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Subspaces of L^1 containing L^1

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Abstract. An analytically defined class of operators on $L^1[0,1]$ called E-operators is introduced. It is proven that a bounded linear operator $T\colon L^1 \to L^1$ is an E-operator if and only if there is a closed linear subspace Y isomorphic (i.e., linearly homeomorphic) to L^1 such that $T|_Y$ is a homeomorphism (into). If T is an E-operator, then there exists a subspace Z isometrically isomorphic to L^1 with $T|_Z$ a homeomorphism (into) and $T(Z)$ complemented in L^1.

As a corollary it is shown that every subspace Y of L^1 isomorphic to L^1 contains a subspace which is isomorphic to L^1 and complemented in the whole space. From this it follows that if a complemented closed linear subspace X of L^1 contains a subspace isomorphic to L^1, then X is isomorphic to L^1.

Another corollary of the main theorems is that if L^1 is isomorphic to an unconditional sum of a sequence of Banach spaces, then one of the spaces is isomorphic to L^1. In particular, L^1 is primary.

It is shown that an operator T on L^1 is an E-operator if and only if $|T|$ is an E-operator.

1. Introduction. This paper contains a study of certain bounded linear operators $T\colon L^1 \to L^1$ called E-operators. This class of operators is defined analytically.

Theorem 4.1 states that an operator on L^1 is an E-operator if and only if the operator carries some subspace isomorphic to L^1 isomorphically. It is shown in Theorem 4.2 that an E-operator actually possesses an apparently stronger property: if $T\colon L^1 \to L^1$ is an E-operator, then there exists a subspace Y of L^1 with Y *isometric* to L^1, with $T|_Y$ an isomorphism, and with TY complemented. (Y is also automatically complemented; see [5].)

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As a corollary of Theorem 4.2 we prove, using a result of Lindenstrauss and Pełczyński [8], that if L^1 is isomorphic to an unconditional sum of a sequence of Banach spaces, then one of the spaces is isomorphic to L^1 (Corollary 5.5). In particular L^1 is primary. B. Maurey [11] adapted unpublished techniques of one of the present authors (Enflo) to give one proof that L^p is primary for all p, $1 \leqslant p < \infty$. The result asserts that if L^p is written as the direct sum of two Banach spaces, then at least one of them is isomorphic to L^p. The question whether this was true was raised by Lindenstrauss and Pełczyński in [9], where they proved that $C[0,1]$ is primary. For $p > 1$, an alternative proof that L^p is primary, based on a result of Casazza and Lin [2], is presented by Alspach, Enflo, and Odell in [1].

Another corollary (Corollary 5.2) of Theorem 4.2 is that any isomorph of L^1 in L^1 contains a subspace isomorphic to L^1 which is complemented in the whole space. This yields some information about complemented subspaces of L^1, for it implies, by the Pełczyński decomposition method [12], that if a complemented subspace X of L^1 contains a subspace isomorphic to L^1, then X itself is isomorphic to L^1 (Corollary 5.3). It is an open question whether every complemented infinite dimensional subspace Z of L^1 is isomorphic either to L^1 or to l_1. Lewis and Stegall's results [6] (see also [15]) show that a complemented infinite dimensional subspace Z has the Radon–Nikodym property if and only if any projection onto Z factors through l_1; hence Z is isomorphic to l_1. Corollary 5.2 implies that if the projection onto Z is an E-operator, then Z is isomorphic to L^1. It is known that there are operators which do not factor through l_1 and yet are not E-operators ([3] and [13]). If such a *projection* exists, then the above open question would be answered in the negative.

It may be possible to reduce some questions about bounded linear operators on L^1 to questions about positive operators by using Proposition 7.1. Proposition 7.1 states that T is an E-operator if and only if $|T|$ is an E-operator. (With every bounded linear operator $T\colon L^1 \to L^1$ can be associated a positive bounded linear operator $|T|$, the absolute value of T. See the remarks preceding Proposition 7.1.) One tends to regard E-operators as "big" in that they carry a big subspace (i.e., one isomorphic to L^1) isomorphically. Proposition 7.1 then implies that if $|T|$ is big in this sense, then T is already big.

As an application of the theorems, we answer affirmatively the following question of A. Pełczyński: Suppose

$$\tilde{S} = \int T_g S T_{-g} dg$$

is an isomorphism on a subspace isomorphic to L^1. Must S have the same property? Here g ranges over points in the circle group G, T_g is translation by g, and S is an operator on $L^1(G)$. Proposition 7.2 gives the affirmative answer. Of course, in view of Theorem 4.1, the result is that if $\tilde{S}$ is an E-operator, then so is S. The fact that the "average" operator $\tilde{S}$ being an E-operator implies that S is an E-operator agrees with the intuition of thinking of an E-operator as "big".

The structure of E-operators as illuminated in Theorem 4.2 and in the more technical Theorem 4.3 forms the basis for most of the other results, including Theorem 4.1. We give now a brief intuitive summary of the methods used to prove Theorems 4.2 and 4.3. One fundamental idea is that a simple way to construct a subspace of L^1 isometric to L^1 is to divide a subset E of $[0,1]$ of positive measure into two subsets, then divide each of those subsets into two subsets, and so on, in such a way that all the subsets eventually become smaller and smaller in measure. The collection of the characteristic functions of these subsets of $[0,1]$ has closed linear span isometric to L^1. (The closed linear span consists of all $\mathscr{A}$-measurable functions in L^1, where $\mathscr{A}$ is the σ-subalgebra of subsets of E generated by all the subsets into which E has been divided. Since $\mathscr{A}$ will be non-atomic, $L^1(\mathscr{A})$ is isometric to L^1 [10]. The operator of conditional expectation with respect to $\mathscr{A}$ is a projection of norm 1 onto $L^1(\mathscr{A})$.)

