\prod_{i=1}^{n-1} L_{i-1}}{\prod_{i=1}^{n-1} L_{p+i-1}}\right)^{1/p};$$

in particular, $a_1(I_n) = \pi L_{n-1}^{-1}$.

(v) *If E and F are complex,*

$$a_p(I_n) = \left(\frac{\prod_{i=1}^{2n-2} L_i}{\prod_{i=1}^{2n-2} L_{p+i}}\right)^{1/p};$$

in particular, $a_1(I_n) = 2L_{2n-1}^{-1}$.

If T is a continuous linear operator of finite rank from a Hilbert space E into a Hilbert space F, and if T_1 is the operator which it induces from $E/T^{-1}(0)$ onto $T(E)$, then it is easily verified that $a_p(T) = a_p(T_1)$ and that $\sigma(T) = \sigma(T_1)$. Thus the first two statements of Theorem 1 are trivial, and the interest in Theorem 1 lies in the evaluation of the constants.

Let us now state the main, infinite-dimensional, result.

THEOREM 2. *Let E and F be infinite-dimensional real or complex Hilbert spaces. A continuous linear operator T from E into F is absolutely p-summing, for any $1 \leqslant p < \infty$, if and only if it is a Hilbert-Schmidt operator. The Hilbert-Schmidt norm $\sigma(T)$ is equivalent to the absolutely p-summing norm $a_p(T)$. Let $k_{G,p}$ and $K_{G,p}$ be respectively the largest and the smallest positive constants for which*

$$k_{G,p}\sigma(T) \leqslant a_p(T) \leqslant K_{G,p}(T).$$

Then $k_{G,p} = \lim_{n\to\infty} k_{G,p,n}$ *and* $K_{G,p} = \lim_{n\to\infty} K_{G,p,n}$.

In particular,

(i) $k_{G,2} = K_{G,2} = 1$ (*i.e.*, $\sigma(T) = a_2(T)$ *for all Hilbert-Schmidt operators* T).

(ii) *If* $1 \leqslant p < 2$, $k_{G,p} = 1$; *if* $p > 2$, $K_{G,p} = 1$.

(iii) *If E and F are real,* $K_{G,1} = \sqrt{\pi/2}$; *if E and F are complex* $K_{G,1} = 2/\sqrt{\pi}$.

(iv) *If p is an integer greater than 2, and E and F are real,*

$$k_{G,p} = \frac{1}{\sqrt{2\pi}}\Big(\prod_{i=1}^{p} L_{i-1}\Big)^{1/p};$$

if E and F are complex,

$$k_{G,p} = \frac{1}{\sqrt{\pi}}\Big(\prod_{i=1}^{p} L_i\Big)^{1/p}.$$

Note that $k_{G,p} \to 0$ as $p \to \infty$; thus the collection of all absolutely p-summing norms is not uniformly equivalent. Grothendieck [1] showed that if E and F are real, $K_{G,1} = \sqrt{\pi/2}$. Note that the constants are different in the real and complex cases; this appears to have been overlooked by Grothendieck.

The proofs of Theorems 1 and 2 will be broken down into a collection of Propositions. The proofs of these depend only upon the elementary properties of Hilbert-Schmidt and absolutely p-summing operators, together with the following fundamental Theorem ([4], Theorem 2, and [2], Proposition 3.1):

THEOREM A. *Let M be a weakly closed subset of the unit ball of the dual of a normed space E with the property that*

$$\|x\| = \sup_{x'\in M} |x'(x)|$$

for all x in E. If a linear mapping T from E into a normed space F is absolutely p-summing, there is a positive Radon measure μ on M of total mass 1 such that

$$\|Tx\|^p \leqslant (a_p(T))^p \int_M |x'(x)|^p \, d\mu(x')$$

for all x in E.

Conversely, if T is a continuous linear mapping from E into F, and if there exists a positive Radon measure μ on M of total mass 1 such that

$$\|Tx\|^p \leqslant C^p \int_M |x'(x)|^p \, d\mu(x')$$

for some constant C and all x in E, then T is absolutely p-summing, and $a_p(T) \leqslant C$.

