STUDIA MATHEMATICA, T. XLIV. (1972)

On Banach spaces X for which $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$

by

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Abstract. The class of Banach spaces X such that every bounded linear operator from every $\mathscr{L}_\infty$-space into X is 2-absolutely summing is investigated. It is shown that this class is larger than the class of all subspaces of $\mathscr{L}_1$-spaces, and that it contains all quotient spaces of subspaces of $\mathscr{L}_p$-spaces for $2 \geqslant p > 1$. A complete characterization (in terms of an unconditional basis) is obtained in order that a Banach space with an unconditional basis belong to this class. In this case it is shown that under a natural uniformity condition the l_p-product of spaces belonging to this class also belongs to the class.

Introduction. The present paper is devoted to a study of Banach spaces X with the property

(∗) *every bounded linear operator from c_0 into X is 2-absolutely summing*

equivalently (cf. Section 2)

every bounded linear operator from every $\mathscr{L}_\infty$-space into X is 2-absolutely summing, in symbols

$$\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X).$$

Another equivalent condition says that every unconditionally convergent series in X is Hilbertian, i.e. it is the image by a linear operator of an unconditionally convergent series in a Hilbert space (cf. Corollary 2.1).

One of the basic facts of the theory of absolutely summing operators is a result essentially discovered by Grothendieck [6] which says that the space L_1 has the property (∗). Hence every subspace of an $\mathscr{L}_1$-space, in particular every $\mathscr{L}_p$-space for $1 \leqslant p \leqslant 2$ has the property (∗) (cf. [16]). In the present paper we show however that the class of Banach spaces satisfying (∗) is larger than the class of all subspaces of $\mathscr{L}_1$-spaces (cf. Examples 5.1 and 5.2).

In Section 3 we deal with Banach spaces satisfying (∗) which have unconditional bases or more generally what we have called local unconditional structures (cf. Definition 3.1). For such spaces the picture is rather clear. We establish several necessary and sufficient conditions in terms of unconditional bases of a space in order that the space satisfies (∗). One of our conditions asserts that a Banach space X with an unconditional basis (e_n) satisfies $(*)$ iff every operator from c_0 into X which takes the nth unit vector of c_0 into a multiple of e_n for $n = 1, 2, \ldots$ is 2-absolutely summing. It is also shown that, roughly speaking, products in the sense of l_p for $1 \leqslant p \leqslant 2$ of Banach spaces satisfying uniformly $(*)$ also satisfy $(*)$.

In Section 4 we study a property which is stronger than $(*)$. This property however is preserved under passing to quotient spaces which is not the case for the property $(*)$. The new property is also related to some properties of Gaussian random variables with values in Banach spaces (cf. [14] and [4]). Our technique unables us to show that every quotient of a subspace of an $\mathscr{L}_p$-space for $1 < p \leqslant 2$ has the property $(*)$ and to prove some further results on l_p-products.

Finally in Section 5 we discuss some examples and some open problems.

1. Notations and preliminaries. We begin with some notation. Let X and Y be (real or complex) Banach spaces. We denote by $B(X, Y)$ the space of all the operators from X into Y with the usual operator norm $\|T\| = \sup_{\|x\|\leqslant 1} \|Tx\|$. By "operator" we always mean a linear and bounded operator.

Let $+\infty > p \geqslant q \geqslant 1$. An operator $T\colon X \to Y$ is said to be (p, q)-*absolutely summing* (cf. [16]) if there is a $C < +\infty$ such that

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^p\Big)^{1/p} \leqslant C \sup_{\|x^*\|\leqslant 1} \Big(\sum_{i=1}^{n} |x^*(x_i)|^q\Big)^{1/q} \tag{1.1}$$

for every finite sequence $x_1, x_2, \ldots, x_n$ in X $(n = 1, 2, \ldots)$, where the supremum in (1.1) is taken over all linear functionals x^* from β the unit ball of the dual X^* of X. The lowest upper bound of those C for which (1.1) holds we denote by $\pi_{p,q}(T)$ and we call it the (p, q)-absolutely summing norm of T. The set of all (p, q)-absolutely summing operators from X into Y (for a fixed pair (p, q)) is a Banach space under the norm $\pi_{p,q}(\cdot)$ with the usual operations of addition of operators and multiplication by scalars. This space is denoted by $\Pi_{p,q}(X, Y)$. If $p = q$ we say "*p-absolutely summing operator*" instead of "(p, p)-absolutely summing", and we use the notation $\pi_p(T)$ instead of $\pi_{p,q}(T)$ and $\Pi_p(X, Y)$ instead of $\Pi_{p,q}(X, Y)$.

An operator $T\colon X \to Y$ is said to be *Hilbertian* if there exists a Hilbert space H and operators $A\colon X \to H$ and $B\colon H \to Y$ such that $T = BA$. The pair (A, B) with this property is called a *Hilbertian factorization of* T. The Hilbertian norm of a Hilbertian operator $T\colon X \to Y$ is defined by

$$h(T) = \inf \|A\|\,\|B\| = \inf \big((\|A\|^2 + \|B\|^2)/2\big)^{1/2}, \tag{1.2}$$

where the infimum is taken over all possible Hilbertian factorizations of T. The set of all Hilbertian operators from X into Y is a Banach space under the norm $h(\cdot)$ and with the usual operations of addition of operators and multiplication by scalars. This space is denoted by $H(X, Y)$. The (p, q)-absolutely summing operators and the Hilbertian operators form Banach ideals (cf. [25]). In particular they have the following properties.

1) If $T \in \Pi_{p,q}(X, Y)$ (resp. $T \in H(X, Y)$), then for all Banach spaces X_1 and Y_1 and all operators $S_1 \in B(X_1, X)$ and $S_2 \in B(Y, Y_1)$ we have $S_2TS_1 \in \Pi_{p,q}(X_1, Y_1)$ (resp. $S_2TS_1 \in H(X_1, Y_1)$) and $\pi_{p,q}(S_2TS_1) \leqslant \pi_{p,q}(T)\|S_1\|\|S_2\|$ (resp. $h(S_2TS_1) \leqslant h(T)\|S_1\|\|S_2\|$).

2) $\pi_{p,q}(T) \geqslant \|T\|$ for $T \in \Pi_{p,q}(X, Y)$; $h(T) \geqslant \|T\|$ for $T \in H(X, Y)$.

3) If $1 \leqslant p_1 < p_2 < +\infty$, then $\Pi_{p_1}(X, Y) \subset \Pi_{p_2}(X, Y)$ and $\pi_{p_1}(T) \geqslant \pi_{p_2}(T)$.

4) If $1 \leqslant p \leqslant 2$, then $\Pi_p(X, Y) \subset H(X, Y)$ and $\pi_p(T) \geqslant h(T)$ (cf. [23]).

The following Proposition is an easy consequence of the Closed Graph Theorem:

PROPOSITION 1.1. *Let $I_1(X, Y)$, and $I_2(X, Y)$ be Banach spaces whose elements are operators from X into Y and the operations of addition and multiplication by scalars are defined to be the usual addition of operators and the usual multiplication of operators by scalars. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the norms in $I_1(X, Y)$ and $I_2(X, Y)$ respectively. Then the condition*

$$\|T\|_1 \leqslant \|T\|_2 \text{ for } T \in I_2(X, Y) \text{ and } I_1(X, Y) = I_2(X, Y) \tag{1.3}$$

implies the existence of a K with $1 \leqslant K < +\infty$ such that

$$\|T\|_1 \leqslant \|T\|_2 \leqslant K\|T\|_1 \quad \text{for } T \in I_1(X, Y). \tag{1.4}$$

DEFINITION 1.1. Under the assumption and the notation of Proposition 1.1 we shall write

$$I_1(X, Y) \underset{K}{=} I_2(X, Y)$$

iff $I_1(X, Y) = I_2(X, Y)$ and there exists a $K \geqslant 1$, such that

$$K^{-1}\|T\|_1 \leqslant \|T\|_2 \leqslant K\|T\|_1.$$

Given Banach spaces X and Y and $C < +\infty$, Y is said to be *C-isomorphic* to X if there is an invertible linear operator $T\colon X \xrightarrow[\text{onto}]{} Y$ with $\|T\|\|T^{-1}\| \leqslant C$.

Let X be a Banach space and let A be any non empty set. Let $1 \leqslant p \leqslant \infty$. By $l_p^A(X)$ we denote the Banach space of all functions $x(\cdot)\colon A \to X$ such that $\|x(\cdot)\|_p < +\infty$, where

$$\|x(\cdot)\|_p = \begin{cases} \Big(\sum_{a\in A} \|x(a)\|^p\Big)^{1/p} & \text{for } 1 \leqslant p < \infty, \\ \sup_{a\in A} \|x(a)\| & \text{for } p = \infty. \end{cases}$$

The operations of addition and multiplication by scalars are defined pointwise.

If X is the field of scalars we write l_p^A instead of $l_p^A(X)$. If A is a finite set, say A has n-elements we write $l_p^n(X)$ (resp. l_p^n for X being the field of scalars). If A is the set of all positive integers we write $l_p(X)$ (resp. l_p). By c_0 we denote the subspace (= closed linear subspace) of l_∞ consisting of the convergent to zero sequences.

Let $1 \leqslant p \leqslant \infty$. We recall (cf. [16], [17]) that a Banach space X is said to be an $\mathscr{L}_p$-space, written $X \in \mathscr{L}_p$, if there exists a $\lambda \geqslant 1$ such that for any finite dimensional subspace E of X there exists a finite dimensional subspace F of X such that $F \supset E$ and F is λ-isomorphic to $l_p^{\dim F}$.

2. General equivalences. The equivalences given below are in fact known and can be deduced from results already stated in various papers (cf. [11], [12], [23]).

PROPOSITION 2.1. *Let X be a Banach space. Then the following conditions are equivalent*

(i) $\Pi_2(Y, X) = B(Y, X)$ *for every* $Y \in \mathscr{L}_\infty$,

(i$_a$) $\Pi_2(Y, X) = B(Y, X)$ *for some infinite dimensional* $Y \in \mathscr{L}_\infty$,

(i$_b$) *there exists* $K \geqslant 1$ *such that* $\Pi_2(l_\infty^n, X) \underset{K}{=} B(l_\infty^n, X)$ *for* $n = 1, 2, \ldots$,

(i$_c$) $\Pi_2(Y, X^{**}) = B(Y, X^{**})$ *for every* $Y \in \mathscr{L}_\infty$,

(ii) $H(Y, X) = B(Y, X)$ *for every* $Y \in \mathscr{L}_\infty$,

(ii$_a$) $H(Y, X) = B(Y, X)$ *for some infinite dimensional* $Y \in \mathscr{L}_\infty$,

(ii$_b$) *there exists* $K \geqslant 1$ *such that* $H(l_\infty^n, X) \underset{K}{=} B(l_\infty^n, X)$ *for* $n = 1, 2, \ldots$,

(ii$_c$) $H(Y, X^{**}) = B(Y, X^{**})$ *for every* $Y \in \mathscr{L}_\infty$,

(iii) $H(X^*, Z) = B(X^*, Z)$ *for every* $Z \in \mathscr{L}_1$,

(iii$_a$) $H(X^*, Z) = B(X^*, Z)$ *for some infinite dimensional* $Z \in \mathscr{L}_1$,

(iii$_b$) *there exists* $K \geqslant 1$ *such that*

$$H(X^*, l_1^n) \underset{K}{=} B(X^*, l_1^n) \quad \text{for } n = 1, 2, \ldots,$$

(iii$_c$) $H(X^{***}, Z) = B(X^{***}, Z)$ *for every* $Z \in \mathscr{L}_1$,

(iv) $\Pi_2(X, l_2) = \pi_1(X, l_2)$,

(iv$_a$) *there exists* $K \geqslant 1$ *such that*

$$\Pi_2(X, l_2^n) \underset{K}{=} \Pi_1(X, l_2^n) \quad \text{for } n = 1, 2, \ldots,$$

(iv$_b$) $\Pi_2(X, Y) = \Pi_1(X, Y)$ *for some infinite dimensional Banach space* Y,

(v) *there exists* $K \geqslant 1$ *such that*

$$\sum_{j=1}^{n} \Big(\sum_{i=1}^{m} |x_i^*(x_j)|^2 \Big)^{\frac{1}{2}} \leqslant K \Big(\sum_{i=1}^{m} \|x_i^*\|^2 \Big)^{\frac{1}{2}} \sup_{\substack{|\lambda(j)|=1 \\ j=1,2,\ldots,n}} \Big\| \sum_{j=1}^{n} \lambda(j) x_j \Big\|$$

for $x_1, x_2, \ldots, x_n$ *in* X *and* $x_1^*, x_2^*, \ldots, x_m^*$ *in* X^* $(n, m = 1, 2, \ldots)$.

Proof. The implication (i) $\Rightarrow$ (i$_a$) is trivial. To prove that (i$_a$) $\Rightarrow$ (i$_b$) observe: 1) if $\Pi_2(Y, X) = B(Y, X)$, then (by Proposition 1.1) $\Pi_2(Y, X) \underset{c}{=} B(Y, X)$ for some $c \geqslant 1$, 2) if P is a projection in Y and $Y_1 = P(Y)$ and if $\Pi_2(Y, X) \underset{c}{=} B(Y, X)$ then $\Pi_2(Y_1, X) \underset{c\|P\|^2}{=} B(Y_1, X)$, 3) if $Y \in \mathscr{L}_p$ for some p with $1 \leqslant p \leqslant \infty$, then there exists a constant $K_1 < +\infty$ depending only on Y such that for every finite dimensional subspace $E \subset Y$ there exists a finite dimensional subspace $Y_1 \subset Y$ such that $E \subset Y_1$; Y_1 is K_1-isomorphic to $l_p^{\dim Y_1}$ and there exists a projection $P\colon Y \xrightarrow[\text{onto}]{} Y_1$ with $\|P\| \leqslant K_1$ (cf. [17]). The implication (i$_b$) $\Rightarrow$ (i$_c$) follows from the fact that (i$_b$) and the Local Reflexivity Principle [17] imply that

$$\Pi_2(l_\infty^n, X^{**}) \underset{K}{=} B(l_\infty^n, X^{**}) \quad \text{for } n = 1, 2, \ldots$$

Now the observation 3) mentioned above implies (i$_c$). The implication (i$_c$) $\Rightarrow$ (i) follows from the fact that (i$_c$) implies that $\Pi_2(Y, Z) = B(Y, Z)$ for any subspace Z of X^{**} in particular for the cannonical image of X in X^{**}.