The proof of Theorem 4.3 shows that if T is an E-operator, then there exist such a set E and such a splitting process for E, generating a σ-subalgebra $\mathscr{A}$ of subsets of E, and there exists a subset F of $[0,1]$ such that by making a single change of signs (by multiplying by a fixed $\{1,-1\}$-valued function s), the operator $sR_F T|L^1(\mathscr{A})$ is almost exactly a non-zero scalar multiple of a positive isometry. Here R_F denotes the operator which restricts functions to the set F.

The proof accomplishes this result by finding a splitting process on E such that when T is applied to the characteristic functions of two disjoint subsets of E in $\mathscr{A}$, the two image functions are almost disjointly supported when restricted to F.

There are actually two senses in which the image functions are "almost" disjointly supported on F. The one needed for the proof of Theorem 4.2 is the well-known concept of relative disjointness. Rosenthal proved in [14] that relatively disjoint collections of functions in L^1 span complemented isomorphs of l_1 (of the appropriate dimension, either finite or infinite). His calculations of the bounds for the distance from l_1 of such isomorphs, and for the norm of a projection onto such isomorphs, are used in the proof of Theorem 4.2.

Another concept of "almost disjoint", stated in Theorem 4.3 (c), is useful for proving Theorem 4.1 and Proposition 7.2.

The method by which almost disjointly supported functions are recognized is to compare the integral of the maximum of the absolute values of the functions with the integral of the sum of the absolute values of the functions. When these two quantities are (almost) equal, the functions are (almost) disjointly supported. The integral of the maximum of functions in L^1 was investigated by L. Dor [5] in connection with the still open problem of whether every subspace of L^1 isomorphic to L^1 is complemented. (He showed that this is true if the subspace is sufficiently close to L^1 in the Banach–Mazur sense.) We use one of this results in the proofs of Theorem 4.1 and Corollary 5.2. In our terminology his result implies that if T is an (into) isomorphism, then T is an E-operator.

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Corollary 5.5 was called to our attention in private correspondence by N. J. Kalton, who has proven independently the same result by a different method.

Format. The format of the paper is as follows: Section 2 gives the definitions of bush, tree, and E-operator. Section 3 presents some preliminary facts about operators and bushes. Section 4 states the main theorems and gives most of their proofs. Section 5 draws corollaries of Theorem 4.2. In Section 6 the proof of Theorem 4.1 is completed. Section 7 contains the propositions about $|T|$ and the average operator $\tilde{S}$, and contains two open problems.

2. Definitions. We deal with $L^1 = L^1([0,1],\lambda)$, the Banach space of equivalence classes of Lebesgue integrable real-valued functions defined on $[0,1]$. λ is Lebesgue measure. The notation $|E|$ will also be used to denote the Lebesgue measure of a measurable subset $E\subset[0,1]$. χ_E denotes the characteristic function of E, where

$$\chi_E(t)=\begin{cases}1 & \text{if} \quad t\in E,\\ 0 & \text{if} \quad t\notin E.\end{cases}$$

R_E denotes the restriction operator on L^1 defined by

$$R_E(f)(t)=f(t)\chi_E(t).$$

If $T\colon L^1\to L^1$ is a bounded linear operator, we sometimes write TE in place of $T(\chi_E)$.

An "isomorphism" is a linear homeomorphism into. If there exists an isomorphism from a Banach space X onto a Banach space Y, then X and Y are "isomorphic", and we write $X\sim Y$. An "isomorph of L^1" is a Banach space isomorphic to L^1.

If ζ is a collection of sets, $\mathscr{A}(\zeta)$ denotes the ring generated by ζ, and $\sigma\mathscr{A}(\zeta)$ denotes the σ-ring generated by ζ.

Definitions.

(A) A *bush* is a sequence of finite partitions of a measurable subset $E_1^0\subset[0,1]$ of positive measure in which each partition refines the preceding partition, and in which the mesh of the partitions tends to zero. In symbols, (E_i^n), $i=1,\ldots,M_n$; $n=0,1,2,\ldots$ is a bush if

(1) $M_0=1$ and $|E_1^0|>0$,

(2) for each n, $\bigcup_{i=1}^{M_n} E_i^n = E_1^0$,

(3) for each n, $E_i^n\cap E_j^n=\emptyset$ if $i\neq j$,

(4) for each n, and each j, $1\leqslant j\leqslant M_{n+1}$, there is an i, $1\leqslant i\leqslant M_n$ with $E_j^{n+1}\subset E_i^n$,

(5) $\max\limits_{1\leqslant i\leqslant M_n}|E_i^n|\to 0$ as $n\to\infty$.

(B) A *tree* is a bush (E_i^n), $1\leqslant i\leqslant M_n$, $n=0,1,2,\ldots$ in which

(1) $M_n=2^n$

and

(2) $E_i^n=E_{2i-1}^{n+1}\cup E_{2i}^{n+1}$ for each n and i, $1\leqslant i\leqslant 2^n$.

(C) Let $T\colon L^1\to L^1$ be a bounded linear operator. T is called an *E-operator* if there exist $\delta>0$ and a bush (E_i^n) with

$$\frac{1}{|E_1^0|}\int \max_{1\leqslant i\leqslant M_n}|T(\chi_{E_i^n})|>\delta$$

for each n. If T is an E-operator and $\delta>0$, T is called an *E-operator of constant* δ if

$$\delta\leqslant \sup \varlimsup_{n\to\infty}\frac{1}{|E_1^0|}\int \max_{1\leqslant i\leqslant M_n}|T(\chi_{E_i^n})|,$$

where the supremum is taken over all bushes (E_i^n).