2. Wallis's formula. We shall need an elementary result on integrals which was originally obtained by Wallis [5]. For completeness and convenience, we shall establish it here.

Let

$$L_n = \int_0^{\pi} \sin^n\theta \, d\theta.$$

Integrating by parts, we obtain $nL_n = (n-1)L_{n-2}$ for $n = 2, 3, \ldots$ Thus

$$L_{2m} = \frac{(2m-1)(2m-3)\ldots 1}{2m(2m-2)\ldots 2}\cdot\pi = \frac{2\pi}{2mL_{2m-1}}$$

and

$$L_{2m+1} = \frac{2m(2m-2)\ldots 2}{(2m+1)(2m-1)\ldots 3}\cdot 2 = \frac{2\pi}{(2m+1)L_{2m}}.$$

Since $L_{2m-1} \geqslant L_{2m} \geqslant L_{2m+1}$ it follows that $(2m+1)L_{2m}^2 \geqslant 2\pi \geqslant 2mL_{2m}^2$

and $$(2m+2)L_{2m+1}^2 \geqslant 2\pi \geqslant (2m+1)L_{2m+1}^2.$$

From these inequalities it follows that $\sqrt{n}L_{n-1} \to \sqrt{2\pi}$ as $n \to \infty$.

3. The absolutely p-summing norm of operators of finite rank. We first determine the absolutely p-summing norm $a_p(I_n)$ of the identity map I_n from an n-dimensional inner-product space E^n onto itself.

PROPOSITION 1(¹). *If E^n is real,*

$$a_p(I_n) = \left(\frac{\prod_{i=1}^{n-1} L_{i-1}}{\prod_{i=1}^{n-1} L_{p+i-1}}\right)^{1/p} \quad \textit{for } 1 \leqslant p < \infty.$$

Proof. We follow the notation used by Lindenstrauss and Pełczyński [2]; we denote the real $(n-1)$-dimensional sphere $\{x \in E^n: \|x\| = 1\}$ by S^n, and denote the normalised rotation-invariant measure on S^n by m. Suppose that μ is a measure of total mass 1 on S^n satisfying the conditions of Theorem A; i.e.

$$\|x\|^p \leqslant (a_p(I_n))^p \int_{S^n} |\langle x, x'\rangle|^p\, d\mu(x')$$

for all x in E^n. Let $U \in O(n)$, the group of $n\times n$ orthogonal matrices. Then if $x \in E^n$,

$$\begin{aligned}\|x\|^p = \|Ux\|^p &\leqslant (a_p(T))^p \int_{S^n} |\langle Ux, x'\rangle|^p\, d\mu(x')\\ &= (a_p(I_n))^p \int_{S^n} |\langle x, U^{-1}x'\rangle|^p\, d\mu(x')\\ &= (a_p(I_n))^p \int_{S^n} |\langle x, x'\rangle|^p\, d\mu_U(x'),\end{aligned}$$

(¹) This result is due to Gordon [6]; his proof is the same as ours.

where μ_U is the measure on S^n defined by

$$\int_{S^n} f(x')\, d\mu_U(x') = \int_{S^n} f(U^{-1}x')\, d\mu(x')$$

for all $f \in C(S^n)$. Let $\nu = \int_{O(n)} \mu_U\, dh(U)$, where h is the normalised Haar measure on $O(n)$. Then

$$\nu(S^n) = \int_{O(n)} \mu_U(S^n)\, dh(U) = 1,$$

so that $\nu = m$, since ν is clearly rotation-invariant. Further, since

$$\|x\|^p \leqslant (a_p(I_n))^p \int_{S^n} |\langle x, x'\rangle|^p\, d\mu_U(x'),$$

we have

$$\|x\|^p = \int_{O(n)} \|x\|^p\, d\mu(U) \leqslant (a_p(I_n))^p \int_{S^n} |\langle x, x'\rangle|^p\, dm\rangle(x').$$

It therefore follows that

$$a_p(I_n) = \inf\{\lambda: \|x\|^p \leqslant \lambda^p \int_{S^n} |\langle x, x'\rangle|^p\, dm(x') \text{ for all } x \text{ in } E^n\}.$$