The proof of the implications (ii) $\Rightarrow$ (ii$_a$) $\Rightarrow$ (ii$_b$) $\Rightarrow$ (ii$_c$) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iii$_a$) $\Rightarrow$ (iii$_b$) $\Rightarrow$ (iii$_c$) $\Rightarrow$ (iii) is analogous.

(i) $\Rightarrow$ (ii). Use the inclusion $\Pi_2(Y, X) \subset H(Y, X) \subset B(Y, X)$ for all Banach spaces Y and X.

(ii$_c$) $\Rightarrow$ (iii). If $Z \in \mathscr{L}_1$, then $Z^* \in \mathscr{L}_\infty$ (cf. [17]). Thus, by (ii$_c$), for every $T \in B(X^*, Z)$ the adjoint T^* is Hilbertian. Thus, by [16], Proposition 5.1, T is Hilbertian.

Similarly (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). The implication follows from a result of Grothendieck (cf. [6], [16]) that every operator from an $\mathscr{L}_\infty$-space into a Hilbert space is 2-absolutely summing.

(i$_b$) $\Rightarrow$ (iv$_b$). Let $S \in \Pi_2(X, Y)$. Fix $x_1, x_2, \ldots, x_n$ in X and define $T\colon l_\infty^n \to X$ by $T\lambda = \sum_{j=1}^{n} \lambda(j) x_j$ for $\lambda = (\lambda(j)) \in l_\infty^n$. Then $\|T\| = \sup_{\|x^*\| \leqslant 1} \sum_{j=1}^{n} |x^*(x_j)|$. It follows from (i$_b$) that $\Pi_2(T) \leqslant K\|T\|$ where $K \geqslant 1$ does not depend on T and on n. It follows from the result of Pietsch [24] on the composition of p-absolutely summing operators that $\pi_1(ST) \leqslant K\pi_2(S)\pi_2(T) \leqslant K\|T\|\pi_2(S)$. Let (δ_j) denote the unit vector basis in l_∞^n. Since

$$\|\xi^*\| = \sum_{j=1}^{n} |\xi^*(\delta_j)| \quad \text{for every } \xi^* \in (l_\infty^n)^* = l_1^n,$$

we get

$$\sum_{j=1}^{n} \|Sx_j\| = \sum_{j=1}^{n} \|ST\delta_j\| \leqslant \pi_1(ST) \sup_{\|\xi^*\| \leqslant 1} \sum_{j=1}^{n} |\xi^*(\delta_j)|$$

$$= \pi_1(ST) \leqslant K\pi_2(S) \sup_{\|x^*\| \leqslant 1} \sum_{j=1}^{n} |x^*(x_j)|.$$

Thus $\pi_1(S) \leqslant K\pi_2(S)$ which shows (iv$_b$) because always $\pi_2(S) \leqslant \pi_1(S)$.

$(\mathrm{iv_b}) \Rightarrow (\mathrm{iv_a})$. It follows from $(\mathrm{iv_b})$ and Proposition 1.1 that there exists $K \geqslant 1$ such that $\Pi_1(X, Y) \underset{K}{=} \Pi_2(X, Y)$. Hence $\Pi_1(X, Z) \underset{K}{=} \Pi_2(X, Z)$ for every subspace Z of Y. Since Y is infinite dimensional, the Dvoretzky Theorem (cf. [2] and [19]) implies that for every $\varepsilon > 0$ and every $n = 1, 2, \dots$ there exists a subspace $Z_{n,\varepsilon}$ of Y such that $Z_{n,\varepsilon}$ is $(1+\varepsilon)$-isomorphic to l_2^n. Thus we get $(\mathrm{iv_a})$.

$(\mathrm{iv_a}) \Rightarrow (\mathrm{iv}) \Rightarrow (\mathrm{iv_b})$. These implications are trivial.

$(\mathrm{v}) \Rightarrow (\mathrm{i_b})$. Any $T \in B(l_\infty^n, X)$ is of the form

$$T\lambda = \sum_{j=1}^{n} \lambda(j) x_j \quad \text{for } \lambda = (\lambda(j)), \tag{2.1}$$

moreover

$$\|T\| = \sup_{|\lambda(j)|=1; j=1,2,\dots,n} \Big\| \sum_{j=1}^{n} \lambda(j) x_j \Big\|. \tag{2.2}$$

Fix $\lambda_i = (\lambda_i(j)) \in l_\infty^n$ for $i = 1, 2, \dots, m$ and choose $\tilde{x}_i^*$ in X^* so that $\|\tilde{x}_i^*\| = 1$ and $\tilde{x}_i^*(T\lambda_i) = \|T\lambda_i\|$ for $i = 1, 2, \dots, m$. Next define $\mu_i \geqslant 0$ $(i = 1, 2, \dots, m)$ so that for $x_i^* = \mu_i \tilde{x}_i^*$ we have

$$\sum_{i=1}^{m} \|x_i^*\|^2 = 1 \quad \text{and} \quad \sum_{i=1}^{m} x_i^*(T\lambda_i) = \Big(\sum_{i=1}^{m} \|T\lambda_i\|^2\Big)^{\frac{1}{2}}. \tag{2.3}$$

By (v), (2.1), (2.2) and (2.3) we get

$$\Big(\sum_{i=1}^{m} \|T\lambda_i\|^2\Big)^{\frac{1}{2}} = \sum_{j=1}^{n} \sum_{i=1}^{m} \lambda_i(j) x_i^*(x_j) \leqslant \sum_{j=1}^{n} \Big(\sum_{i=1}^{m} |\lambda_i(j)|^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{m} |x_i^*(x_j)|^2\Big)^{\frac{1}{2}}$$
$$\leqslant K\|T\| \max_{1\leqslant j\leqslant n} \Big(\sum_{i=1}^{m} |\lambda_i(j)|^2\Big)^{\frac{1}{2}} \leqslant K\|T\| \sup_{\|\xi^*\|\leqslant 1; \xi^* \in l_1^n} \Big(\sum_{i=1}^{m} |\xi^*(\lambda_i)|^2\Big)^{\frac{1}{2}}.$$

Thus $\pi_2(T) \leqslant K\|T\|$.

$(\mathrm{iv_b}) \Rightarrow (\mathrm{v})$. Pick $x_1, x_2, \dots, x_n$ in X and $x_1^*, x_2^*, \dots, x_m^*$ in X^* Let $S\colon X \to l_2^m$ be the operator defined by

$$Sx = (x_i^*(x))_{1\leqslant i\leqslant m} \quad \text{for } x \in X.$$

Fix $z_1, z_2, \dots, z_p$ in X $(p = 1, 2, \dots)$. Then we have

$$\sum_{k=1}^{p} \|Sz_k\|^2 = \sum_{k=1}^{p} \sum_{i=1}^{m} |x_i^*(z_k)|^2 = \sum_{i=1}^{m} \sum_{k=1}^{p} |x_i^*(z_k)|^2$$
$$\leqslant \sum_{i=1}^{m} \|x_i^*\|^2 \sup_{\|x^*\|\leqslant 1} \sum_{k=1}^{p} |x^*(z_k)|^2.$$

Thus $\pi_2(S) \leqslant \big(\sum_{i=1}^{m} \|x_i^*\|^2\big)^{\frac{1}{2}}$.

Now suppose that X satisfies $(\mathrm{iv_b})$ with some $K \geqslant 1$. Then

$$\pi_1(S) \leqslant K\pi_2(S) \leqslant K\Big(\sum_{i=1}^{m} \|x_i^*\|^2\Big)^{\frac{1}{2}}.$$

Hence, remembering that

$$\sup_{\|x^*\|\leqslant 1} \sum_{j=1}^{m} |x^*(x_j)| = \sup_{\substack{|\lambda(j)|=1\\ j=1,2,\dots,n}} \Big\| \sum_{j=1}^{n} \lambda(j) x_j \Big\|,$$

we get

$$\sum_{j=1}^{n} \Big(\sum_{i=1}^{m} |x_i^*(x_j)|^2\Big)^{\frac{1}{2}} = \sum_{j=1}^{n} \|Sx_j\| \leqslant \pi_1(S) \sup_{\|x^*\|\leqslant 1} \sum_{j=1}^{n} |x^*(x_j)|$$
$$\leqslant K\Big(\sum_{i=1}^{m} \|x_i^*\|^2\Big)^{\frac{1}{2}} \sup_{\substack{|\lambda(j)|=1\\ j=1,2,\dots,n}} \Big\| \sum_{j=1}^{n} \lambda(j) x_j \Big\|.$$

Thus X satisfies (v) and this completes the proof.

Definition 2.1. If a Banach space X satisfies one of the equivalent conditions of Proposition 2.1 we write

$$\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X).$$

More precisely we write

$$\Pi_2(\mathscr{L}_\infty, X) \underset{K}{=} B(\mathscr{L}_\infty, X)$$

iff $\Pi_2(l_\infty^n, X) \underset{K}{=} B(l_\infty^n, X)$ for $n = 1, 2, \dots$

Definition 2.2. An unconditionally convergent series $\sum x_n$ in a Banach space X is said to be Hilbertian if there exists an operator $S\colon l_2 \to X$ and an unconditionally convergent series $\sum z_n$ in l_2 such that $Sz_n = x_n$.

Corollary 2.1. *Let X be a Banach space. Then $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$ iff every unconditionally convergent series in X is Hilbertian.*

Proof. It is well known (cf. [1], [21]) that a series $\sum x_n$ in a Banach space X is unconditionally convergent iff there exists a compact operator $T\colon c_0 \to X$ which maps the unit vector basis of c_0 onto the sequence (x_n). Hence if $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$, then, by (ii) and the fact that every operator from c_0 into l_2 is compact it follows that every unconditionally convergent series in X is Hilbertian.

Conversely, assume that every unconditionally convergent series in X is Hilbertian. This property is in view of the previous observation equivalent to the fact that every compact operator from c_0 into X is Hilbertian. Hence the natural embedding of the space $K(c_0, X)$ of all compact operators from c_0 into X under the operator norm into the space

$H(c_0, X)$ is continuous (by the Closed Graph Theorem) which implies the condition (ii$_b$) of Proposition 2.1.

Remark 1. The condition (v) of Proposition 2.1 is in fact a finite dimensional version of Kwapień's condition: "Every 2-nuclear operator from X into l_2 is 1-absolutely summing". Kwapień ([11], [12]) proved the equivalence of this condition with the condition (i) of Proposition 2.1 using the Pietsch–Persson Duality Theorem (cf. [23]).

Remark 2. The equivalence (i) $\Leftrightarrow$ (v) can be generalized. One can get a similar necessary and sufficient condition in order that every operator from every $\mathscr{L}_\infty$ space into a Banach space X be (p, q)-absolutely summing for fixed pair (p, q) with $p \geqslant q$ (cf. [26]).

3. Banach spaces X with unconditional bases and with local unconditional structures for which $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$. Let $(e_a, e_a^*)_{a\in\mathscr{A}}$ be a biorthogonal system in a Banach space X. Then $(e_a)_{a\in\mathscr{A}}$ is said to be an unconditional basis for X if for every $x \in X$ and every scalars ε_a with $|\varepsilon_a| = 1$ $(a \in \mathscr{A})$

$$\inf_{\mathscr{B}} \Big\| \sum_{a\in\mathscr{B}} e_a^*(x)\, e_a - x \Big\| = 0,$$

$$\sup_{\mathscr{B}} \Big\| \sum_{a\in\mathscr{B}} \varepsilon_a e_a^*(x)\, e_a \Big\| < +\infty,$$

where the infimum and the supremum is taken over all finite non-empty subsets of $\mathscr{A}$.

It is well known that the last inequality implies (by a standard Baire category argument) that there exists $C \geqslant 1$ such that

$$\sup_{|\varepsilon_a|=1} \sup_{\mathscr{B}} \Big\| \sum_{a\in\mathscr{B}} \varepsilon_a e_a^*(x)\, e_a \Big\| \leqslant C\|x\|.$$

The greatest lower bound of those C is called the *unconditional characteristic* of the basis (e_a) and will be denoted by $\gamma\big((e_a)\big)$.

Definition 3.1. A family $\{E_\iota\}_{\iota\in J}$ of finite dimensional spaces is said to be a *local unconditional structure* for a Banach space X if each E_ι has an unconditional basis with the unconditional characteristic 1 and there exists a $C \geqslant 1$ such that for any finite dimensional subspace $F \subset X$ there exist an index $\iota \in J$ and an operator $T_{\iota,F}\colon E_\iota \to X$ such that $T_{\iota,F}(E_\iota) \supset F$ and

$$\|e\| \leqslant \|T_{\iota,F}e\| \leqslant C\|e\| \quad \text{for } e \in E_\iota,$$

and moreover for each $\iota \in J$ there exists an operator $T_\iota\colon E_\iota \to X$ with $\|e\| \leqslant \|T_\iota e\| \leqslant C\|e\|$ for $e \in E_\iota$.

The greatest lower bound of those C is called the *unconditional characteristic of the pair* $(X, \{E_\iota\}_{\iota\in J})$ and will be denoted by $\gamma(X, \{E_\iota\})_{\iota\in J}$.

The concept of a local unconditional structure is a generalization of the concept of an unconditional basis. We have

Proposition 3.1. *If X is a Banach space with an unconditional basis $(e_a)_{a\in\mathscr{A}}$, then X has a local unconditional structure $\{E_\iota\}_{\iota\in J}$ with $\gamma(X, \{E_\iota\}_{\iota\in J}) = \gamma\big((e_a)\big)$.*

Proof. Let J be a family of all non empty finite subsets of $\mathscr{A}$. For $\iota \in J$ let E_ι be the linear space of $\{e_a\}_{a\in\iota}$ equipped with the norm $\big|\big|\big|\sum_{a\in\iota} t_a e_a\big|\big|\big| = \sup_{|\varepsilon_a|=1;\, a\in\iota} \big\|\sum_{a\in\iota} \varepsilon_a t_a e_a\big\|$. It follows from the standard stability argument that $\{E_\iota\}_{\iota\in J}$ is the desired local unconditional structure.