Remark. It is shown in Section 3 (see the remark after Lemma 3.2) that for any bush (E_i^n) the limit

$$\lim_{n\to\infty}\frac{1}{|E_1^0|}\int \max_{1\leqslant i\leqslant M_n}|T(\chi_{E_i^n})|$$

exists; hence, in the above definition, the limit superior could be replaced by limit.

Remark. The letter E in the term E-operator is an abbreviation for "Enflo". The phrase "Enflo operator" was coined by H. P. Rosenthal after Enflo began the study of these operators several years ago.

3. Properties of operators on L^1 with respect to bushes. For the whole of this section, let $T\colon L^1\to L^1$ be a bounded linear operator and let (E_i^n), $i=1,2,\ldots,M_n$; $n=1,2,\ldots$ be a bush.

For $n=0,1,2,\ldots$ define

$$g_n=\max_{1\leqslant i\leqslant M_n}|TE_i^n|.$$

(Here we have used the notation TE_i^n in place of $T(\chi_{E_i^n})$.) Then $g_n\in L^1$. Define the L^1-valued measure v_n on the finite algebra $\mathscr{A}(E_1^n,\ldots,E_{M_n}^n)$ by

$$v_n(E)=\sum_{E_i^n\subset E}|TE_i^n|,\qquad E\in\mathscr{A}(E_1^n,\ldots,E_{M_n}^n).$$

LEMMA 3.1. *Given $E\in\mathscr{A}\big((E_i^n)_{n,i}\big)$,*

(1) $v_n(E)$ is defined for sufficiently large n. For such n

(2) $v_{n+1}(E)\geqslant v_n(E)$;

(3) $\|v_n(E)\|\leqslant\|T\|\cdot|E|$.

(4) As $n\to\infty$, $v_n(E)$ converges both a.e. and in the L^1 norm to a function $v(E)\in L^1$.

(5) v is a σ-additive positive L^1-valued measure on $\mathscr{A}\big((E_i^n)\big)$ which extends to a σ-additive positive L^1-valued measure on $\sigma\mathscr{A}\big((E_i^n)\big)$. (We denote this extension by v still.)

(6) For each C in $\sigma\mathscr{A}\big((E_i^n)\big)$, $|TC|\leqslant v(C)$ a.e.

Proof. (1) is clear. (2) and (3) are applications of the triangle inequality. (4) follows from the monotone convergence theorem.

For (5), note that v is clearly finitely additive. Hence using (3), it is σ-additive on $\mathscr{A}\big((E_i^n)\big)$ with $v(E)\leqslant\|T\|\cdot|E|$. Therefore v extends to $\sigma\mathscr{A}\big((E_i^n)\big)$.

For (6), suppose first that $B\in\mathscr{A}\big((E_i^n)\big)$. Then $|TB|\leqslant v(B)$ a.e. by the triangle inequality. Now suppose $C\in\sigma\mathscr{A}\big((E_i^n)\big)$. Then choose $B_m\in\mathscr{A}\big((E_i^n)\big)$ with $|B_m\triangle C|\to 0$, that is $\chi_{B_m}\to\chi_C$ in L^1, as $m\to\infty$. Then $TB_m\to TC$ in L^1. By passing to a subsequence if necessary, we may assume that $TB_m\to TC$ a.e. and, similarly, that $v(B_m)\to v(C)$ a.e. ∎

LEMMA 3.2. *The sequence g_n converges a.e. and in the L^1 norm to a function g in L^1.*

Proof. We shall show below that for all n,

$$g_{n+1}-v_{n+1}(E_1^0)\leqslant g_n-v_n(E_1^0).$$

Hence $\lim\limits_{n\to\infty}\big(g_n-v_n(E_1^0)\big)$ exists, being the limit of a decreasing sequence.

By Lemma 3.1 (4), $\lim\limits_{n\to\infty}v_n(E_1^0)$ exists and is finite a.e. Hence $g=\lim\limits_{n\to\infty}g_n$ exists a.e. Furthermore

$$0\leqslant g_n\leqslant v_n(E_1^0)\leqslant v(E_1^0)\in L^1,$$

so by the dominated convergence theorem, $g_n\to g$ in the L^1 norm, and $g\in L^1$.

We now show that for fixed $t\in[0,1]$,

$$g_{n+1}(t)-v_{n+1}(E_1^0)(t)\leqslant g_n(t)-v_n(E_1^0)(t).$$

By the definition of g_{n+1}, there is a j with $g_{n+1}(t)=|(TE_j^{n+1})(t)|$. By the properties of bushes, there is an i with $E_j^{n+1}\subset E_i^n$. Then

$$\begin{aligned}g_{n+1}(t)-g_n(t)&\leqslant|TE_j^{n+1}(t)|-|TE_i^n(t)|\\&=v_{n+1}(E_j^{n+1})(t)-v_n(E_i^n)(t)\\&\leqslant v_{n+1}(E_i^n)(t)-v_n(E_i^n)(t)\\&\leqslant v_{n+1}(E_1^0)(t)-v_n(E_1^0)(t)\end{aligned}$$

since $v_{n+1}-v_n\geqslant 0$ (so $v_{n+1}(E_1^0\setminus E_i^n)-v_n(E_1^0\setminus E_i^n)\geqslant 0$). ∎

Remark. Lemma 3.2 implies that in the definition of an "E-operator of constant δ" the limit superior could be replaced by limit, or by limit inferior.

4. The main theorems. Our first major result is

THEOREM 4.1. *Let $T\colon L^1\to L^1$ be a bounded linear operator. T is an E-operator if and only if there exists a subspace Y of L^1 with Y isomorphic to L^1 and with $T|_Y$ an isomorphism (into).*

The proof of Theorem 4.1 depends on the next two theorems which are our other main results.