But since m is rotation-invariant, this means that

$$a_p(I_n) = \left(\int_{S^n} |\langle x, x'\rangle|^p\, dm(x')\right)^{-1/p},$$

where x is any unit vector in E^n, and we can choose an orthonormal basis in such a way that $x = (1, 0, \ldots, 0)$. In order to evaluate the integral, we use polar coordinates $\varphi = (\varphi_1, \ldots, \varphi_{n-1})$ (cf. [2], p. 278). For any integrable function g on S^n we have the formula

$$\int_{S^n} g(x')\, dm(x') = |S^n|^{-1} \int_{I^{n-1}} g(x'(\varphi)) J(\varphi)\, d\varphi,$$

where $d\varphi$ is Lebesgue measure on the $(n-1)$-dimensional interval

$$I^{n-1} = \{\varphi: 0 \leqslant \varphi_1 < 2\pi;\ 0 \leqslant \varphi_i \leqslant \pi \text{ for } i = 2, 3, \ldots, n-1\},$$

and where

$$J(\varphi) = \prod_{i=2}^{n-1} (\sin^{i-1}\varphi_i), \qquad |S^n| = 2\prod_{i=1}^{n-1} L_{i-1},$$

and

$$x'(\varphi) = (x_i'(\varphi), \ldots, x_n'(\varphi)).$$

In particular,

$$\langle x, x'\rangle = x_1'(\varphi) = \prod_{i=1}^{n-1} \sin\varphi_i,$$

so that

$$\int_{S^n} |\langle x, x'\rangle|^p\, dm(x') = |S^n|^{-1} \int_{I^{n-1}} \prod_{i=1}^{n-1} |\sin^p\varphi_i|\, J(\varphi)\, d\varphi = \frac{\prod_{i=1}^{n-1} L_{p+i+1}}{\prod_{i=1}^{n-1} L_{i-1}}.$$

Corollary. *If p is a positive integer,*

$$a_p(I_n) = \left(\frac{\prod_{i=1}^{p} L_{i-1}}{\prod_{i=1}^{p} L_{n+i-2}}\right)^{1/p};$$

in particular, $a_1(I_n) = \pi L_{n-1}^{-1}$ and $a_2(I_n) = \sqrt{n}$ (²).

Proposition 2. *If E^n is complex,*

$$a_p(I_n) = \left(\frac{\prod_{i=1}^{2n-2} L_i}{\prod_{i=1}^{2n-2} L_{p+i}}\right)^{1/p} \quad \textit{for } 1 \leqslant p < \infty.$$

Proof. E^n is isometrically isomorphic to a real $2n$-dimensional space F^{2n}; we denote the real $(2n-1)$-dimensional sphere $\{x \in E^n : \|x\| = 1\}$ by S^{2n}, and denote the normalised rotation-invariant measure by m. Then, arguing as in the real case,

$$a_p(I_n) = \Big(\int_{S^{2n}} |\langle x, x'\rangle|^p\, dm(x')\Big)^{-1/p},$$

where x is any unit vector in E^n. If we choose complex orthonormal coordinates in such a way that $x = (1, 0, \ldots, 0)$, and if $x' \in S^{2n}$ has real polar coordinates $(\varphi_1, \ldots, \varphi_{2n-1})$, then

$$\langle x, x'\rangle = \prod_{i=1}^{2n-1} \sin\varphi_i + i\cos\varphi_1 \prod_{i=1}^{2n-1} \sin\varphi_i,$$

(²) This result is due to Mayer [7]; his proof is rather different.

so that

$$|\langle x, x'\rangle| = \prod_{i=2}^{2n-1} \sin\varphi_i.$$

Thus

$$\int_{S^{2n}} |\langle x, x'\rangle|^p\, dm(x') = |S^{2n}|^{-1} \int_{I^{2n-1}} \prod_{i=2}^{2n-1} (\sin^{p+i-1}\varphi_i)\, d\varphi$$
$$= \frac{\pi \prod_{i=2}^{2n-1} L_{p+i-1}}{\prod_{i=1}^{2n-1} L_{i-1}} = \frac{\prod_{i=1}^{2n-2} L_{p+i}}{\prod_{i=1}^{2n-2} L_i}.$$

Corollary. *If p is a positive integer,*

$$a_p(I_n) = \left(\frac{\prod_{i=1}^{p} L_i}{\prod_{i=1}^{p} L_{2n-2+i}}\right)^{1/p};$$

in particular, $a_1(I_n) = 2L_{2n-1}^{-1}$ and $a_2(I_n) = \sqrt{n}$.