Next we recall the following.

Definition 3.2. Let $(e_a)_{a\in\mathscr{A}}$ be an unconditional basis in a Banach space X and let $\{X_a\}_{a\in\mathscr{A}}$ be a family of Banach spaces indexed by the same set of indices. The product of the spaces X_a in the sense of the basis $(e_a)_{a\in\mathscr{A}}$ written $[X_a]_{(e_a)}$ is the Banach space of all sequences $\big(x(a)\big)_{a\in\mathscr{A}}$ such that $x(a) \in X_a$ for $a \in \mathscr{A}$ and there exists an $x \in X$ such that $e_a^*(x) = x(a)$ for $a \in \mathscr{A}$ i.e. the series $\sum_{a\in\mathscr{A}} \|x(a)\| e_a$ converges in X. We define

$$|||x||| = \Big\| \sum_{a\in\mathscr{A}} \|x(a)\| e_a \Big\|.$$

The operations of addition and multiplication by scalars are defined coordinatwise.

If $(e_a)_{a\in\mathscr{A}}$ is the unit vector basis of the space $l_p^{\mathscr{A}}$, then we shall write $[X_a]_{l_p^{\mathscr{A}}}$, instead of $[X_a]_{(t_a)}$.

Next we show that products in the sense of unconditional bases preserve local unconditional structures and unconditional bases.

Proposition 3.2. *Let $(e_a)_{a\in\mathscr{A}}$ be an unconditional basis in a Banach space X. Let $\{X_a\}_{a\in\mathscr{A}}$ be a family of Banach spaces with unconditional bases $(e_\beta^{(a)})_{\beta\in\mathscr{A}_a}$ respectively $\beta \in \mathscr{A}_a$ $(a \in \mathscr{A})$. Suppose that $\sup_{a\in\mathscr{A}} \gamma\big((e_\beta^{(a)})\big) = C < +\infty$. For $\beta \in \mathscr{A}_a$ and $a \in \mathscr{A}$ define $f_{a\beta} \in [X_a]_{(e_a)}$ by $f_{a\beta}(a') = 0$ for $a \neq a'$ and $f_{a\beta}(a) = e_\beta^{(a)}$. Then $(f_{a\beta})_{\beta\in\mathscr{A}_a; a\in\mathscr{A}}$ forms an unconditional basis in the product $[X_a]_{(e_a)}$ with $\gamma\big((f_{a\beta})\big) \leqslant C\gamma\big((e_a)\big)$.*

Proof. Observe first that for any non-empty finite subset $\mathscr{B} \subset \mathscr{A}$ and any scalar sequences $(s_a)_{a\in\mathscr{B}}$, $(t_a)_{a\in\mathscr{B}}$ such that $|s_a| \leqslant C|t_a|$ for $a \in \mathscr{B}$ we have

$$(3.0) \qquad \Big\| \sum_{a\in\mathscr{B}} s_a e_a \Big\| \leqslant C\gamma\big((e_a)\big) \Big\| \sum_{a\in\mathscr{B}} t_a e_a \Big\|.$$

To see this define ϱ_a so that $s_a = C\varrho_a t_a$ for $a \in \mathscr{B}$. Clearly $|\varrho_a| \leqslant 1$ for

$\alpha\in\mathscr{B}$. Any point of the cube $\{(\varrho_\alpha)_{\alpha\in\mathscr{B}}\colon |\varrho_\alpha|\leqslant 1\}$ is a finite convex combinations of the extreme points of the cube i.e. of points $(\varepsilon_\alpha)_{\alpha\in\mathscr{B}}$ with $|\varepsilon_\alpha|=1$ for $\alpha\in\mathscr{B}$. Thus there are non negative $\lambda_1,\lambda_2,\ldots,\lambda_N$ with $\sum\lambda_i=1$ and $(\varepsilon_\alpha^i)_{\alpha\in\mathscr{B}}$ with $|\varepsilon_\alpha^i|=1$ for $\alpha\in B$ and for $i=1,2,\ldots,N$ such that $\varrho_\alpha=\sum_{i=1}^N\lambda_i\varepsilon_\alpha^i$ for $\alpha\in\mathscr{B}$. Thus

$$\Big\|\sum_{\alpha\in\mathscr{B}}s_\alpha e_\alpha\Big\|=C\Big\|\sum_{\alpha\in\mathscr{B}}\varrho_\alpha t_\alpha e_\alpha\Big\|=C\Big\|\sum_{i=1}^N\lambda_i\sum_{\alpha\in\mathscr{B}}\varepsilon_\alpha^i t_\alpha e_\alpha\Big\|$$
$$\leqslant C\max_{1\leqslant i\leqslant N}\Big\|\sum_{\alpha\in\mathscr{B}}\varepsilon_\alpha^i t_\alpha e_\alpha\Big\|\leqslant C\gamma\big((e_\alpha)\big)\Big\|\sum_{\alpha\in\mathscr{B}}t_\alpha e_\alpha\Big\|.$$

Now pick any non empty finite sets $\mathscr{B}\subset\mathscr{A}$ and $\mathscr{B}_\alpha\subset\mathscr{A}_\alpha$ for $\alpha\in\mathscr{B}$. Let $(t_{\alpha\beta})_{\beta\in\mathscr{B}_\alpha,\alpha\in\mathscr{B}}$ and $(\varepsilon_{\alpha\beta})_{\beta\in\mathscr{B}_\alpha,\alpha\in\mathscr{B}}$ be any sequence of scalars such that $|\varepsilon_{\alpha\beta}|=1$ for $\beta\in\mathscr{B}_\alpha$ and $\alpha\in\mathscr{B}$. Let us set $t_\alpha=\big\|\sum_{\beta\in\mathscr{B}_\alpha}t_{\alpha\beta}e_\beta^{(\alpha)}\big\|$ and $s_\alpha=\big\|\sum_{\beta\in\mathscr{B}_\alpha}\varepsilon_{\alpha\beta}t_{\alpha\beta}e_\beta^{(\alpha)}\big\|$. Since $\gamma\big((e_\beta^{(\alpha)})\big)\leqslant C$, we have $|s_\alpha|\leqslant C|t_\alpha|$ for $\alpha\in\mathscr{B}$. Thus, by (3.0),

$$\Big|\Big|\Big|\sum\varepsilon_{\alpha\beta}t_{\alpha\beta}f_{\alpha\beta}\Big|\Big|\Big|=\Big\|\sum_{\alpha\in\mathscr{B}}s_\alpha e_\alpha\Big\|$$
$$\leqslant C\,\gamma\big((e_\alpha)\big)\Big\|\sum_{\alpha\in\mathscr{B}}t_\alpha e_\alpha\Big\|$$
$$=C\,\gamma\big((e_\alpha)\big)\Big|\Big|\Big|\sum t_{\alpha\beta}f_{\alpha\beta}\Big|\Big|\Big|.$$

This inequality together with an easy observation that linear combinations of $f_{\alpha\beta}$ $(\beta\in\mathscr{A}_\alpha,\ \alpha\in\mathscr{A})$ are dense in the product $[X_\alpha]_{(e_\alpha)}$ show that $(f_{\alpha\beta})$ is the desired unconditional basis.

Proposition 3.3: *Let $(e_\alpha)_{\alpha\in\mathscr{A}}$ be an unconditional basis in a Banach space X. Let $\{X_\alpha\}_{\alpha\in\mathscr{A}}$ be a family of Banach spaces and let $\{E_\iota^{(\alpha)}\}_{\iota\in I_\alpha}$ be a local unconditional structure for X_α with $\gamma(X_\alpha,\{E_\iota^{(\alpha)}\}_{\iota\in I_\alpha})\leqslant C$ for all α in $\mathscr{A}$. Then the product $[X_\alpha]_{(e_\alpha)}$ has a local unconditional structure with the unconditional characteristic $\leqslant C\gamma\big((e_\alpha)\big)$.*

Proof. Let $\mathscr{T}$ be the set of all sequences $\tau=(\iota_\alpha)_{\alpha\in\mathscr{B}}$ where $\mathscr{B}$ is a non empty finite subset of $\mathscr{A}$ and $\iota_\alpha\in I_\alpha$ for $\alpha\in\mathscr{B}$. To any $\tau\in\mathscr{T}$ assign the finite dimensional space E_τ defined as follows. The elements of E_τ are all sequences $\big(x(\alpha)\big)_{\alpha\in\mathscr{B}}$ such that $x(\alpha)\in E_{\iota_\alpha}^{(\alpha)}$ for $\alpha\in\mathscr{B}$, the operations of addition and multiplication by scalars are defined coordinatewise and the norm is defined by $\big|\big|\big|\big(x(\alpha)\big)\big|\big|\big|=\sup_{|\varepsilon_\alpha|=1}\big\|\sum_{\alpha\in\mathscr{B}}\varepsilon_\alpha\|x(\alpha)\|e_\alpha\big\|$. By the definition of a local unconditional structure, each $E_{\iota_\alpha}^{(\alpha)}$ has an unconditional basis with the unconditional characteristic one. Thus, by Proposition 3.2, E_τ has an unconditional basis with the unconditional characteristic one because $E_\tau=[E_\alpha]_{(e_\alpha^\sim)_{\alpha\in\mathscr{B}}}$ where $(e_\alpha^\sim)_{\alpha\in\mathscr{B}}$ denotes the unit vector basis in the space of sequences $(t_\alpha)_{\alpha\in\mathscr{B}}$ equipped with the norm $\|(t_\alpha)_{\alpha\in\mathscr{B}}\|=\sup_{|\varepsilon_\alpha|=1}\big\|\sum_{\alpha\in\mathscr{B}}t_\alpha\varepsilon_\alpha e_\alpha\big\|$.

We omit the routine checking that the family $\{E_\tau\}_{\tau\in\mathscr{T}}$ is a local unconditional structure for the product $[X_\alpha]_{(e_\alpha)}$ with the unconditional characteristic $\leqslant C\gamma\big((e_\alpha)\big)$.

Next we pass to a characterization of those Banach spaces X with local unconditional structures for which $\Pi_2(\mathscr{L}_\infty,X)=B(\mathscr{L}_\infty,X)$. The following result is the main technical tool of this section.

Proposition 3.4. *Let $(e_j,e_j^*)_{1\leqslant j\leqslant n}$ be a biorthogonal system in an n-dimensional Banach space E. Assume that the unconditional characteristic $\gamma\big((e_j)\big)=1$,* i.e.

$$\Big\|\sum_{j=1}^{\infty}t_je_j\Big\|=\Big\|\sum_{j=1}^{n}|t_j|e_j\Big\|\quad\textit{for any scalars } t_1,t_2,\ldots,t_n. \tag{3.1}$$

Then for any $K\geqslant 1$ the following implications hold (I) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) $\Rightarrow$ (V) $\Rightarrow$ (VI), *where*

(I) $\Pi_2(l_\infty^n,E)\underset{K}{=}B(l_\infty^n,E)$,

(II) $\pi_2(D)\leqslant K\|D\|$ *for any operator $D\colon l_\infty^n\to E$ which is diagonal with respect to the basis (e_j), i.e. there exist scalars $d_1,d_2,\ldots,d_n$ such that*

$$D(\lambda)=\sum_{j=1}^{n}d_j\lambda(j)e_j\quad\textit{for } \lambda=(\lambda(j))\in l_\infty^n, \tag{3.2}$$

(III) $\Big(\sum_{i=1}^m\|x_i\|^2\Big)^{\frac12}\leqslant K\Big\|\sum_{j=1}^n\Big(\sum_{i=1}^m|e_j^*(x_i)|^2\Big)^{\frac12}e_j\Big\|$ *for any $x_1,x_2,\ldots,x_m$ in E* $(m=1,2,\ldots)$,

(IV) $\Big\|\sum_{j=1}^n\Big(\sum_{i=1}^m|x_i^*(e_j)|^2\Big)^{\frac12}e_j^*\Big\|\leqslant K\Big(\sum_{i=1}^m\|x_i^*\|^2\Big)^{\frac12}$ *for any $x_1^*,x_2^*,\ldots,x_m^*$ in E^** $(m=1,2,\ldots)$,

(V) *for any $m=1,2,\ldots$ and for any $x_1^*,x_2^*,\ldots,x_m^*$ in the unit ball of E^* and scalars $\lambda_1,\lambda_2,\ldots,\lambda_m$ with $\sum_{i=1}^m|\lambda_i|^2=1$ there exist $x^*\in E^*$ with $\|x^*\|\leqslant K$ and scalars $\lambda_i(j)$ $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$ such that*

$$\max_{1\leqslant j\leqslant n}\sum_{i=1}^m|\lambda_i(j)|^2\leqslant 1\quad\textit{and}\quad\lambda_ix_i^*(e_j)=\lambda_i(j)x^*(e_j) \tag{3.3}$$
$$(i=1,2,\ldots,m;\ j=1,2,\ldots,n),$$

(VI) $\Pi_2(l_\infty^s,E)\underset{KK_G}{=}B(l_\infty^s,E)$ *for $s=1,2,\ldots$ where K_G is the Grothendieck constant,* i.e.

$$K_G=\inf\{C\,|\,\Pi_2(l_\infty^s,l_1)\underset{C}{=}B(l_\infty^s,l_1)\}\quad\textit{for } s=1,2,\ldots$$

Proof. (I) $\Rightarrow$ (II). Obvious.