THEOREM 4.2. *Suppose T is an E-operator of constant δ, and $0<\varepsilon<\frac{1}{2}$. Then there exists a purely non-atomic σ-ring $\mathscr{A}_\sigma$ of Lebesgue measurable sets such that*

1. $L^1(\lambda|\mathscr{A}_\sigma)$ is isometric to L^1;

2. $T|L^1(\lambda|\mathscr{A}_\sigma)$ is an isomorphism; for $f\in L^1(\lambda|\mathscr{A}_\sigma)$,

$$\|Tf\|\geqslant\frac{(1-\varepsilon)^2}{1+\varepsilon}\,\delta\|f\|;$$

3. The image $T\big(L^1(\lambda|\mathscr{A}_\sigma)\big)$ is complemented; it is the range of a projection of norm at most

$$\frac{\|T\|}{\delta}\,\frac{(1+\varepsilon)}{(1-\varepsilon)^2}.$$

Theorem 4.2 follows immediately from the more technical

THEOREM 4.3. *Suppose T is an E-operator of constant δ, and $0 < \varepsilon < \frac{1}{2}$. Then there exists a tree (A_i^n), $i = 1, \ldots, 2^n$; $n = 0, 1, \ldots$ of measurable subsets of $[0, 1]$ with*

(a) $$|A_i^n| = \frac{|A_1^0|}{2^n}$$

and there exists a tree (F_i^n) of measurable subsets of $[0, 1]$ such that for each n, and i, $1 \leqslant i \leqslant 2^n$,

(b) $$(1-\varepsilon)\,\delta\,|A_i^n| \leqslant \int_{F_1^0} |TA_i^n| \leqslant (1+\varepsilon) \int_{F_i^n} |TA_i^n|,$$

and

(c) $$\sum_{\substack{j=1\\ j\neq i}}^{2^n} |TA_j^n(t)| \leqslant \varepsilon |TA_i^n(t)|$$

for almost all $t \in F_i^n$; and such that

(d) *if $B_1, \ldots, B_m$ are disjoint members of $\sigma\mathscr{A}\big((A_i^n)_{n,i}\big)$, then*

$$\int_{F_1^0} \max_{1\leqslant j\leqslant m} |TB_j| \geqslant (1-\varepsilon)\,\delta\,\Big|\bigcup_{j=1}^{m} B_j\Big|.$$

Remark. Notice that conclusions 1 and 2 of Theorem 4.2 assert a strong form of the direct implication claimed in Theorem 4.1. The proof of the other direction of Theorem 4.1 will be given in Section 6.

In this section we first show that Theorem 4.3 implies Theorem 4.2. Then we prove Theorem 4.3.

Proof of Theorem 4.2 assuming the truth of Theorem 4.3. We actually need only conclusions (a) and (b) stated in Theorem 4.3 in order to derive Theorem 4.2. (Conclusions (c) and (d) will be used in Sections 6 and 7.)

Suppose there exist trees (A_i^n) and (F_i^n) with properties (a) and (b). Let $\mathscr{A}_\sigma = \sigma\mathscr{A}\big((A_i^n)_{n,i}\big)$. Then by (a) $\mathscr{A}_\sigma$ is purely non-atomic, so by Maharam's Theorem [10], $L^1(\lambda|\mathscr{A}_\sigma)$ is isometric to L^1. This gives conclusion 1 of Theorem 4.2.

The right-hand inequality of (b) implies that the collection $\{TA_1^n, \ldots, TA_{2^n}^n\}$ is a relative disjoint collection (see [14]). It follows that the finite sequence $TA_1^n, \ldots, TA_{2^n}^n$ is a basic sequence equivalent to the usual basis of $l_1^{2^n}$. Because of (a) and (b) and the fact that ε and δ are independent of n, the constant of equivalence for this basic sequence is bounded by a constant independent of n. Specifically, for any scalars a_i, $1 \leqslant i \leqslant 2^n$,

$$\Big\|\sum_{i=1}^{2^n} a_i TA_i^n\Big\| \geqslant \frac{(1-\varepsilon)^2}{1+\varepsilon}\,\delta\,\Big\|\sum a_i \chi_{A_i^n}\Big\|.$$

This gives conclusion 2 of Theorem 4.2.

Another property implied by the relative disjointness of $TA_1^n, \ldots, TA_{2^n}^n$ is that their linear span is complemented with the projection constant bounded by

$$\frac{\|T\|}{\delta}\,\frac{(1+\varepsilon)}{(1-\varepsilon)^2}.$$

It follows by a familiar compactness argument (see [4]) that the closed linear span of $(TA_i^n)_{n,i}$ is complemented. This is conclusion 3 of Theorem 4.2. ■

Proof of Theorem 4.3. We shall use the functions g_n and g, and the L^1-valued measures v_n and v defined in Section 3. The assumption that T is an E-operator of constant δ means that there is a bush (E_i^n) with

(1) $$\frac{1}{|E_1^0|}\int g_n > \delta(1-\tfrac{1}{4}\varepsilon)$$

for infinitely many values of n. Lemma 3.2 says that $g_n \to g$ in L^1. Hence (1) is actually true for all sufficiently large n and

$$\frac{1}{|E_1^0|}\int g \geqslant \delta(1-\tfrac{1}{4}\varepsilon).$$

Hence we may use Egoroff's Theorem to find a subset $F \subset [0, 1]$ such that

(2) $$\frac{1}{|E_1^0|}\int_F g > \delta(1-\tfrac{1}{2}\varepsilon),$$

(3) $$g_n \to g \text{ uniformly on } F,$$

and

(4) $$v_n(E_1^0) \to v(E_1^0) \text{ uniformly on } F.$$

We may also assume

(5) $$\inf_{t\in F} g(t) > 0.$$

Choose β so small that

(6) $$(1-\tfrac{1}{2}\varepsilon)(1-6\beta) \geqslant (1-\varepsilon);\quad (1-6\beta) \geqslant 1/(1+\varepsilon);\ \text{and}\quad 6\beta/(1-6\beta) \leqslant \varepsilon.$$

Then using (3), (4) and (5), choose N so large that for $n \geqslant N$,

(7) $$|g_n - g| < \beta g \quad \text{on } F$$

and

(8) $$0 \leqslant v(E) - v_n(E) < \beta g \quad \text{on } F$$

(for any E on which v_n is defined).