We now consider an invertible linear operator T mapping an n-dimensional inner product space E^n onto an n-dimensional inner product space F^n. We can find orthonormal bases $(e_1, \ldots, e_n)$ for E^n and $(f_1, \ldots, f_n)$ for F^n such that T is given by $Te_i = \lambda_i f_i$, where each λ_i is real and positive.

Proposition 3. *If E^n and F^n are real,*

$$a_p(T) = a_p(I_n)\Big(\int_{S^n} (\lambda_1^2 x_1^2 + \ldots + \lambda_n^2 x_n^2)^{p/2}\, dm(x)\Big)^{1/p} \quad \textit{for } 1 \leqslant p < \infty.$$

Proof. There exists a measure μ on S^n of total mass 1 such that

$$\|Tx\|^p \leqslant (a_p(T))^p \int_{S^n} |\langle x, x'\rangle|^p\, d\mu(x') \quad \text{for all } x \text{ in } E^n.$$

If $U \in O(n)$,

$$\|TUx\|^p \leqslant (a_p(T))^p \int_{S^n} |\langle Ux, x'\rangle|^p\, d\mu(x')$$
$$= (a_p(T))^p \int_{S^n} |\langle x, x'\rangle|^p\, d\mu_U(x').$$

Integrating over $O(n)$, we obtain

$$\int_{O(n)} \|TUx\|^p\, dh(U) \leqslant (a_p(T))^p \int_{S^n} |\langle x, x'\rangle|^p\, dm(x').$$

Since

$$\int_{O(n)} \|TUx\|^p\,dh(U) = \int_{S^n} \|Ty\|^p\,dm(y) = \int_{S^n} (\lambda_1^2 x_1^2 + \ldots + \lambda_n^2 x_n^2)^{p/2}\,dm(x)$$

and since

$$\int_{S^n} |\langle x, x'\rangle|^p\,dm(x') = \big(a_p(I_n)\big)^{-p},$$

it follows that

$$a_p(T) \geqslant a_p(I_n)\Big(\int_{S^n} (\lambda_1^2 x_1^2 + \ldots + \lambda_n^2 x_n^2)^{p/2}\,dm(x)\Big)^{1/p}.$$

Now let R^n be the $(n-1)$-dimensional sphere $\{x \in F^m\colon \|x\| = 1\}$ and let m again denote the rotation-invariant measure on R^n. $T'(R^n)$ is an ellipsoid in E^n, and T' maps R^n homeomorphically onto $T'(R^n)$. The mapping T' defines a measure μ on $T'(R^n)$, given by

$$\int_{T'(R^n)} f(\omega')\,d\mu(\omega') = \int_{R^n} f(T'y')\,dm(y')$$

for $f \in C(T'(R^n))$; $\mu\big(T'(R^n)\big) = m(R^n) = 1$. The mapping $K\colon \omega' \to \omega'/\|\omega'\|$ maps $T'(R^n)$ homeomorphically onto S^n, and we define a measure ν on S^n by setting

$$\int_{S^n} f(x')\,d\nu(x') = \int_{T'(R^n)} f(K\omega')\|\omega'\|^p\,d\mu(\omega') \quad \text{for } f \in C(S^n).$$