(II) $\Rightarrow$ (III). Suppose that we are given $x_1,x_2,\ldots,x_m$ in E. Let $d_j=\Big(\sum_{i=1}^m|e_j^*(x_i)|^2\Big)^{\frac12}$ for $j=1,2,\ldots,n$. Then the sequence (d_j) defines by

(3.2) a diagonal operator. It follows from (3.1) that $\|D\| = \left\|\sum_{j=1}^{n} d_j e_j\right\|$. Define for $i = 1, 2, \ldots, m$ a sequence $\lambda_i = (\lambda_i(j)) \in l_\infty^n$ by $\lambda_i(j) = d_j^{-1} e_j^*(x_i)$ for $d_j \neq 0$ and $\lambda_i(j) = 0$ for $d_j = 0$ $(j = 1, 2, \ldots, n)$. Then

$$\sup_{\|\xi^*\|=1} \sum_{i=1}^{m} |\xi^*(\lambda_i)|^2 = \max_{1\leqslant j\leqslant n} \sum_{i=1}^{m} |\lambda_i(j)|^2 \leqslant 1.$$

Thus, by (II), we get

$$\Big(\sum_{i=1}^{m} \|x_i\|\Big)^{\frac12} = \Big(\sum_{i=1}^{m} \|D\lambda_i\|^2\Big)^{\frac12} \leqslant \pi_2(D) \leqslant K\Big\|\sum_{j=1}^{n} d_j e_j\Big\|.$$

(III) $\Rightarrow$ (IV). Fix $x_1^*, x_2^*, \ldots, x_m^*$ in E^* and $x = \sum_{j=1}^{m} t_j e_j$ in E. Next choose scalars $\lambda_i(j)$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ so that for $j = 1, 2, \ldots, n$

$$\sum_{i=1}^{m} |\lambda_i(j)|^2 = 1 \quad\text{and}\quad \sum_{i=1}^{m} \lambda_i(j) x_i^*(e_j) = \Big(\sum_{i=1}^{m} |x_i^*(e_j)|^2\Big)^{\frac12}. \tag{3.4}$$

Then, remembering that $t_j = e_j^*(x)$ we have

$$\Big|\Big(\sum_{j=1}^{n}\Big(\sum_{i=1}^{m}|x_i^*(e_j)|^2\Big)^{\frac12} e_j^*\Big)(x)\Big| = \Big|\sum_{j=1}^{n}\sum_{i=1}^{m}\lambda_i(j)x_i^*(e_j)t_j\Big| = \sum_{i=1}^{m} x_i^*\Big(\sum_{j=1}^{n}\lambda_i(j)t_j e_j\Big)$$

$$\leqslant \sum_{i=1}^{m}\|x_i^*\|\Big\|\sum_{j=1}^{n}\lambda_i(j)t_j e_j\Big\| \leqslant \Big(\sum_{i=1}^{m}\|x_i^*\|^2\Big)^{\frac12}\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{n}\lambda_i(j)t_j e_j\Big\|^2\Big)^{\frac12}.$$

It follows from (III) and (3.4) that

$$\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{n}\lambda_i(j)t_j e_j\Big\|^2\Big)^{\frac12} \leqslant K\Big\|\sum_{j=1}^{n}\Big(\sum_{i=1}^{m}|\lambda_i(j)|^2|t_j|^2\Big)^{\frac12} e_j\Big\|$$

$$= K\Big\|\sum_{j=1}^{n}|t_j| e_j\Big\| = K\|x\|.$$

Hence for any $x \in E$

$$\Big|\Big(\sum_{j=1}^{n}\Big(\sum_{i=1}^{m}|x_i^*(e_j)|^2\Big)^{\frac12} e_j^*\Big)(x)\Big| \leqslant K\Big(\sum_{i=1}^{m}\|x_i^*\|^2\Big)^{\frac12}\|x\|$$

which is equivalent to (IV).

(IV) $\Rightarrow$ (V). Given $x_1^*, x_2^*, \ldots, x_m^*$ in the unit ball of E^* and scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$ with $\sum_{i=1}^{m}|\lambda_i|^2 = 1$. Define

$$x^* = \sum_{j=1}^{n}\Big(\sum_{i=1}^{m}|\lambda_i x_i^*(e_j)|^2\Big)^{\frac12} e_j^*$$

and

$$\lambda_i(j) = \lambda_i x_i^*(e_j)/x^*(e_j) \quad \text{for } x^*(e_j) \neq 0,$$
$$\lambda_i(j) = 0 \quad \text{for } x^*(e_j) = 0.$$

Then $\sum_{i=1}^{m}|\lambda_i(j)|^2 \leqslant 1$ for $j = 1, 2, \ldots, n$ and $\lambda_i(j)x_i^*(e_j) = \lambda_i x^*(e_j)$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Moreover, by (IV), we get

$$\|x^*\| = K\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{n}\lambda_i x_i^*(e_j)e_j^*\Big\|^2\Big)^{\frac12} \leqslant K\Big(\sum_{i=1}^{m}|\lambda_i|^2\|x_i^*\|^2\Big)^{\frac12} \leqslant K.$$

(V) $\Rightarrow$ (VI). Let $T\colon l_\infty^s \to E$ be defined by $T\mu = \sum_{k=1}^{s}\mu(k)z_k$ for $\mu = (\mu(k)) \in l_\infty^s$. For $i = 1, 2, \ldots, m$ fix $\mu_i = (\mu_i(k))$ in l_∞^s and pick $x_1^*, x_2^*, \ldots, x_m^*$ in E^* with $\|x_i^*\| = 1$ and scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$ with $\sum_{i=1}^{m}|\lambda_i|^2 \leqslant 1$ so that

$$x_i^*(T\mu_i) = \|T\mu_i\|;\quad \sum_{i=1}^{m}\lambda_i x_i^*(T\mu_i) = \Big(\sum_{i=1}^{m}\|T\mu_i\|^2\Big)^{\frac12}.$$

By (V), there are scalars $\lambda_i(j)$ for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ and an $x^* \in E^*$ with $\|x^*\| \leqslant K$ such that (3.3) is satisfied. Let us put

$$a_{jk} = e_j^*(z_k)x^*(e_j) \quad (j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, s).$$

By (3.1), for any ε_j and η_k with $|\varepsilon_j| = |\eta_k| = 1$ for $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, s$ we have

$$\Big|\sum_{j=1}^{n}\sum_{k=1}^{s} a_{jk}\varepsilon_j\eta_k\Big| = \Big(\sum_{j=1}^{n}\varepsilon_j x^*(e_j)e_j^*\Big)\Big(\sum_{k=1}^{s}\eta_k z_k\Big) \leqslant \|x^*\|\|T\| \leqslant K\|T\|.$$

Hence, by the Grothendieck inequality (cf. [6], [16]) and (3.3), we get

$$\Big(\sum_{i=1}^{m}\|T\mu_i\|^2\Big)^{\frac12} = \sum_{i=1}^{m}\lambda_i x_i^*(T\mu_i) = \sum_{i=1}^{m}\lambda_i x_i^*\Big(\sum_{k=1}^{s}\mu_i(k)z_k\Big)$$

$$= \sum_{i=1}^{m}\lambda_i x_i^*\Big[\sum_{k=1}^{s}\mu_i(k)\Big(\sum_{j=1}^{n}e_j^*(z_k)e_j\Big)\Big]$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{s}\sum_{i=1}^{m}\mu_i(k)\lambda_i x_i^*(e_j)e_j^*(z_k) = \sum_{j=1}^{n}\sum_{k=1}^{s}a_{jk}\sum_{i=1}^{m}\lambda_i(j)\mu_i(k)$$

$$\leqslant K_G K\|T\|\max_{1\leqslant k\leqslant s}\Big(\sum_{i=1}^{m}|\mu_i(k)|^2\Big)^{\frac12}$$

$$= K_G K\|T\|\sup_{\|\xi^*\|=1}\Big(\sum_{i=1}^{m}|\xi^*(\mu_i)|^2\Big)^{\frac12}.$$

Hence $\pi_2(T) \leqslant K_G K\|T\|$ and this completes the proof.

Our next result is an immediate consequence of Proposition 3.4, the definition of a local unconditional structure and the fact that the property $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$ is a "local" property of a Banach space X.

THEOREM 3.1. *Let $\{E_\iota\}_{\iota\in J}$ be a local unconditional structure for an infinite dimensional Banach space X.*

Then $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$ if and only if there exists $K \geqslant 1$ such that for any unconditional basis $(e_j^{(\iota)})$ for E_ι with $\gamma(e_j^{(\iota)}) = 1$ one of the conditions (I), (II), (III), (IV), (V) *is satisfied (equivalently all the conditions* (I)–(V) *are satisfied) with (e_j) replaced by $(e_j^{(\iota)})$ and with $n = \dim E_\iota$.*

THEOREM 3.2. *Let X be a Banach space with an unconditional basis $(e_a)_{a\in\mathscr{A}}$. Then $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$ if and only if there exists a constant $K \geqslant 1$ such that for any finite non-empty subset $\mathscr{B}$ of $\mathscr{A}$ one of the conditions* (I), (II), (III), (IV), (V) *is satisfied (equivalently all the conditions* (I)–(V) *are satisfied) for $E = \operatorname{span}(e_a)_{a\in\mathscr{B}}$ and with (e_j) replaced by $(e_a)_{a\in\mathscr{B}}$.*

Proof. Let J be the family of all non empty finite subsets of $\mathscr{A}$. Observe that if for some equivalent norm on X there is a constant $K \geqslant 1$ such that with this constant for every $\mathscr{B}\in J$ one of the conditions (I), (II), (III), (IV), (V) is satisfied for every space $E_{\mathscr{B}} = \operatorname{span}(e_a)_{a\in\mathscr{B}}$ with (e_j) replaced by $(e_a)_{a\in\mathscr{B}}$, then the same condition is satisfied by any equivalent norm on X perhaps with a different constant K. Now renorm the space X so that in the new norm, say $\|\cdot\|_1$, the basis $(e_a)_{a\in\mathscr{A}}$ has the unconditional characteristic one. Then the family $\{E_{\mathscr{B}}\}_{\mathscr{B}\in J}$ is a local unconditional structure for X (cf. proof of Proposition 4.1). Thus, by Theorem 3.1, $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$ if and only if there exists K such that one of the conditions (I), (II), (III), (IV), (V) is satisfied for every space $E_{\mathscr{B}} = \operatorname{span}(e_a)_{a\in\mathscr{B}}$ with respect to the new norm $\|\cdot\|_1$ $(\mathscr{B}\in J)$.

Remark 1. It clearly follows that conditions (I)–(V) hold for a given unconditional basis iff they hold for every unconditional basis in the space.

Remark 2. The conditions (I)–(V) are stated in the "finite dimensional language". One can easily obtain many other conditions in order that a Banach space X with an unconditionial basis (e_j) have the property that $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$. We list some of them. Here by a *diagonal operator* from X (resp. X^*) into l_p $(1 \leqslant p \leqslant \infty)$ or c_0 we mean a map D such that $Dx = (\lambda_j e^*(x))$ for $x\in X$ (resp. $Dx^* = (\lambda_j x^*(e_j))$ for $x^*\in X^*$ where (λ_j) is a fixed scalar sequence depending only on D and (e^*) denote the sequence of coefficient functionals of the basis (e_j). Similarly diagonal operators from l_p or c_0 into X and X^* are defined.

(VII). *Any operator $T\in B(X, l_2)$ which can be factored $T = DU$ through l_∞ with D a diagonal operator can also be factored $T = VD_1$ through l_1 with D_1 a diagonal operator*

(VIII). *Every diagonal operator from c_0 into X is 2-absolutely summing*

(IX). *Every diagonal operator from X^* into l_1 is Hilbertian.*

Remark 3. Proposition 3.4 admits the following generalization

PROPOSITION 3.5. *Let $2 \leqslant p < +\infty$ and let $p^* = \dfrac{p}{p-1}$. Then under the notation and assumption of Proposition 3.4 for any $K \geqslant 1$ the following implications hold* $(\mathrm{I_p}) \Rightarrow (\mathrm{II_p}) \Rightarrow (\mathrm{III_p}) \Rightarrow (\mathrm{IV_p}) \Rightarrow (\mathrm{V_p}) \Rightarrow (\mathrm{VI_p})$, *where*

$(\mathrm{I_p})$ $\Pi_{p,2}(l_\infty^n, E) \underset{K}{=} B(l_\infty^n, E)$,

$(\mathrm{II_p})$ *$\pi_{p,2}(D) \leqslant K\|D\|$ for every operator $D\colon l_\infty^n \to E$ which is diagonal with respect to the basis (e_j),*

$(\mathrm{III_p})$ $\left(\sum_{i=1}^m \|x_i\|^p\right)^{1/p} \leqslant K \left\| \sum_{j=1}^n \left(\sum_{i=1}^m |e_j^*(x_i)|^2\right)^{\frac12} e_j \right\|$ *for any $x_1, x_2, \ldots, x_m$ in E $(m = 1, 2, \ldots)$,*

$(\mathrm{IV_p})$ $\left\| \sum_{j=1}^n \left(\sum_{i=1}^m |x_i^*(e_j)|^2\right)^{\frac12} e_j^* \right\| \leqslant K\left(\sum_{i=1}^m \|x_i^*\|^{p^*}\right)^{1/p^*}$ *for any $x_1^*, x_2^*, \ldots, x_m^*$ in E^* $(m = 1, 2, \ldots)$,*

$(\mathrm{V_p})$ *for any $m = 1, 2, \ldots$ and for any $x_1^*, x_2^*, \ldots, x_m^*$ in the unit ball of E^* and scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$ with $\sum_{i=1}^m |\lambda_i|^{p^*} = 1$ there exist $x^*\in E^*$ with $\|x^*\| \leqslant K$ and scalars $\lambda_i(j)$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ such that $\max_{1\leqslant j\leqslant n} \sum_{i=1}^m |\lambda_i(j)|^2 \leqslant 1$ and $\lambda_i x_i^*(e_j) = \lambda_i(j) x^*(e_j)$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$,*

$(\mathrm{VI_p})$ *$\Pi_{p,2}(l_\infty^s, E) \underset{KK_G}{=} B(l_\infty^s, E)$ for $s = 1, 2, \ldots$, where K_G is the Grothendieck constant.*

The proof of Proposition 3.5 is almost the same as the proof of Proposition 3.4. In a few places the Schwartz inequality should be replaced by the Hölder inequality with exponents p and p^*.

Clearly using Proposition 3.5 instead of Proposition 3.4 one can generalize Theorems 3.1 and 3.2 and the conditions (VII)–(IX) in Remark 2 above to obtain analogous characterizations of those Banach spaces X with an unconditional basis for which $\Pi_{p,2}(Y, X) = B(Y, X)$ for every $Y\in\mathscr{L}_\infty$ $(2 \leqslant p < +\infty)$.