The next stage is to select a "subbush", by choosing one of the sets E_i^N, $i = 1, \ldots, M_N$ and all of its subsets in the original bush, with the property that the image functions of disjoint elements in this subbush are almost disjoint when restricted to a fixed set (to be called F_0).

For each i, $1 \leqslant i \leqslant M_N$, let

$$G_i = \{t \in F \mid \text{for infinitely many values of } n, \text{ there exists } E_j^n \subset E_i^N \text{ with } g_n(t) = |TE_j^n(t)|\}.$$

Then (since M_N is finite),

$$\bigcup_{i=1}^{M_N} G_i = F.$$

Thus there exists $i_0 \leqslant M_N$ such that

$$\int_{G_{i_0}} g > \delta(1-\tfrac{1}{2}\varepsilon)|E_{i_0}^N|. \tag{9}$$

(If not, $\int_{G_i} g \leqslant \delta(1-\frac{1}{2}\varepsilon)|E_i^N|$ for all i; summation over i contradicts (2)).

Let $F_0 = G_{i_0}$, $B_1^0 = E_{i_0}^N$ and $B_i^n = B_1^0 \cap E_i^n$ (for $n > 0$). We now restrict our attention to the bush (B_i^n). We begin by showing that for $t \in F_0$,

$$g(t) \leqslant v(B_1^0)(t) \leqslant (1+2\beta)g(t). \tag{10}$$

(Intuitively this means that on F_0 the images of disjoint elements of the bush (B_i^n) are almost disjointly supported.)

To see the left-hand inequality in (10), let E_j^n be a subset of B_1^0 such that $g_n(t) = |TE_j^n(t)|$ (see the definition of G_i). Then

$$g_n(t) = |TE_j^n(t)| = v_n(E_j^n)(t) \leqslant v(B_1^0)(t).$$

Since this is true for infinitely many values of n, the left-hand inequality of (10) follows.

For the right-hand inequality of (10), note that for $t \in F_0 \subset F$,

$$\begin{aligned} v(B_1^0)(t) &\leqslant v_N(B_1^0)(t) + \beta g(t) && \text{(by (8))} \\ &= |TB_1^0(t)| + \beta g(t) && \text{(since } B_1^0 = E_{i_0}^N) \\ &\leqslant g_N(t) + \beta g(t) \\ &\leqslant (1+2\beta)g(t) && \text{(by (7))}. \end{aligned}$$

This gives (10).

LEMMA 4.4. *For each C in $\sigma\mathscr{A}((B_i^n))$ let*

$$\Phi(C) = \{t \in F_0\colon |TC(t)| > (1-6\beta)v(B_1^0)(t)\}.$$

Then Φ is a Boolean σ-homomorphism of $\sigma\mathscr{A}((B_i^n))$ onto a σ-algebra of subsets of F_0 (modulo sets of measure 0) such that if $C_1, \ldots, C_m$ are disjoint members of $\sigma\mathscr{A}((B_i^n))$, then for each i

$$\sum_{j\neq i} |TC_j(t)| \leqslant \frac{6\beta}{1-6\beta}|TC_i(t)| \tag{11}$$

for almost all $t \in \Phi(C_i)$. Moreover, if $C_m \in \sigma\mathscr{A}((B_i^n))$ and $|C_m| \to 0$, then $|\Phi(C_m)| \to 0$.

Proof. The bulk of the proof is to show that Φ takes a finite $\sigma\mathscr{A}((B_i^n))$-partition of B_1^0 to a partition of F_0. This is equivalent to showing that Φ is a Boolean algebra homomorphism. Then it is shown that Φ is a σ-homomorphism.

Let $C_1, \ldots, C_m$ be disjoint members of $\sigma\mathscr{A}((B_i^n))$. If $t \in \Phi(C_i)$, we have

$$v(C_i)(t) \geqslant |TC_i(t)| \geqslant (1-6\beta)v(B_1^0)(t).$$

Hence

$$\sum_{j\neq i} |TC_j(t)| \leqslant v(B_1^0 \setminus C_i)(t) \leqslant 6\beta v(B_1^0)(t) \leqslant \frac{6\beta}{1-6\beta}|TC_i(t)|. \tag{12}$$

This gives (11), and also shows that if $j \neq i$, then $|TC_j(t)| \leqslant 6\beta v(B_1^0)(t)$. By (6), $t \notin \Phi(C_j)$. Hence

$$\Phi(C_1), \ldots, \Phi(C_m) \text{ are (essentially) disjoint.} \tag{13}$$

To complete the demonstration that Φ takes partitions of B_1^0 to partitions of F_0, we must show that $\bigcup_{i=1}^{m} \Phi(C_i) = F_0$ if $\bigcup_{i=1}^{m} C_i = B_1^0$. This requires several steps.

Define

$$\psi(C) = \left\{t \in F_0 \,|\, |TC(t)| \geqslant \frac{1-4\beta}{1+2\beta}v(B_1^0)(t)\right\}.$$

Notice that $\dfrac{1-4\beta}{1+2\beta} > 1-6\beta$ so $\psi(C) \subset \Phi(C)$.