In particular, for fixed x in E^n,

$$\begin{aligned}\int_{S^n} |\langle x, x'\rangle|^p\,d\nu(x') &= \int_{T'(R^n)} |\langle x, K\omega'\rangle|^p \|\omega'\|^p\,d\mu(\omega') \\ &= \int_{T'(R^n)} |\langle x, \omega'\rangle|^p\,d\mu(\omega') \\ &= \int_{R^n} |\langle x, T'y'\rangle|^p\,dm(y') \\ &= \int_{R^n} |\langle Tx, y'\rangle|^p\,dm(y') \\ &= \|Tx\|^p\big(a_p(I_n)\big)^{-p},\end{aligned}$$

by Proposition 1. From this it follows that

$$a_p(T) \leqslant a_p(I_n)\cdot\big(\nu(S^n)\big)^{1/p}.$$

Now

$$\begin{aligned}\nu(S^n) &= \int_{T'(R^n)} \|\omega'\|^p\,d\mu(\omega') \\ &= \int_{R^n} \|T'y'\|^p\,dm(y') \\ &= \int_{R^n} (\lambda_1^2 y_1'^2 + \ldots + \lambda_n^2 y_n'^2)^{p/2}\,dm(y').\end{aligned}$$

This completes the proof of the proposition.

In the same way, it can be shown that

PROPOSITION 4. *If E^n and F^m are complex,*

$$a_p(T) = a_p(I_n)\Big(\int_{S^{2n}} (\lambda_1^2 x_1^2 + \lambda_1^2 x_2^2 + \ldots + \lambda_n^2 x_{2n}^2)^{p/2}\,dm(x)\Big)^{1/p} \quad \text{for } 1 \leqslant p < \infty.$$

The details will be omitted.

PROPOSITION 5. *Let T be a continuous operator of finite rank n from a Hilbert space E into a Hilbert space F.*

(i) *If* $1 \leqslant p < 2$, $\sigma(T) \leqslant a_p(T) \leqslant \dfrac{a_p(I_n)}{\sqrt{n}}\,\sigma(T)$.

(ii) $\sigma(T) = a_2(T)$.

(iii) *If* $p > 2$, $\dfrac{a_p(I_n)}{\sqrt{n}}\,\sigma(T) \leqslant a_p(T) \leqslant \sigma(T)$.

We may suppose that T is an invertible linear operator mapping an n-dimensional inner-product space E^n onto an n-dimensional inner-product space F^m, as was remarked after the statement of Theorem 1. Thus (ii) follows directly from Propositions 3 and 4. If $1 \leqslant p < 2$, $a_p(T) \geqslant a_2(T)$ ([4], Satz 5), which establishes the first inequality of (i). Also (in the real case)

$$\Big(\int_{S^n} (\lambda_1^2 x_1^2 + \ldots + \lambda_n^2 x_n^2)^{p/2}\,dm(x)\Big)^{1/p} \leqslant \Big(\int_{S^n} (\lambda_1^2 x_1^2 + \ldots + \lambda_n^2 x_n^2)\,dm(x)\Big)^{1/2} = \frac{1}{\sqrt{n}}\Big(\sum_{i=1}^n \lambda_i^2\Big)^{1/2} = \frac{1}{\sqrt{n}}\,\sigma(T),$$

since the inclusion mapping $L^2(S^n, m) \to L^p(S^n, m)$ is norm-decreasing. This gives the second inequality of (i). The complex case, and (iii), are proved similarly.

COROLLARY. (i) *If* $1 \leqslant p < 2$, $a_p(I_n)/\sqrt{n} \leqslant a_p(I_{n+1})/\sqrt{n+1}$.

(ii) *If* $p > 2$, $a_p(I_n)/\sqrt{n} \geqslant a_p(I_{n+1})/\sqrt{n+1}$.

Proof. (i) Let $E = F$ be an $(n+1)$-dimensional inner-product space, and let $(e_1, \ldots, e_{n+1})$ be an orthonormal basis. If $0 < \varepsilon < 1$, let E_ε be the linear operator from E into F defined by setting

$$\begin{aligned} T_\varepsilon e_i &= e_i \quad \text{for } 1 \leqslant i < n, \\ T_\varepsilon e_n &= (1-\varepsilon^2)^{1/2} e_n, \\ T_\varepsilon e_{n+1} &= \varepsilon e_n. \end{aligned}$$

Then $\sigma(T_\varepsilon) = \sqrt{n}$, so that

$$a_p(T_\varepsilon) \leqslant \frac{a_p(I_{n+1})}{\sqrt{n+1}} \cdot \sqrt{n}.$$

Let $P = \lim_{\varepsilon \to 0} T_\varepsilon$. P is a partial isometry of rank n from E into F, so that $a_p(P) = a_p(I_n)$. Since a_p is a continuous norm on $L(E, F)$, the result follows. The proof of (ii) is exactly similar.