THEOREM 3.3. *For every Banach space X with an unconditional basis $(e_a)_{a\in\mathscr{A}}$ the following conditions are equivalent*

(a) $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$,

(b) *If $\{X_a\}_{a\in\mathscr{A}}$ is a family of Banach spaces with the property that there exist constants K and C such that $\Pi_2(\mathscr{L}_\infty, X_a) \underset{K}{=} B(\mathscr{L}_\infty, X_a)$ for $a\in\mathscr{A}$ and each X has a local unconditional structure $\{E_\iota^{(a)}\}_{\iota\in J_a}$ with $\gamma(X_a, \{E_\iota^{(a)}\}_{\iota\in J_a}) \leqslant C$ for $a\in\mathscr{A}$, then*

$$\Pi_2(\mathscr{L}_\infty, [X_a]_{(e_a)}) = B(\mathscr{L}_\infty, [X_a]_{(e_a)}),$$

(c) *There exists a family* $\{X_a\}_{a\in\mathscr{A}}$ *such that*

$$\Pi_2(\mathscr{L}_\infty, [X_a]_{(e_a)}) = B(\mathscr{L}_\infty, [X_a]_{(e_a)}).$$

Proof. (a) $\Rightarrow$ (b). Let $\mathscr{T}$ be the set of all sequences $\tau = (\iota_a)_{a\in\mathscr{B}}$ where $\iota_a \in J_a$ for $a\in\mathscr{B}$ and B is any non-empty finite subset of $\mathscr{A}$. Given a $\tau = (\iota_a)_{a\in\mathscr{B}} \in \mathscr{T}$ we define E_τ to be the Banach space of all sequences $(x(a))_{a\in\mathscr{B}}$ such that $x(a)\in E_{\iota_a}^{(a)}$ for $a\in\mathscr{B}$ with norm given by

$$\|(x(a))_{a\in\mathscr{B}}\| = \sup_{|\varepsilon(a)|=1;\,a\in\mathscr{B}} \Big\|\sum_{a\in\mathscr{B}} \varepsilon(a)\|x(a)\|e_a\Big\|.$$

By Proposition 3.2, the family $\{E_\tau\}_{\tau\in\mathscr{T}}$ is a local unconditional structure for the product $[X_a]_{(e_a)}$. To complete the proof of this implication, by Theorem 3.1, it is enough to show that the family $\{E_\tau\}_{\tau\in\mathscr{T}}$ satisfies the condition (III) of Proposition 3.4 with some constant independent on τ. To this end fix a $\tau = (\iota_a)_{a\in\mathscr{B}}\in\mathscr{T}$ and a basis $(e_j^{(a)})_{1\leqslant j\leqslant n(a)}$ (where $n(a) = \dim E_{\iota_a}^{(a)}$) in $E_{\iota_a}^{(a)}$ with $\gamma((e_j^{(a)})) = 1$. For $1\leqslant j\leqslant n(a)$ and $a\in\mathscr{B}$ we define $f_{j,a}\in E_\tau$ by $f_{j,a}(a') = 0$ for $a'\neq a$ and $f_{j,a}(a) = e_j^{(a)}$. Then (cf. Proposition 3.1), $(f_{j,a})$ is an unconditional basis in E_τ with $\gamma((f_{j,a})) = 1$. Next fix $x_i = (x_i(a))_{a\in\mathscr{B}}\in E_\tau$ for $i = 1, 2, \ldots, m$. Let $x_i(a) = \sum_{j=1}^{n(a)} t_{ja}^{(i)} e_j^{(a)}$ for $a\in\mathscr{B}$. Since for all $a\in\mathscr{A}$, $\Pi_2(\mathscr{L}_\infty, X_a) \underset{K}{=} B(\mathscr{L}_\infty, X_a)$ and $\gamma(X_a, \{E_\iota^a\}_{\iota\in J_a})\leqslant C$ it follows from Proposition 3.4 and the definition of a local unconditional structure that there exists a constant K_1 depending only on the constants K and C such that

$$K_1\Big\|\sum_{j=1}^{n(a)}\Big(\sum_{i=1}^{m}|t_{ja}^{(i)}|^2\Big)^{\frac12} e_j^{(a)}\Big\| \geqslant \Big(\sum_{i=1}^{m}\|x_i(a)\|^2\Big)^{\frac12} \quad \text{for } a\in\mathscr{B}. \tag{3.5}$$

Thus

$$\Big(\sum_{i=1}^{m}\|x_i\|^2\Big)^{\frac12} = \Big(\sum_{i=1}^{m}\sup_{|\varepsilon(a)|=1}\Big\|\sum_{a\in\mathscr{B}}\varepsilon(a)\|x_i(a)\|e_a\Big\|^2\Big)^{\frac12}$$

$$\leqslant \gamma((e_a))\Big(\sum_{i=1}^{m}\Big\|\sum_{a\in\mathscr{B}}\|x_i(a)\|e_a\Big\|^2\Big)^{\frac12}$$

$$\leqslant \gamma((e_a))K_2\Big\|\sum_{a\in\mathscr{B}}\Big(\sum_{i=1}^{m}\|x_i(a)\|^2\Big)^{\frac12}e_a\Big\|,$$

because the assumption $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$ implies, by Theorem 3.2 the existence of K_2 such that

$$\Big(\sum_{i=1}^{m}\Big\|\sum_{a\in\mathscr{B}}s_a^i e_a\Big\|^2\Big)^{\frac12} \leqslant K_2\Big\|\sum_{a\in\mathscr{B}}\Big(\sum_{i=1}^{m}|s_a^i|^2\Big)^{\frac12}e_a\Big\|$$

for all scalars s_a^i, $i = 1, 2, \ldots, m$; $a\in\mathscr{B}$ and every non empty finite subset $\mathscr{B}$ of $\mathscr{A}$. Now using (3.0) and (3.5) we get

$$\Big(\sum_{i=1}^{m}\|x_i\|^2\Big)^{\frac12} \leqslant K_1K_2[\gamma((e_a))]^2\Big\|\sum_{a\in\mathscr{B}}\Big\|\sum_{j=1}^{n(a)}\Big(\sum_{i=1}^{m}|t_{ja}^{(i)}|^2\Big)^{\frac12}e_j^{(a)}\Big\|e_a\Big\|$$

$$\leqslant K_1K_2[\gamma((e_a))]^2\Big\|\sum_{a\in\mathscr{B}}\sum_{j=1}^{n(a)}\Big(\sum_{i=1}^{m}|t_{ja}^{(i)}|^2\Big)^{\frac12}f_{j,a}\Big\|.$$

Thus the basis $(f_{j,a})$ of the space E_τ satisfies the condition (III) with the constant $K_1K_2[\gamma((e_a))]^2$ independent on τ.

(b) $\Rightarrow$ (c). This implication is trivial.

(c) $\Rightarrow$ (a). Let R_a denote any one dimensional subspace of X_a. Clearly the product $[R_a]_{(e_a)}$ can be regarded as a subspace of $[X_a]_{(e_a)}$. Thus, by (c), $\Pi_2(\mathscr{L}_\infty, [R_a]_{(e_a)}) = B(\mathscr{L}_\infty, [R_a]_{(e_a)})$. To complete the proof let us observe that the product $[R_a]_{(e_a)}$ is isometrically isomorphic to X.

Next we pass to the case of p-products.

Corollary 3.1. *Let* $\{X_a\}_{a\in A}$ *be an infinite family of Banach spaces satisfying the hypothesis* (b) *of Theorem* 3.3. *Then* $\Pi_2(\mathscr{L}_\infty, [X_a]_{l_p^{\mathscr{A}}}) = B(\mathscr{L}_\infty, [X_a]_{l_p^{\mathscr{A}}})$ *if and only if* $1\leqslant p\leqslant 2$.

Proof. For $1\leqslant p\leqslant 2$ the desired conclusion follows from the inequality

$$\Big(\sum_i\Big(\sum_j|t_{ij}|^p\Big)^{2/p}\Big)^{1/2}\leqslant\Big(\sum_j\Big(\sum_i|t_{ij}|^2\Big)^{p/2}\Big)^{1/p}$$

for any scalars t_{ij} $(i, j = 1, 2, \ldots)$.

Now if $p > 2$ then for $t_{ij} = \delta_i^j$ $(i, j = 1, 2, \ldots)$ and for $n = 1, 2, \ldots$ we have

$$\Big(\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|t_{ij}|^p\Big)^{2/p}\Big)^{\frac12} = n^{\frac12}$$

while

$$\Big(\sum_{j=1}^{n}\Big(\sum_{i=1}^{n}|t_{ij}|^2\Big)^{p/2}\Big)^{1/p} = n^{1/p}.$$

Thus the condition (III) is not satisfied.

A useful partial generalization of Corollary 3.1 is the following

Theorem 3.4. *Let* $1\leqslant p\leqslant 2$. *Let* (Ω, Σ, μ) *be any measure space. Let* $L_p(X; \Omega, \Sigma, \mu)$ *denote the space of all* μ*-strongly measurable and Bochner integrable* X*-valued function* f *on* Ω *such that*

$$\|f\|_p = \Big(\int_\Omega\|f(\omega)\|^p\mu(d\omega)\Big)^{1/p} < +\infty.$$

Suppose that X has a local unconditional structure and that $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$.

Then $\Pi_2(\mathscr{L}_\infty, L_p(X; \Omega, \Sigma, \mu)) = B(\mathscr{L}_\infty, L_p(X; \Omega, \Sigma, \mu))$.

Proof. Let $\{E_\iota\}_{\iota\in J}$ be a local unconditional structure for X. Let $\mathscr{T}$ be the family of all finite sequences $\tau = (\iota_1, \iota_2, \ldots, \iota_{n(\tau)})$ such that $\iota_j \in J$ for $1 \leqslant j \leqslant n(\tau)$. For $\tau = (\iota_1, \iota_2, \ldots, \iota_{n(\tau)}) \in \mathscr{T}$ let $F_\tau = (E_{\iota_1} \times E_{\iota_2} \times \ldots \times E_{\iota_{n(\tau)}})_{l_p^{n(\tau)}}$. Using the standard technique of approximation by simple functions and an easy stability argument one can show that $\{F_\tau\}_{\tau\in\mathscr{T}}$ is a local unconditional structure for the space $L_p(X; \Omega, \Sigma, \mu)$. The desired conclusion follows from Theorem 3.3 and Theorem 3.1.

COROLLARY 3.2. *Let $(\Omega_i, \Sigma_i, \mu_i)$ be measure spaces and let $1 \leqslant p_i \leqslant 2$ for $i = 1, 2, \ldots, n$; $n = 1, 2, \ldots$ Let Y be the space of all scalar valued functions f on the product $\Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$ which are $\mu_1 \times \mu_2 \times \ldots \times \mu_n$-measurable and such that*

$$\|f\| = \Big\{\int\limits_{\Omega_n}\Big[\int\limits_{\Omega_{n-1}} \ldots \Big(\int\limits_{\Omega_2}\Big(\int\limits_{\Omega_1}|f(s_1, s_2, \ldots, s_n)|^{p_1}\mu(ds_1)\Big)^{p_2/p_1}\mu(ds_2)\Big)^{p_3/p_2}\ldots\Big]^{p_n/p_{n-1}}\mu(ds)\Big\}^{1/p_n} < +\infty.$$

Then $\Pi_2(\mathscr{L}_\infty, Y) = \mathscr{B}(\mathscr{L}_\infty, Y)$.

4. Banach spaces whose duals have subquadratic Gaussian averages. By a *probability space* we mean a measure space (Ω, Σ, μ) where μ is a non negative measure of total mass 1. Instead of μ-measurable function we shall often use the term "*random variable*".

THEOREM 4.1. *Let X be a Banach space. Suppose that there exist a probability space (Ω, Σ, μ), a sequence of functions (f_n) in $L_1(\Omega, \Sigma, \mu)$ and a constant $C > 0$ such that*

$$C^{-1}\Big(\sum_{i=1}^m |a_i|^2\Big)^{\frac12} \leqslant \int\limits_\Omega \Big|\sum_{i=1}^m a_i f_i(\omega)\Big|\,\mu(d\omega) \tag{4.1}$$

for any scalars $a_1, a_2, \ldots, a_m$ $(m = 1, 2, \ldots)$

$$\int\limits_\Omega \Big\|\sum_{i=1}^m x_i^* f_i(\omega)\Big\|\,\mu(d\omega) \leqslant C\Big(\sum_{i=1}^m \|x_i^*\|^2\Big)^{\frac12}. \tag{4.2}$$

for any $x_1^*, x_2^*, \ldots, x_m^*$ in X^* $(m = 1, 2, \ldots)$. Then $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$.

Proof. Fix $x_1, x_2, \ldots, x_n$ in X and $x_1^*, x_2^*, \ldots, x_m^*$ in X^* $(n, m = 1, 2, \ldots)$ and put

$$b = \sup_{|\lambda(j)|=1;\, j=1,2,\ldots,n} \Big\|\sum_{j=1}^n \lambda(j) x_j\Big\|.$$

Then for any $\omega \in \Omega$ choosing $\lambda_\omega(j)$ with $|\lambda_\omega(j)| = 1$ so that

$$\sum_{i=1}^m f_i(\omega) x_i^*(x_j)\lambda_\omega(j) = \Big|\Big(\sum_{i=1}^m f_i(\omega) x_i^*\Big)(x_j)\Big| \quad \text{for } j = 1, 2, \ldots, n$$

we get

$$\sum_{j=1}^n \Big|\Big(\sum_{i=1}^m f_i(\omega) x_i^*\Big)(x_j)\Big| = \Big(\sum_{i=1}^m f_i(\omega) x_i^*\Big)\Big(\sum_{j=1}^n \lambda_\omega(j) x_j\Big) \leqslant b\,\Big\|\sum_{i=1}^m f_i(\omega) x_i^*\Big\|.$$

Now, by (4.1) and (4.2), we obtain

$$\begin{aligned}\sum_{j=1}^n\Big(\sum_{i=1}^m |x_i^*(x_j)|^2\Big)^{\frac12} &\leqslant C\sum_{j=1}^n \int\limits_\Omega \Big|\Big(\sum_{i=1}^m f_i(\omega)x_i^*\Big)(x_j)\Big|\,\mu(d\omega)\\ &= C\int\limits_\Omega \sum_{j=1}^n\Big|\Big(\sum_{i=1}^m f_i(\omega)x_i^*\Big)(x_j)\Big|\,\mu(d\omega)\\ &\leqslant Cb\int\limits_\Omega \Big\|\sum_{i=1}^m f_i(\omega)x_i^*\Big\|\,\mu(d\omega) \leqslant C^2 b\Big(\sum_{i=1}^m \|x_i^*\|^2\Big)^{\frac12}.\end{aligned}$$

Thus X satisfies the condition (v) of Proposition 2.1 and this completes the proof.