We now show that

$$\text{if } B_1, \ldots, B_m \in \mathscr{A}((B_i^n)) \text{ with } \bigcup_{i=1}^{m} B_i = B_1^0, \text{ then } \bigcup_{i=1}^{m} \psi(B_i) = F_0. \tag{14}$$

For suppose $t \in F_0$. Choose $n \geqslant N$ so large that $B_i \in \mathscr{A}(B_1^n, \ldots, B_{M_n}^n)$ for $1 \leqslant i \leqslant m$. (So $|g_n(t) - g(t)| < \beta g(t)$.) Then by making n larger if necessary, $g_n(t) = |TE_j^n(t)|$ for some $E_j^n (= B_j^n)$ contained in some B_i (by the definition of $F_0 = G_{i_0}$). Then

$$|TB_i(t)| \geqslant |TB_j^n(t)| - \sum_{k\neq j} |TB_k^n(t)| = 2|TB_i^n(t)| - \sum_k |TB_k^n(t)|$$
$$= 2|TB_j^n(t)| - v_n(B_1^0)(t) = 2g_n(t) - v_n(B_1^0)(t)$$
$$\geqslant (2-2\beta)g(t) - v(B_1^0)(t) \geqslant \frac{2-2\beta}{1+2\beta} v(B_1^0)(t) - v(B_1^0)(t) \qquad \text{(by (10))}$$
$$= \frac{1-4\beta}{1+2\beta} v(B_1^0)(t).$$

Hence $t \in \psi(B_i)$. So

$$\bigcup_{i=1}^m \psi(B_i) = F_0.$$

Note that if $C, B_k \in \sigma\mathscr{A}((B_i^n))$ and $|B_k \Delta C| \to 0$ as $k \to \infty$, then for some subsequence k_i, $\lim |\psi(B_{k_i}) \setminus \Phi(C)| = 0$. For there exists a subsequence with $|TB_{k_i}| \to |TC|$ a.e. Since $1-6\beta < \dfrac{1-4\beta}{1+2\beta}$, $\chi_{\psi(B_{k_i})\setminus\Phi(C)} \to 0$ a.e. By the dominated convergence theorem

$$\chi_{\psi(B_{k_i})\setminus\Phi(C)} \to 0 \text{ in } L^1, \quad \text{i.e.} \quad |\psi(B_{k_i})\setminus\Phi(C)| \to 0.$$

We now show that if $C_1, \ldots, C_m \in \sigma\mathscr{A}((B_i^n))$ with $\bigcup_{i=1}^m C_i = B_1^0$, then $\bigcup_{i=1}^m \Phi(C_i) = F_0$ a.e. Given $\eta > 0$, choose $B_1, \ldots, B_m \in \mathscr{A}((B_i^n))$ with $\bigcup_{i=1}^m B_i = B_1^0$ and $|\psi(B_i)\setminus\Phi(C_i)| < \eta/m$ for $1 \leqslant i \leqslant m$. Then

$$\Big|F_0 \setminus \bigcup_{i=1}^m \Phi(C_i)\Big| = \Big|\bigcup_{i=1}^m \psi(B_i) \setminus \bigcup_{i=1}^m \Phi(C_i)\Big| \qquad \text{(by (14))}$$
$$= \Big|\bigcup_{i=1}^m \big[\psi(B_i) \setminus \bigcup_{j=1}^m \Phi(C_j)\big]\Big|$$
$$\leqslant \sum_{i=1}^m |\psi(B_i)\setminus\Phi(C_i)| < \eta.$$

Since η was arbitrary, $F_0 = \bigcup_{i=1}^m \Phi(C_i)$ a.e.

This together with (13) shows that finite $\sigma\mathscr{A}((B_i^n))$-partitions of B_1^0 are sent by Φ to (essential) partitions of F_0.

Next note that $\Phi(C) \subset \{t \in F_0 |\ |TC(t)| > (1-6\beta) \inf_{s\in F_0} g(s)\}$ by (10). Thus if $|C_m| \to 0$, then $|TC_m| \to 0$ in L^1, and hence $|\Phi(C_m)| \to 0$.

To finish the proof of the lemma, suppose (C_i), $i = 1, 2, \ldots$ is a disjoint sequence in $\sigma\mathscr{A}((B_i^n))$. Then $\big|\Phi\big(\bigcup_{i=m}^\infty C_i\big)\big| \to 0$ as $m \to \infty$, so for some subsequence m_k, $\chi_{\Phi(\bigcup_{i=m_k}^\infty C_i)} \to 0$ a.e. Then

$$\chi_{\Phi(\bigcup_{i=1}^\infty C_i)} = \sum_{i=1}^{m_k-1} \chi_{\Phi(C_i)} + \chi_{\Phi(\bigcup_{i=m_k}^\infty C_i)} \to \sum_{i=1}^\infty \chi_{\Phi(C_i)} \quad \text{a.e.}$$

So $\Phi\big(\bigcup_{i=1}^\infty C_i\big) = \bigcup_{i=1}^\infty \Phi(C_i)$ a.e., and Φ is a σ-homomorphism. ■

We are now in a position to construct the trees (A_i^n) and (F_i^n) with the properties listed in the conclusion of Theorem 4.3.