Remark. These inequalities can also be deduced, for integral values of p, from the formulae in the corollaries to Propositions 1 and 2. Direct verification, for non-integral p, seems to be more difficult.

Proof of Theorem 1. Theorem 1 now follows easily from Propositions 1-5, and their corollaries. Suppose that $1 \leqslant p < 2$, and that T is a continuous linear operator of rank r. Then

$$\begin{aligned} \sigma(T) \leqslant a_p(T) &\leqslant \frac{a_p(I_r)}{\sqrt{r}} \sigma(T) \qquad \text{(Proposition 5)} \\ &\leqslant \frac{a_p(I_n)}{\sqrt{n}} \sigma(T) \qquad \text{(Corollary to Proposition 5).} \end{aligned}$$

On the other hand, if T is an operator of rank 1, $\sigma(T) = a_p(T) = \|T\|$, while if T is a partial isometry of rank n, $a_p(T) = a_p(I_n)$, whereas $\sigma(T) = \sqrt{n}$. Thus, the constants are best possible. The proofs in the cases $p = 2$ and $p > 2$ are exactly similar.

4. The infinite-dimensional case.

First we need two easy results, whose proofs we shall give, for the sake of completeness. Let $R(E, F)$ denote the space of continuous linear operators of finite rank from E into F, let $S(E, F)$ denote the space of Hilbert-Schmidt operators from E into F, and let $A_p(E, F)$ denote the space of absolutely p-summing operators from E into F. Let $\|T\|$ denote the operator norm of T.

PROPOSITION 6. *A continuous linear operator T from E into F is a Hilbert-Schmidt operator if and only if*

$$\sigma'(T) = \sup\{\sigma(ST) \colon S \in R(F, F),\ \|S\| \leqslant 1\} < \infty.$$

Further $\sigma(T) = \sigma'(T)$.

Suppose first that $T \in S(E, F)$. Since $\sigma(ST) \leqslant \|S\| \sigma(T)$, $\sigma'(T) \leqslant \sigma(T) < \infty$. We can write

$$Tx = \sum_{i=1}^{\infty} \lambda_i \langle x, e_i \rangle f_i,$$

where (e_i) and (f_i) are orthonormal sequences in E and F respectively, and

$$\sum_{i=1}^{\infty} |\lambda_i|^2 = (\sigma(T))^2.$$

Let

$$P_n y = \sum_{i=1}^{n} \langle y, f_i \rangle f_i.$$

Then $P_n \in R(F, F)$, $\|P_n\| = 1$, and

$$(\sigma(P_n T))^2 = \sum_{i=1}^{n} |\lambda_i|^2,$$

so that $\sigma(T) = \sigma'(T)$. Conversely, suppose that $\sigma'(T) < \infty$. If (e_i) and (f_i) are orthonormal sequences in E and F respectively, and if

$$P_n y = \sum_{i=1}^{n} \langle y, f_i \rangle f_i,$$

then

$$\sum_{i=1}^{n} |\langle Te_i, f_i \rangle|^2 = \sum_{i=1}^{\infty} |\langle P_n Te_i, f_i \rangle|^2 \leqslant \sigma'(T),$$

so that $T \in S(E, F)$.

PROPOSITION 7. *A continuous linear operator T from E into F is absolutely p-summing if and only if*

$$a_p'(T) = \sup\{a_p(ST) \colon S \in R(F, F),\ \|S\| \leqslant 1\} < \infty.$$

Further $a_p(T) = a_p'(T)$.