It is obvious from Definition 2.1 that if a Banach space X has the property that $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$ then any subspace of X has the same property. In general this property is not preserved for quotient spaces of X. For example let $X = l_1$. Then $\Pi_2(\mathscr{L}_\infty, l_1) = B(\mathscr{L}_\infty, l_1)$ (Grothendieck [6] cf. also [16]). This is clearly not true for quotients of l_1 because every separable Banach space is a quotient of l_1 (in particular c_0) and obviously

$$\Pi_2(\mathscr{L}_\infty, c_0) \neq B(\mathscr{L}_\infty, c_0).$$

However we have:

COROLLARY 4.1. *If a Banach space X satisfies the assumptions of Theorem 4.1, then every quotient space Y of X has the same property (with the same sequence (f_n) and constant C). Consequently $\Pi_2(\mathscr{L}_\infty, Y) = B(\mathscr{L}_\infty, Y)$.*

Proof. If Y is a quotient of X, then Y^* is isometrically isomorphic to a subspace of X^*. Hence the inequality (3.2) is satisfied for any $y_1^*, y_2^*, \ldots, y_m^*$ in Y^* $(m = 1, 2, \ldots)$.

Our next result gives some more information on the class of Banach spaces for which the assertion of Corollary 3.1 is true. Recall that a Banach space X is said to be uniformly l_1^n if for some $C > 1$ (equivalently for any $C > 1$) there exists for $n = 1, 2, \ldots$ an operator $T_{n,c}\colon l_1^n \to X$ such that

$$\|\xi\| \leqslant \|T_{n,c}\xi\| \leqslant C\|\xi\| \quad \text{for} \quad \xi \in l_1^n$$

(cf. [5], [7]).

PROPOSITION 4.1. *If X is an infinite diemensional Banach space such that $\Pi_2(\mathscr{L}_\infty, Y) = B(\mathscr{L}_\infty, Y)$ for every quotient space Y of X, then X is not uniformly l_1^n.*

Proof. If X_1 is a subspace of a uniformly l_1^n space X and if $\dim X/X_1 < +\infty$, then X_1 is uniformly l_1^n. Hence using the technique of [8] and [10] one can construct a sequence (Z_n) of subspaces of X such that

(4.3) there exists an operator $S_n\colon Z_n \to l_1^{2^n}$ such that

$$2^{-1}\|x\| \leqslant \|S_n x\| \leqslant \|x\| \quad \text{for } x \in Z_n,$$

(4.4) if $Z = \operatorname{closure}\big(\bigcup_{n=1}^{\infty} Z_n\big)$, then (Z_n) is a Schauder decomposition of Z. Precisely there are projections $P_n\colon Z\to Z_n$ such that $P_nP_m = P_mP_n = 0$ for $n\neq m$ and $\|P_n\|\leqslant 3$ for $n = 1, 2, \ldots$

Next observe that l_∞^n is a quotient space of $l_1^{2^n}$ because the unit ball of l_∞^n has exactly 2^n extreme points. Let $h_n\colon l_1^{2^n}\to l_\infty^n$ denote a quotient map and let $Q_n = h_nS_n\colon Z_n\to l_\infty^n$. Then

$$\|Q_n\|\leqslant 1 \text{ and } Q_n\{x\in Z_n\colon \|x\|\leqslant 1\}\supset\{\lambda\in l_\infty^n\colon \|\lambda\|\leqslant 2^{-1}\}. \tag{4.5}$$

Let $E_n = \ker Q_n$ $(n = 1, 2, \ldots)$ and let $E = \operatorname{closure}\big(\bigcup_{n=1}^{\infty} E_n\big)$. Let h denote the quotient map of Z onto Z/E. Define the map $A_n\colon l_\infty^n\to Z/E$ by

$$A_n\lambda = hx \text{ for any } x\in Z \text{ such that } Q_nP_nx = \lambda \quad (\lambda\in l_\infty^n).$$

One can easily check that A is a well defined operator and, by (4.5),

$$3^{-1}\|\lambda\|\leqslant \|P_n\|^{-1}\|\lambda\|\leqslant\|A_n\lambda\|\leqslant 2\|\lambda\| \quad \text{for } \lambda\in l_\infty^n. \tag{4.6}$$

Let $1_{l_\infty^n}$ denote the identity map on l_∞^n. Then $\pi_2(1_{l_\infty^n}) = \sqrt{n}$. It follows from (4.6) that $\|A_n\|\leqslant 2$ while $\pi_2(A_n)\geqslant 3^{-1}\pi_2(1_{l_\infty^n}) = 3^{-1}\sqrt{n}$. Hence $\Pi_2(\mathscr{L}_\infty, Z/E)\neq B(\mathscr{L}_\infty, Z/E)$ because the condition ($\mathrm{i_b}$) of Proposition 2.1 is not satisfied. Since the quotient Z/E of a subspace $Z\subset X$ is isometrically isomorphic to a subspace of the quotient $Y = X/E$ we infer that $\Pi_2(\mathscr{L}_\infty, Y)\neq B(\mathscr{L}_\infty, Y)$ and this completes the proof of the proposition.

Next we shall discuss Banach spaces satisfying the condition (4.2) for (f_i) being a sequence of independent Gaussian random variables (cf. [27]). In the sequel we shall denote by (γ_i) a sequence of independent Gaussian random variables on a probability space (Ω, Σ, μ) each of which is distributed by the rule

$$\mu\{\omega\in\Omega\colon \gamma(\omega) < t\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-s^2/2}\,ds.$$

Let Z be a Banach space and let $1\leqslant p<\infty$. By the pth Gaussian average of the vectors $z_1, z_2, \ldots, z_n$ in Z $(n = 1, 2, \ldots)$ we mean the pth root of the expectation

$$E\Big(\Big\|\sum_{i=1}^{n}\gamma_iz_i\Big\|^p\Big) = \int_\Omega\Big\|\sum_{i=1}^{n}\gamma_i(\omega)z_i\Big\|^p\mu(d\omega)$$
$$= (2\pi)^{-\frac{n}{2}}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty}\Big\|\sum_{i=1}^{n}s_iz_i\Big\|^p e^{-\frac{s_1^2+\ldots+s_n^2}{2}}\,ds_1\ldots ds_n.$$

Recall that, by a result of Landau and Shepp [14] and Fernique [4] there is a universal constant g_p such that for any Banach space Z

$$E\Big(\Big\|\sum_{i=1}^{n}\gamma_iz_i\Big\|\Big)\leqslant\Big(E\Big(\Big\|\sum_{i=1}^{n}\gamma_iz_i\Big\|^p\Big)\Big)^{\frac{1}{p}}\leqslant g_pE\Big(\Big\|\sum_{i=1}^{u}\gamma_iz_i\Big\|\Big). \tag{4.7}$$

for any $z_1, z_2, \ldots, z_n$ in Z $(n = 1, 2, \ldots)$.

DEFINITION 4.1. A Banach space Z is said to have a *subquadratic Gaussian average* if there exists a constant G_Z such that

$$E\Big(\Big\|\sum_{i=1}^{n}\gamma_iz_i\Big\|\Big)\leqslant G_Z\Big(\sum_{i=1}^{n}\|z_i\|^2\Big)^{\frac12}$$

for any $z_1, z_2, \ldots, z_n$ in Z $(n = 1, 2, \ldots)$.

The following two facts are well known and follow directly from the properties of the Gaussian variables (γ_i).

COROLLARY 4.2. *Any real or complex Hilbert space H has a subquadratic Gaussian average. Moreover $G_H\leqslant 1$.*

COROLLARY 4.3.

$$E\Big(\Big|\sum_{i=1}^{n}a_i\gamma_i\Big|\Big) = \frac{\sqrt2}{\sqrt\pi}\Big(\sum_{i=1}^{n}a_i^2\Big)^{\frac12}$$

for $a_1, a_2, \ldots, a_n$ real $(n = 1, 2, \ldots)$;

$$\frac{1}{\sqrt\pi}\sqrt{\sum_{i=1}^{n}|a_i|^2}\leqslant E\Big(\Big|\sum_{i=1}^{n}a_i\gamma_i\Big|\Big)\leqslant\frac{2}{\sqrt\pi}\sqrt{\sum_{i=1}^{n}|a_i|^2}$$

for any complex $a_1, a_2, \ldots, a_n$ $(n = 1, 2, \ldots)$.

Combining Theorem 4.1 with Corollary 4.2 we get immediately

COROLLARY 4.4. *Let X be a Banach space such that X^* has a subquadratic Gaussian average. Then $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$.*

Our next result concerns products of Banach spaces with subquadratic Gaussian averages.

PROPOSITION 4.2. *Let $2\leqslant p<\infty$ and let $\{Z_a\}_{a\in\mathscr{A}}$ be a family of Banach spaces. Suppose that each Z_a has a subquadratic Gaussian average and assume that $\sup_{a\in\mathscr{A}} G_{Z_a} = G < +\infty$.*

Then the product $X_p = [Z_a]_{l_p^{\mathscr{A}}}$ has a subquadratic Gaussian average and $G_{X_p} \leqslant g_p G$.

Proof. Fix $z^1, z^2, \ldots, z^n$ in X_p. Let $z^i = (z_a^i)_{a\in\mathscr{A}}$ for $i = 1, 2, \ldots, n$. Using (4.7) we get

$$E\Big(\Big\|\sum_{i=1}^n \gamma_i z^i\Big\|_p\Big) \leqslant \Big(E\Big(\Big\|\sum_{i=1}^n \gamma_i z^i\Big\|_p^p\Big)\Big)^{1/p} = \Big(E\Big(\sum_{a\in\mathscr{A}}\Big\|\sum_{i=1}^n \gamma_i z_a^i\Big\|^p\Big)\Big)^{1/p}$$

$$= \Big(\sum_{a\in\mathscr{A}} E\Big(\Big\|\sum_{i=1}^n \gamma_i z_a^i\Big\|^p\Big)\Big)^{1/p} \leqslant \Big(\sum_{a\in\mathscr{A}} g_p^p\Big(E\Big(\Big\|\sum_{i=1}^n \gamma_i z_a^i\Big\|\Big)\Big)^p\Big)^{1/p}$$

$$\leqslant g_p\Big(\sum_{a\in\mathscr{A}} G_{Z_a}^p\Big(\sum_{i=1}^n \|z_a^i\|^2\Big)^{p/2}\Big)^{1/p} \leqslant g_p G\Big(\sum_{a\in\mathscr{A}}\Big(\sum_{i=1}^n \|z_a^i\|^2\Big)^{p/2}\Big)^{1/p}.$$

Putting $r = p/2 \geqslant 1$ and $a_a^i = \|z_a^i\|^2$ for $a \in \mathscr{A}$, $i = 1, 2, \ldots, n$, by the triangle inequality in $l_r^{\mathscr{A}}$, we get

$$\Big(\sum_{a\in\mathscr{A}}\Big(\sum_{i=1}^n \|z_a^i\|^2\Big)^{p/2}\Big)^{2/p} = \Big(\sum_{a\in\mathscr{A}}\Big(\sum_{i=1}^n a_a^i\Big)^r\Big)^{1/r} \leqslant \sum_{i=1}^n\Big(\sum_{a\in\mathscr{A}}(a_a^i)^r\Big)^{1/r}$$

$$= \sum_{i=1}^n\Big(\sum_{a\in\mathscr{A}}\|z_a^i\|^p\Big)^{2/p} = \sum_{i=1}^n \|z^i\|_p^2.$$

Hence

$$E\Big(\Big\|\sum_{i=1}^n \gamma_i z_i\Big\|_p\Big) \leqslant g_p G\Big(\sum_{i=1}^n \|z^i\|_p^2\Big)^{\frac{1}{2}}.$$

Hence $G_{X_p} \leqslant g_p G$ and this completes the proof.

COROLLARY 4.5. *Let $\{X_a\}_{a\in\mathscr{A}}$ be a family of Banach spaces such that X_a^* has a subquadratic Gaussian average. If $\sup_{a\in\mathscr{A}} G_{X_a^*} = G < +\infty$, then $\Pi_2(\mathscr{L}_\infty, [X_a]_{l_q^{\mathscr{A}}}) = B(\mathscr{L}_\infty, [X_a]_{l_q^{\mathscr{A}}})$ for any q with $1 < q \leqslant 2$.*

Proof. Use the fact that $([X_a]_{l_q^{\mathscr{A}}})^* = [X_a^*]_{l_{q^*}^{\mathscr{A}}}$ where $q^* = q/(q-1)$ and apply Proposition 4.2.