Define a measure μ: $\sigma\mathscr{A}((B_i^n)) \to \boldsymbol{R}^3$ by

$$\mu(C) = \Big(|C|, \int_{\Phi(C)} v(B_1^0), \int_{F_0} v(C)\Big).$$

This measure is non-atomic since $|C|$, $|\Phi(C)|$, and $\|v(C)\|$ all tend to 0 as $|C| \to 0$. Hence, by Liapunov's convexity theorem (see [7]), the range of μ is convex. Note that

$$\mu(B_1^0) = \Big(|B_1^0|, \int_{F_0} v(B_1^0), \int_{F_0} v(B_1^0)\Big).$$

Thus we can find a tree (A_i^n) with

$$A_1^0 = B_1^0$$

and

$$\mu(A_i^n) = \frac{1}{2^n} \mu(A_1^0),$$

i.e., with

$$|A_i^n| = \frac{1}{2^n} |A_1^0| \tag{15}$$

and

$$\int_{\Phi(A_i^n)} v(A_1^0) = \frac{1}{2^n} \int_{F_0} v(A_1^0) = \int_{F_0} v(A_i^n). \tag{16}$$

Define $F_i^n = \Phi(A_i^n)$. (In particular, $F_1^0 = F_0 = G_{i_0}$.) We now verify conclusions (b), (c), and (d) listed in the statement of Theorem 4.3.

$$\int_{F_i^n} |TA_i^n| \geqslant (1-6\beta) \int_{\Phi(A_i^n)} v(A_1^0)$$
$$= (1-6\beta) \frac{1}{2^n} \int_{F_1^0} v(A_1^0) \qquad \text{(by 16))}$$

$$\geqslant (1-6\beta)\frac{1}{2^n}\,\delta\left(1-\frac{\varepsilon}{2}\right)|A_1^0| \qquad \text{(by (9) and (10))}$$

$$\geqslant (1-\varepsilon)\,\delta\,\frac{|A_1^0|}{2^n}. \qquad \text{(by (6))}$$

This gives the left-hand inequality in (b).

For the right-hand inequality in (b),

$$\int_{F_i^n} |TA_i^n| \geqslant (1-6\beta)\int_{\Phi(A_i^n)} v(A_1^0)$$

$$= (1-6\beta)\int_{F_1^0} v(A_i^n) \qquad \text{(by (16))}$$

$$\geqslant (1-6\beta)\int_{F_1^0} |TA_i^n| \geqslant \frac{1}{1+\varepsilon}\int_{F_1^0} |TA_i^n| \qquad \text{(by (6))}.$$

Inequality (c) is immediate from (12) and (6). For (d), notice that from (15) and (16) in follows that for any $B \in \sigma\mathscr{A}((A_i^n))$,

$$\int_{\Phi(B)} v(A_1^0) = \frac{|B|}{|A_1^0|}\int_{F_1^0} v(A_1^0).$$

Now let $B_1, \ldots, B_m$ be disjoint members of $\sigma\mathscr{A}((A_i^n))$. Then since $\Phi(B_i)$ are disjoint,

$$\int_{F_1^0}^{m} \max_{1\leqslant j\leqslant m} |TB_j| \geqslant \sum_{i=1}^{m}\int_{\Phi(B_i)} |TB_i|$$

$$\geqslant (1-6\beta)\sum_{i=1}^{m}\int_{\Phi(B_i)} v(A_1^0)$$

$$= (1-6\beta)\int_{\Phi(\bigcup_{i=1}^{m} B_i)} v(A_1^0)$$

$$= (1-6\beta)\frac{\left|\bigcup_{i=1}^{m} B_i\right|}{|A_1^0|}\int_{F_1^0} v(A_1^0)$$

$$\geqslant (1-\varepsilon)\,\delta\left|\bigcup_{i=1}^{m} B_i\right| \qquad \text{(by (10), (9) and (6)).} \ \blacksquare$$

5. Corollaries of Theorem 4.2. We have proven Theorem 4.3, and hence have proven Theorem 4.2 and the direct implication in Theorem 4.1. Before proving the reverse implication in Theorem 4.1, we deduce some corollaries of Theorem 4.2. Recall first the result of Dor [5]:

PROPOSITION 5.1 (Dor). *If $T\colon L^1 \to L^1$ is an (into) isomorphism, then for any partition (E_i), $i = 1, \ldots, M$, of a set E of positive measure, we have*

$$\frac{1}{\|T^{-1}\|^2} \leqslant \frac{1}{|E|}\int \max_{1\leqslant i\leqslant M} |TE_i|.$$

COROLLARY 5.2. *Let $Z = L^1$. If X is a subspace of Z and $X \sim L^1$, then there exists a subspace Y of X with $Y \sim L^1$ and with Y complemented in Z.*

Proof. Let $T\colon L^1 \to Z$ be any isomorphism onto X. Proposition 5.1 implies that T is an E-operator. Theorem 4.2 gives the result. $\blacksquare$

COROLLARY 5.3. *If a complemented subspace X of L^1 contains a subspace isomorphic to L^1, then $X \sim L^1$.*

Proof. Apply Corollary 5.2 and Pełczyński's decomposition method [12]. $\blacksquare$

Before stating the next corollary we make the following

Remark. If $T_1 + T_2$ is an E-operator, then either T_1 or T_2 must be an E-operator. For

$$\int \max_i |(T_1+T_2)E_i^n| \leqslant \int \max_i |T_1E_i^n| + \int \max_i |T_2E_i^n| \tag{17}$$

for each n and for any bush (E_i^n). If neither T_1 nor T_2 is an E-operator, then as $n\to\infty$ the limit of the right-hand side, and hence the left-hand side, of (17) is 0. Since this is true for all bushes, T_1+T_2 cannot be an E-operator. A continuous version of this observation is given in Proposition 7.2.