Suppose first that $T \in A_p(E, F)$. Since $a_p(ST) \leqslant \|S\| a_p(T)$, $a_p'(T) \leqslant a_p(T) < \infty$. Given $\varepsilon > 0$, there exist vectors $x_1, \ldots, x_n$ in E such that

$$\sum_{i=1}^{n} \|Tx_i\|^p \geqslant \left((a_p(T))^p - \varepsilon\right) \sup_{\|a\| \leqslant 1} \left(\sum_{i=1}^{n} |\langle x_i, a \rangle|^p \right);$$

let P be the orthogonal projection of F onto the span of $Tx_1, \ldots, Tx_n$. Then $PTx_i = Tx_i$ for each i, so that $(a_p(PT))^p \geqslant (a_p(T))^p - \varepsilon$. Since ε is arbitrary, $a_p'(T) = a_p(T)$. Conversely, suppose that $a_p'(T) < \infty$. Given any finite set $x_1, \ldots, x_n$ of vectors in E, let P be the orthogonal projection of F onto the span of $Tx_1, \ldots, Tx_n$. Then

$$\begin{aligned}\sum_{i=1}^{n} \|Tx_i\|^p &= \sum_{i=1}^{n} \|PTx_i\|^p \\ &\leqslant (a_p(PT))^p \sup_{\|a\|\leqslant 1} \Big(\sum_{i=1}^{n} |\langle x_i, a\rangle|^p \Big) \\ &\leqslant (a_p'(T))^p \sup_{\|a\|\leqslant 1} \Big(\sum_{i=1}^{n} |\langle x_i, a\rangle|^p \Big).\end{aligned}$$

Thus, $T \in A_p(E, F)$.

Proof of Theorem 2. It follows from Propositions 6 and 7 that in order to prove the first two statements of Theorem 2 it is sufficient to show that the norms σ and a_p are equivalent on $R(E, F)$. Suppose that E and F are real (the proof in the complex case is exactly similar). σ and a_2 are of course equal on $R(E, F)$. Next, it follows from Proposition 5 and its corollary that, if $1 \leqslant p < 2$,

$$\sigma(T) \leqslant a_p(T) \leqslant \Big(\lim_{n\to\infty} \frac{a_p(I_n)}{\sqrt{n}} \Big) \sigma(T).$$

But $a_p(I_n) \leqslant a_1(I_n)$, and

$$\lim_{n\to\infty} \frac{a_1(I_n)}{\sqrt{n}} = \sqrt{\frac{\pi}{2}}.$$

Thus σ and a_p are equivalent on $R(E, F)$. Similarly, if $p > 2$,

$$\Big(\lim_{n\to\infty} \frac{a_p(I_n)}{\sqrt{n}} \Big) \sigma(T) \leqslant a_p(T) \leqslant \sigma(T);$$

further, if k is an integer greater than p, $a_k(I_n) \leqslant a_p(I_n)$, and

$$\lim_{n\to\infty} \frac{a_k(I_n)}{\sqrt{n}} = \frac{1}{\sqrt{2\pi}} \Big(\prod_{i=1}^{k} L_{i-1} \Big)^{1/k},$$

so that σ and a_p are again equivalent on $R(E, F)$. Thus $S(E, F) = A_p(E, F)$ for all $p \geqslant 1$, and the norms σ and a_p are equivalent on $S(E, F)$. Finally, the fact that the constants in Theorem 2 are best possible follows easily by considering operators of rank one, and a sequence of partial isometrices of finite, but increasing, rank.

References

[1] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Matem. São Paulo 8 (1956), p. 1-79.

[2] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications*, Studia Math. 29 (1968), p. 275-326.

[3] A. Pełczyński, *A characterization of Hilbert-Schmidt operators*, ibidem 28 (1967), pp. 355-360.

[4] A. Pietsch, *Absolut p-summierende Abbildungen in normierten Räumen*, ibidem 28 (1967), p. 333-353.

[5] J. Wallis, *Arithmetica infinitorum*, Oxford 1656.

[6] Y. Gordon, *On p-absolutely summing constants of Banach spaces*, Israel J. Math. 7(1969), p. 151-163

[7] A. Meyer, *Grösste Polygone mit gegebenen Seitenvektoren*, Comm. Math. Helvetici 10(1938), p. 288-301.