Recall that for any measure space (Ω, Σ, μ) and any Banach space X and p with $1 \leqslant p < +\infty$ we denote by $L_p(X; \Omega, \Sigma, \mu)$ the space of all μ-measurable and Bochner integrable functions $f\colon \Omega \to X$ such that $\|f\|_p = \big(\int_\Omega \|f(\omega)\|^p \mu(d\omega)\big)^{1/p} < +\infty$. Since every finite dimensional subspace $F \subset L_p(X; \Omega, \Sigma, \mu)$ is for every $\varepsilon > 0$ $(1+\varepsilon)$-isomorphic to a subspace of $l_p(X)$, Proposition 4.2 immediately implies

COROLLARY 4.6. *If Z has a subquadratic Gaussian average, then $L_p(Z; \Omega, \Sigma, \mu)$ has a subquadratic Gaussian average for $2 \leqslant p < +\infty$.*

COROLLARY 4.7. *If X is a Banach space such that X^* has a subquadratic Gaussian average, then $\Pi_2(\mathscr{L}_\infty, Y) = B(\mathscr{L}_\infty, Y)$ for every quotient space Y of the space $L_q(X, \Omega, \Sigma, \mu)$ where (Ω, Σ, μ) is any measure space and $1 < q \leqslant 2$.*

Combining Corollary 4.2 (applied to the field of scalars) with Proposition 4.2 and Corollary 4.6 we infer that for any measure space (Ω, Σ, μ) and for any p with $2 \leqslant p < \infty$ the space $L_p(\Omega, \Sigma, \mu)$ has a subquadratic Gaussian average. Thus applying Corollaries 4.6 and 4.7 by an easy induction we get

COROLLARY 4.8. *Let $(\Omega_i, \Sigma_i, \mu_i)$ be measure spaces and let $1 < p_i \leqslant 2$ $(i = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$.*

Then a) *the dual space Z^* of the space Z of all scalar valued functions f on the product $\Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$ which are $(\mu_1 \times \mu_2 \times \ldots \times \mu_n)$-measurable and such that*

$$\|f\| = \Big\{\int_{\Omega_n}\Big[\int_{\Omega_{n-1}}\ldots\Big(\int_{\Omega_2}\Big(\int_{\Omega_1}|f(s_1, s_2, \ldots, s_n)|^{p_1}\mu(ds_1)\Big)^{\frac{p_2}{p_1}}\mu(ds_2)\Big)^{\frac{p_3}{p_2}}\ldots\Big]^{\frac{p_n}{p_{n-1}}} \times$$

$$\times\, \mu(ds_n)\Big\}^{\frac{1}{p_n}} < +\infty$$

has a subquadratic Gaussian average,

b) *If Y is a subspace of a quotient of Z, then $\Pi_2(\mathscr{L}_\infty, Y) = B(\mathscr{L}_\infty, Y)$.*

Next we show that, in a certain sense condition (4.2) has its weakest form when (f_i) is chosen to be a sequence of independent Gaussian variables. We use an argument essentially due to Kwapień [13].

PROPOSITION 4.3. *Let (f_i) be a sequence of real valued independent random variable on a probability space (Ω, Σ, μ). Assume that the Lindeberg conditions for the Central Limit Theorem* (cf. [27], Chapt. VIII), *are satisfied*, i.e.

(4.8)
$$f_i \in L_2(\Omega, \Sigma, \mu) \quad \textit{and} \quad \|f_i\|_2^2 = \int_\Omega |f_i(\omega)|^2 \mu(d\omega) = 1 \ (i = 1, 2, \ldots),$$

(4.9)
$$\lim_{n=\infty} n^{-1}\sum_{i=1}^n \int_{\{\omega\in\Omega: |f_i(\omega)| > \varepsilon\sqrt{n}\}} |f_i(\omega)|^2 \mu(d\omega) = 0 \text{ for every } \varepsilon > 0.$$

Let Z be a Banach space such that

(4.10)
$$\textit{there exists } C > 0 \textit{ such that } \int_\Omega \Big\|\sum_{i=1}^n f_i(\omega) z_i\Big\| \mu(d\omega) \leqslant C\sqrt{\sum_{i=1}^n \|z_i\|^2}$$
$$\textit{for } z_1, z_2, \ldots, z_n \text{ in } Z\ (n = 1, 2, \ldots).$$

Then Z has a subquadratic Gaussian average.

Proof. Fix a positive integer n and $z_1, z_2, \ldots, z_n$ in Z. By the Central Limit Theorem (cf. [27], Chapt. VIII) the joint distribution of the variables $f_{i,m} = m^{-\frac{1}{2}}\left(\sum_{j=1}^{m} f_{m(i-1)+j}\right)$, where $i = 1, 2, \ldots, n$ tends, as $m \to \infty$, to the joint distribution of the independent Gaussian variables $\gamma_1, \gamma_2, \ldots \ldots, \gamma_n$ i.e.

$$(4.11)\quad \lim_{m=\infty} \int_{\Omega} \varphi\big(f_{1,m}(\omega), f_{2,m}(\omega), \ldots f_{n,m}(\omega)\big)\mu(d\omega)$$
$$= \frac{1}{(\sqrt{2\pi})^n} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \varphi(s_1, s_2, \ldots, s_n) e^{-\frac{1}{2}(s_1^2+s_2^2+\ldots+s_n^2)} ds_1 ds_2 \ldots ds_n$$

for any bounded continuous function $\varphi\colon R_1^n \to R$.

Consider the Banach space B of all continuous functions $\varphi\colon R^n \to R$ such that $\lim\limits_{\sum_{i=1}^n |s_i| \to \infty} \dfrac{\varphi(s_1, s_2, \ldots, s_n)}{\sum_{i=1}^n |s_i|^2} = 0$ with the norm

$$\|\varphi\|_B = \max\left(\sup_{\sum_{i=1}^n |s_i| \leqslant 1} |\varphi(s_1, s_2, \ldots, s_n)|,\ \sup_{\sum_{i=1}^n |s_i| \geqslant 1} \frac{\varphi(s_1, s_2, \ldots s_n)}{\sum_{i=1}^n |s_i|^2}\right).$$

Let us set $F_m(\varphi) = \int_{\Omega} \varphi\big(f_{1,m}(\omega), f_{2,m}(\omega), \ldots, f_{n,m}(\omega)\big)\mu(d\omega)$. One can easily check that

$$|F_m(\varphi)| \leqslant \|\varphi\|_B \Big(1 + \sum_{i=1}^n \int_{\Omega} f_{i,m}^2(\omega)\mu(d\omega)\Big) \leqslant (n+1)\|\varphi\|_B$$

for $\varphi \in B$ and for $m = 1, 2, \ldots$ Hence

$$\lim_{m=\infty} F_m(\varphi) = \frac{1}{(\sqrt{2\pi})^n} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \varphi(s_1, s_2, \ldots, s_n) e^{-\frac{1}{2}(s_1^2+s_2^2+\ldots+s_n^2)} ds_1 \ldots ds_n$$

for any $\varphi \in B$ because, by (4.11), the limit exists on a dense subset of B. In particular we have

$$\lim_{m=\infty} \int_{\Omega} \Big\| \sum_{i=1}^n f_{i,m}(\omega) z_i \Big\| \mu(d\omega) = E\Big(\Big\| \sum_{i=1}^n \gamma_i z_i \Big\|\Big),$$

because the function φ defined by $\varphi(s_1 \ldots s_n) = \big\| \sum_{i=1}^n s_i z_i \big\|$ belongs to B. On the other hand, by (4.10), we have

$$\int_{\Omega} \Big\| \sum_{i=1}^n f_{i,m}(\omega) z_i \Big\| \mu(d\omega) = \int_{\Omega} \Big\| \sum_{i=1}^n \sum_{j=1}^m f_{m(i-1)+j}(\omega) \frac{z_i}{\sqrt{m}} \Big\| \mu(d\omega)$$
$$\leqslant C \Big(\sum_{i=1}^n \sum_{j=1}^m \frac{\|z_i\|^2}{m} \Big)^{\frac{1}{2}} = C \Big(\sum_{i=1}^n \|z_i\|^2 \Big)^{\frac{1}{2}} \quad \text{for } m = 1, 2, \ldots$$

Thus

$$E\Big(\Big\| \sum_{i=1}^n \gamma_i z_i \Big\|\Big) \leqslant C \Big(\sum_{i=1}^n \|z_i\|^2 \Big)^{\frac{1}{2}}$$

and this completes the proof.

COROLLARY 4.9. *Let (r_n) be the Rademacher orthogonal system i.e. $r_n(t) = \operatorname{sign} \sin 2^n \pi t$ for $0 \leqslant t \leqslant 1$; $n = 1, 2, \ldots$ and let Z be a Banach space such that for some $C > 0$*

$$\int_0^1 \Big\| \sum_{i=1}^n r_i(t) z_i \Big\| dt \leqslant C \Big(\sum_{i=1}^n \|z_i\|^2 \Big)^{\frac{1}{2}}$$

for $z_1, z_2, \ldots, z_n$ in Z; $n = 1, 2, \ldots$ Then Z has a subquadratic Gaussian average.

Remark 1. Let us say that the Banach space Z has a subquadratic Rademacher average if it satisfies the hypotheses of Corollary 4.9. It follows from Theorem 4, p. 12 of [29] that Z has this property if and only if for any sequence (z_i) of vectors of Z, if $\Sigma\|z_i\|^2 < \infty$ then $\Sigma r_i(t) z_i$ converges almost everywhere. In the language of probability theory, the Rademacher functions are simply a concrete representation of a sequence of $\{1, -1\}$-valued symmetric independent random variables defined on some probability space. Thus to say that $\Sigma r_i(t) z_i$ converges almost everywhere is to say that $\Sigma \pm z_i$ converges for almost all choices of the signs ± 1. We also note that it follows from the above reference that the analogy of (4.7) holds for series $\Sigma r_i(t) z_i$ in place of those of the form $\Sigma \gamma_i z_i$. Of course, by Corollary 4.9, if a Banach space has a subquadratic Rademacher average, it has a subquadratic Gaussian average. It follows from known inequalities (cf. [20]) that for $2 \leqslant p < \infty$, that L^p has a subquadratic Rademacher average.

Remark 2. Corollary 4.9 remains true if we replace the Rademacher functions by any lacunary sequence $(\sin n_i 2\pi t)$ or $(\cos n_i 2\pi t)$, where $n_{i+1}/n_i \geqslant q > 1$ for $i = 1, 2, \ldots$ The proof is analogous to the proof of Proposition 4.4. with the exeption that the Central Limit Theorem is replaced by [28], Chap. XVI, Theorem 5.5.

5. Remarks examples and unsolved problems. We begin with examples disproving the conjecture that every Banach space X for which $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$ is isomorphic to a subspace of some L_1 space.

EXAMPLE 5.1. Let $E = l_2(l_1)$. Then

(a) $\Pi_2(\mathscr{L}_\infty, E) = B(\mathscr{L}_\infty, E)$,

(b) E is not isomorphic to any subspace of any $\mathscr{L}_1$-space.

Proof (a). It follows immediately from Theorem 4.4.

(b). Since E is separable, it is enough to show that E is not isomorphic to any subspace of $L_1 = L_1([0;1], \Sigma, \lambda)$, where λ is the Lebesgue measure and Σ the field of all Lebesgue measurable subsets of $[0;1]$ (cf. [17]). Following [9] for any $\delta > 0$ let

$$Z_\delta = \{f \in L_1 \colon \lambda\{t \in [0;1] \colon |f(t)| \leqslant \delta \|f\|\} \geqslant 1-\delta\}.$$

Recall (cf. [9]) that

(5.1) If G is a subspace of L_1 isomorphic to l_1, then there is a sequence (g_n) in G such that $\|g_n\| = 1$ and $g_n \notin Z_\delta$ for $\delta \geqslant 1/n$ $(n = 1, 2, \ldots)$.

(5.2) If (h_n) is a sequence in L_1 such that $\|h_n\| = 1$ for $n = 1, 2, \ldots$ and for every $\delta > 0$ there exists an index $n(\delta)$ such that $h_{n(\delta)} \notin Z$, then there exists an increasing sequence of indices (n_k) such that the sequence (h_{n_k}) is equivalent to the unit vector basis of l_1.

Assume to the contrary that there exists an isomorphism $T \colon E \xrightarrow[\text{into}]{} L_1$ and let $F = T(E)$. Let us set

$$F_m = T(E_m), \quad \text{where } E_m = \{e = (e(j)) \in E \colon e(j) = 0 \text{ for } j \neq m\} \quad (m = 1, 2, \ldots).$$

Clearly each F_m is isomorphic to l_1. Thus, by (5.1), there exist $f_{n,m} \in F_m$ such that $\|f_{n,m}\| = 1$ and $f_{n,m} \notin Z_\delta$ for $\delta \geqslant 1/n$ $(n = 1, 2, \ldots)$. Let us set $h_n = f_{n,n}$ for $n = 1, 2, \ldots$ Clearly the sequence (h_n) satisfies the assumption of (5.2). Thus there exists an increasing sequence (n_k) of the indices such that (h_{n_k}) is equivalent to the unit vector basis of l_1. Thus the sequence $(T^{-1}h_{n_k})$ has the same property. Clearly $T^{-1}h_{n_k} \in E_{n_k}$ and $\|T\|^{-1} \leqslant \|T^{-1}h_{n_k}\| \leqslant \|T^{-1}\|$ for $k = 1, 2, \ldots$ Hence it follows from the definition of E_n and the definition of the product in the sense of l_2 that the sequence $(T^{-1}h_{n_k})$ is equivalent to the unit vector basis of l_2, a contradiction.

EXAMPLE 5.2. *Let* $V = (l_1^1 \times l_1^2 \times \ldots \times l_1^n \times \ldots)_{l_2}$. *Then* (a′) $\Pi_2(\mathscr{L}_\infty, V) = B(\mathscr{L}_\infty, V)$, (b′) V *is a reflexive space which is not isomorphic to a subspace of* L_1.

Proof. Since V is isometric to a subspace of $E = (l_1 \times l_1 \times \ldots)_{l_2}$, (a′) follows from (a). Clearly V is reflexive. Finally for any $\varepsilon > 0$ and any finite dimensional subspace E_0 of E there exists an operator $T \colon E_0 \to V$ such that $\|e\| \leqslant \|Te\| \leqslant (1+\varepsilon)\|e\|$ for $e \in E_0$. Thus, by [16], if V were isomorphic to a subspace of L_1, then E would have the same property.

Remark. Part (b′) of Example 5.2 (and from (b′) part (b) of Example 5.1) follows immediately from a result of [26] that a reflexive space which is uniformly l_1^n is not isomorphic to any subspace of L_1. The argument given in the present paper was discovered earlier. It is simpler but less general

DEFINITION 5.1. A Banach space X is said to have the *Orlicz property* if the identity operator on X is (2,1)-absolutely summing.

The following fact is well known and easy to proof.

PROPOSITION 5.1. *For every Banach space X the following conditions are equivalent*

(α) *X has the Orlicz property,*

(β) *For every Banach space Y every operator $T \colon Y \to X$ is (2,1)-absolutely summing,*

(γ) *There exists an infinite dimensional $\mathscr{L}_\infty$-space, say Y, such that every operator $T \colon Y \to X$ is (2,1)-absolutely summing*

(δ) $\sum_{m=1}^{\infty} \|x_m\|^2 < +\infty$ *whenever* $\sum_{m=1}^{\infty} x_m$ *is an unconditionally convergent series in X.*

Remark. Orlicz [22] discovered that for $1 \leqslant p \leqslant 2$ the space L_p satisfies (δ). This justifies the terminology "the Orlicz property".

The Orlicz property is weaker than the property that $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$. We have

PROPOSITION 5.2. *If X is a Banach space for which $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$, then X has the Orlicz property.*

Proof. If a series $\sum_{j=1}^{\infty} x_j$ is unconditionally convergent, then there exists a bounded linear operator $T \colon c_0 \to X$ such that $T\delta_j = x_j$ $(j = 1, 2, \ldots)$ where δ_j denotes the jth unit vector in c_0. Since $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$ there is a constant $K \geqslant 1$ independent of T such that $\pi_2(T) \leqslant K\|T\|$. We have

$$\sum_{j=1}^{\infty} \|x_j\|^2 = \sum_{j=1}^{\infty} \|T\delta_j\|^2 \leqslant [\pi_2(T)]^2 \sup_{\|\xi^*\| \leqslant 1} \sum_{j=1}^{\infty} |\xi^*(\delta_j)|^2$$
$$\leqslant K^2 \|T\|^2 \sup_{\|\xi^*\| \leqslant 1} \Big(\sum_{j=1}^{\infty} |\xi^*(\delta_j)|\Big)^2 = K^2 \|T\|^2 < +\infty.$$

Hence, by Proposition 5.1, X has the Orlicz property.

PROBLEM 5.1. *Let X be a Banach space with the Orlicz property. Is it true that $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$?*

We do not know the answer to Problem 5.1 even in the case where X has an unconditional basis. This special case is closely related to the concept of block Besselian basis. Recall that a normalized basis (e_n) in a Banach space X is said to be Besselian (Hilbertian) if the convergence of a series $\sum_{n=1}^{\infty} t_n e_n$ implies $\sum_{n=1}^{\infty} |t_n|^2 < +\infty$ (if $\sum_{n=1}^{\infty} |t_n|^2 < +\infty$ implies that the series $\sum_{n=1}^{\infty} t_n e_n$ converges).

DEFINITION 5.2. A basis (e_n) in a Banach space X is said to be *block Besselian* (*block Hilbertian*) if there exists a constant $K>0$ such that

$$\sup\Big(\sum_{k=0}^{\infty}\Big\|\sum_{j=p_k+1}^{p_{k+1}} e_j^*(x)e_j\Big\|^2\Big)^{\frac12}\leqslant K\|x\| \quad \text{for } x\in X$$

(resp. $\inf\Big(\sum_{k=0}^{\infty}\Big\|\sum_{j=p_k+1}^{p_{k+1}} e_j^*(x)e_j\Big\|^2\Big)^{\frac12}\geqslant K^{-1}\|x\|$ for $x\in X$). The supremum (resp. the infimum) is taken over all increasing sequences (p_k) of the indices with $p_0=0$. Here (e_j^*) denotes the sequence of the coordinate functionals of the basis (e_j).

The name block Besselian (resp. block Hilbertian) is justyfying by the fact that a basis (e_n) is block Besslian (resp. block Hilbertian) if and only if every normalized block basic sequence with respect to the basis (e_n) is Besselian (resp. Hilbertian). The concept of block Besselian and block Hilbertian bases are in full duality. We have

PROPOSITION 5.3. *Let (e_n) be a basis for a Banach space X. Let (e_n^*) denote the sequence of coordinate functionals of the basis. Then (e_n) is block Besselian (resp. block Hilbertian) basis for X if and only if (e_n^*) is a block Hilbertian (resp. block Besselian) basis for a subspace of X^*.*

We omit a routine proof of this result.

There is a simple relation between the Orlicz property and the concept of block Besselian basis

PROPOSITION 5.4. *Let X be a Banach space with the Orlicz property. Then every unconditional basis for X is block Besselian*

Proof. Obvious.

PROBLEM 5.2. *Let X be a Banach space with a block Besselian unconditional basis. Does X have the Orlicz property? In particular is it true that if there exists one unconditional block Besselian basis for X, then every unconditional basis for X is block Besselian?*

Our next examples show that the concept of block Besselian (Hilbertian) basis is essentially stronger than the concept of Besselian (Hilbertian) basis

EXAMPLE 5.3. *Let $\infty>p>2>q>1$ and let $a=q^{-1}-p^{-1}$. Let $X_{p,q}$ be the space of all scalar-valued sequences $x=(x(n))$ such that*

$$\|x\|=\max\Big[\Big(\sum_{n=1}^{\infty}|x(n)|^p\Big)^{1/p},\Big(\sum_{n=1}^{\infty}|x(n)n^{-a}|^q\Big)^{1/q}\Big]<+\infty.$$

Then (α) *the unit vectors form a normalized unconditional Hilbertian basis for $X_{p,q}$*

(β) *$X_{p,q}$ contains a subspace isomorphic to l_q,*

(γ) *$X_{p,q}$ has no block Hilbertian basis.*

Proof (α) Clearly the unit vectors form a normalized unconditional basis for $X_{p,q}$. Let $x=(x(n))\in l_2$. Then $\big(\sum_{n=1}^{\infty}|x(n)|^p\big)^{1/p}\leqslant\big(\sum_{n=1}^{\infty}|x(n)|^2\big)^{\frac12}<+\infty$ because $p>2$. Let $r=2/(2-q)$; $r^*=\frac{2}{q}$. By the Hölder inequality

$$\Big(\sum_{n=1}^{\infty}|x(n)n^{-a}|^q\Big)^{1/q}\leqslant\Big(\sum_{n=1}^{\infty}|x(n)|^2\Big)^{\frac12}\Big(\sum_{n=1}^{\infty}n^{-aqr}\Big)^{1/rq}<+\infty$$

because $aqr=(q^{-1}-p^{-1})(q^{-1}-2^{-1})^{-1}>1$. Thus $x\in X_{p,q}$ and this means that the unit vector basis is Hilbertian.

(β) For $k=1,2,\ldots$ we define $y_k\in X_{p,q}$ by

$$y_k(n)=\begin{cases}2^{-\frac{(k-1)}{p}} & \text{for } 2^{k-1}\leqslant n<2^k,\\ 0 & \text{otherwise.}\end{cases}$$

Then for every finite sequence of scalars $(c_k)_{1\leqslant k\leqslant m}$ we have

$$\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{m}c_k y_k(n)n^{-a}\Big|^q=\sum_{k=1}^{m}|c_k|^q\sum_{n=2^{k-1}}^{2^k-1}2^{-\frac{(k-1)q}{p}}n^{-aq}.$$

Since $\dfrac{q}{p}+aq=1$, we get

$$1\geqslant\sum_{n=2^{k-1}}^{2^k-1}2^{-\frac{(k-1)q}{p}}n^{-aq}\geqslant 2^{-aq}.$$

Thus

$$\Big(\sum_{k=1}^{m}|c_k|^q\Big)^{1/q}\geqslant\Big(\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{m}c_k y_k(n)n^{-a}\Big|^q\Big)^{1/q}\geqslant 2^{-a}\Big(\sum_{k=1}^{m}|c_k|^q\Big)^{1/q}.$$

Moreover

$$\Big(\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{m}c_k y_k(n)\Big|^p\Big)^{1/p}=\Big(\sum_{k=1}^{m}|c_k|^p\sum_{n=2^{k-1}}^{2^k-1}\big(2^{-\frac{(k-1)}{p}}\big)^p\Big)^{1/p}$$
$$=\Big(\sum_{k=1}^{m}|c_k|^p\Big)^{1/p}\leqslant\Big(\sum_{k=1}^{m}|c_k|^q\Big)^{1/q}.$$

Hence

$$\Big(\sum_{k=1}^{m}|c_k|^q\Big)^{1/q}\geqslant\Big\|\sum_{k=1}^{m}c_k y_k\Big\|\geqslant 2^{-a}\Big(\sum_{k=1}^{m}|c_k|^q\Big)^{1/q}.$$

This shows that the closed linear subspace of X spanned by the sequence (y_k) is isomorphic to l_q.

(γ). It follows from (β) and a result of [1] that every basis for X contains a block basic sequence equivalent to the unit vector basis of l_q.

Since $q < 2$, the unit vector basis of l_q is not Hilbertian. Thus X has no block Hilbertian basis.

By Proposition 5.3 and the fact that the coordinate functionals of a Hilbertian basis form a Besselian basic sequence, we get

EXAMPLE 5.4. *If $\infty > p > 2 > q > 1$, then the space $X^*_{p,q}$ is reflexive, has a normalized unconditional (shrinking and boundedly-complete) Besselian basis, but no basis of $X^*_{p,q}$ is block Besselian.*

Remark. If we let $q = 2$ in the definition of $X_{p,q}$ we get an $\mathscr{L}_p$ space $X_{p,w}$ first introduced in [30].

Finally we list some open problems.

PROBLEM 5.3. *Let X be a Banach space such that $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$ and $\Pi_2(\mathscr{L}_\infty, X^*) = B(\mathscr{L}_\infty, X^*)$. Is X isomorphic to a Hilbert space?*

Let us observe that if X has an unconditional basis or X has sufficiently many Boolean algebras of projections then the answer on Problem 5.3 is positive. In fact it is then true that if X has sufficiently many Boolean algebras of projections and both X and X^* have the Orlicz property then X is isomorphic to a Hilbert space (cf. [18]).

PROBLEM 5.4. *Let X be a Banach space for which $\Pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$ Let $(e_a)_{a\in\mathscr{A}}$ be an unconditional basis for X. Let $\{X_a\}_{a\in\mathscr{A}}$ be a family of Banach space such that there exists $K \geqslant 1$ with the property that $\Pi_2(\mathscr{L}_\infty, X_a) \underset{K}{=} B(\mathscr{L}_\infty, X_a)$ for $a \in \mathscr{A}$. Is it true that $\Pi_2(\mathscr{L}_\infty, [X_a]_{(e_a)}) = B(\mathscr{L}_\infty, [X_a)_{(e_a)})$? In particular is it true that $\Pi_2(\mathscr{L}_\infty, [X_a]_{l_2^{\mathscr{A}}}) = B(\mathscr{L}_\infty, [X_a]_{l_2^{\mathscr{A}}})$?*

Partial answers on this problem give Theorem 3.3 and Corollary 4.5.

PROBLEM 5.5. *If a Banach space has a subquadratic Gaussian average, does it has a subquadratic Rademacher average?*

PROBLEM 5.6. *Consider the following four properties of a Banach space X*

(A) *$\Pi_2(\mathscr{L}_\infty, Y) = B(\mathscr{L}_\infty, Y)$ for every quotient space Y of X,*

(B) *X^* has a subquadratic Gaussian average,*

(C) *X^* has a subquadratic Rademacher average,*

(D) *for every sequence (x_i^*) in X^*, if $\Sigma\|x_i^*\|^2 < \infty$ then $\Sigma \pm x_i^*$ converges for some choice of sings $\pm$.*

It follows from the results of this paper that for an arbitrary Banach space X, (C) $\Rightarrow$ (B) $\Rightarrow$ (A); it is trivial that (C) $\Rightarrow$ (D). Which, if any, of the remaining implications between (A), (B), (C), (D) are valid?

It would be interesting to investigate any relationships between properties (A), (B), (C), (D) and geometric properties of Banach spaces like modulus of smoothness, modulus of convexity and superreflexivity (cf. Lindenstrauss [15] and Enflo [3]).

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Received January 20, 1972 (484)

STUDIA MATHEMATICA, T. XLIV. (1972)

Addendum and corrigendum to the paper

"Some applications of Zygmund's lemma to non-linear differential equations in Banach and Hilbert spaces"

(Studia Math., 44 (1972), pp. 335-344)

by

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1. By using an idea of Diaz and Weinacht [1], Theorem 2 of the above paper can be strengthened by the replacement of the condition (1.4), viz

$$\operatorname{re}\langle f(t,y)-f(t,z),\,y-z\rangle \leqslant \tfrac{1}{2}g(t,\|y-z\|^2), \tag{1}$$

by the condition

$$\operatorname{re}\langle f(t,y)-f(t,z),\,y-z\rangle \leqslant \|y-z\|\,g(t,\|y-z\|), \tag{2}$$

where g satisfies the (usual Kamke) condition (A) of §1. In particular, when $g(t,x)=x/(t-t_0)$ (which gives Nagumo's condition), the replacement of (1) by (2) removes the factor $\frac{1}{2}$ on the right of (1).

The proof of the new version of Theorem 2 follows similar lines to that of the original version, but we now take $\sigma_{m,n}(t)=\|\varphi_m(t)-\varphi_n(t)\|$, where, for each n, φ_n is an ε_n-approximate solution of the equation $y'=f(t,y)$ such that $\varphi_n(t_0)=y_0$. If $\varphi_m(t)\neq\varphi_n(t)$, then

$$\begin{aligned}\sigma'_{m,n}(t) &= \frac{d}{dt}\{\|\varphi_m(t)-\varphi_n(t)\|^2\}^{\frac{1}{2}}\\ &= \frac{\operatorname{re}\langle\varphi'_m(t)-\varphi'_n(t),\,\varphi_m(t)-\varphi_n(t)\rangle}{\|\varphi_m(t)-\varphi_n(t)\|}\\ &= \frac{\operatorname{re}\langle f(t,\varphi_m(t))-f(t,\varphi_n(t)),\,\varphi_m(t)-\varphi_n(t)\rangle}{\|\varphi_m(t)-\varphi_n(t)\|}+\\ &\quad+\frac{\operatorname{re}\langle\varphi_m(t)-f(t,\varphi_m(t))-\varphi_n(t)+f(t,\varphi_n(t)),\,\varphi_m(t)-\varphi_n(t)\rangle}{\|\varphi_m(t)-\varphi_n(t)\|}\\ &\leqslant g(t,\sigma_{m,n}(t))+\varepsilon_m+\varepsilon_n.\end{aligned} \tag{3}$$