COROLLARY 5.4. *L^1 is primary; i.e., if $L^1 \sim X \oplus Y$, then either $X \sim L^1$ or $Y \sim L^1$ (or both).*

Proof. Consider X and Y as complementary subspaces in L^1 with projections P onto X and $I-P$ onto Y. Since these two operators sum to the identity operator, which is certainly an E-operator, one of them (let us say P) is an E-operator. Theorem 4.2 then asserts the existence of a complemented subspace of X (the range of P) isomorphic to L^1. Pełczyński's decomposition method [12] implies that $X \sim L^1$. $\blacksquare$

A generalization of Corollary 5.4 is

COROLLARY 5.5. *Suppose L^1 is isomorphic to an unconditional sum of a sequence of Banach spaces X_i. Then there is a j such that $X_j \sim L^1$.*

Proof. By a result of Lindenstrauss and Pełczyński [8] the hypothesis implies that the l_1 sum of the spaces X_i is isomorphic to L^1. Hence we may regard X_j as a subspace of $L^1[1/2^i, 1/2^{i-1}]$, $i = 1, 2, \ldots$ Let X denote the l_1 sum of these X_i, so $X \subset L^1[0,1]$. We are given that $X \sim L^1$. Let $T\colon L^1 \to L^1$ be an isomorphism from L^1 onto X. By Proposition 5.1, T is an E-operator, i.e., for all n

$$\int \max_{1\leqslant i\leqslant 2^n} |TE_i^n| \geqslant \delta$$

for some $\delta > 0$, where $E_i^n = \left(\frac{i-1}{2^n}, \frac{i}{2^n}\right]$. Hence there is a k such that

$$\int_{[1/2^k,1]} \max_{1\leqslant i\leqslant 2^n} |TE_i^n| \geqslant \delta/2$$

for all sufficiently large n (since by Lemma 3.2 $\max\limits_{1\leqslant i\leqslant 2^n} |TE_i^n|$ converges in L^1 as $n \to \infty$). Thus the operator $R_{[1/2^k,1]}T$ is an E-operator. By Theorem 4.2 its range, which is $(\sum_{i=1}^k \oplus X_i)_{l_1}$ (a complemented subspace of $X \sim L^1$) contains a complemented subspace isomorphic to L^1. Thus by Pełczyński's decomposition method $(\sum_{i=1}^k \oplus X_i)_{l_1} \sim L^1$. Since L^1 is primary (Corollary 5.4), there exists j, $1 \leqslant j \leqslant k$, with $X_j \sim L^1$. ∎

6. The proof of Theorem 4.1. This section is devoted to the proof of the converse implication in Theorem 4.1, and to one simple corollary.

LEMMA 6.1. *Let $S\colon L^1 \to L^1$ be a bounded linear operator. Then given $f_1, \ldots, f_m$ and $h_1, \ldots, h_m$ in L^1,*

$$\left|\int \max_i |Sf_i| - \int \max_i |Sh_i|\right| \leqslant \|S\| \sum_i \|f_i - h_i\|.$$

Proof.

$$|Sf_i| \leqslant |Sh_i| + |S(f_i - h_i)| \leqslant \max_j |Sh_j| + \sum_j |S(f_j - h_j)|.$$

So

$$\max_i |Sf_i| \leqslant \max_j |Sh_j| + \sum_j |S(f_j - h_j)|.$$

Integration gives the result. ∎

The next lemma, a consequence of Lemma 6.1, shows that the class of E-operators is invariant under the natural isometries of L^1 determined by a change of sign and density.

R_E denotes the operator on L^1 which restricts functions to the set E.

LEMMA 6.2. *Let $S\colon L^1 \to L^1$ be a bounded linear operator. Suppose there is a function $f \in L^1$ and a bush (F_i^n), $i = 1, \ldots, M_n$; $n = 0, 1, \ldots$, such that*

$$\overline{\lim_{n\to\infty}} \int \max_{1\leqslant i\leqslant M_n} |SR_{F_i^n}(f)| > 0.$$

Then S is an E-operator.

Proof. By approximating f sufficiently closely by a step function

$$g = \sum_{j=1}^m c_j \chi_{G_j}$$

we have by Lemma 6.1 that

$$0 < \overline{\lim_{n\to\infty}} \int \max_{1\leqslant i\leqslant M_n} \left|SR_{F_i^n}\left(\sum_{j=1}^m c_j \chi_{G_j}\right)\right|$$
$$= \overline{\lim_{n\to\infty}} \int \max_{1\leqslant i\leqslant M_n} \left|\sum_{j=1}^m c_j SR_{G_j}(\chi_{F_i^n})\right|.$$

This shows that the operator $\sum_{j=1}^m c_j SR_{G_j}$ is an E-operator. Then there exists a j with SR_{G_j} an E-operator (see the Remark following Corollary 5.3). Indeed

$$0 < \lim_{n\to\infty} \int \max_i |SR_{G_j}(F_i^n)| = \lim_{n\to\infty} \int \max_i |S(G_j \cap F_i^n)|,$$

which shows that S is an E-operator. ∎

Proof of Theorem 4.1. The direct implication is given by Theorem 4.2. To prove the reverse, let $S\colon L^1 \to L^1$ be a bounded linear operator and assume that there is a subspace X isomorphic to L^1 with $S|_X$ an isomorphism. We may assume without loss of generality that $\|S\| = 1$.

Let $T\colon L^1 \to L^1$ be an isomorphism of L^1 onto X with $\|T\| = 1$. Then $ST\colon L^1 \to L^1$ is an isomorphism, and by Dor's result (Proposition 5.1) there is a number $\Delta > 0$ with

$$\int \max_i |ST(B_i)| \geqslant \Delta^2 \left|\bigcup_i B_i\right| \tag{18}$$

for any finite disjoint collection (B_i). Also by Dor's result T is an E-operator. Choose $\delta > 0$ such that T is of constant δ, but not of constant $\delta + \frac{1}{2}(\Delta^2/2)^2$. Let $\varepsilon > 0$ be such that $\varepsilon < \Delta^2/8$ and

$$(1-\varepsilon)\delta + \left(\frac{\Delta^2}{2}\right)^2 > \delta + \frac{1}{2}\left(\frac{\Delta^2}{2}\right)^2.$$

Find trees (A_i^n) and (F_i^n) with properties (a), (b), (c) and (d) listed in Theorem 4.3.

We claim that

$$\lim_{n\to\infty}\int \max_{1\leqslant i\leqslant 2^n} |SR_{F_1^0}(TA_i^n)| \geqslant \frac{\Delta^2}{2}|A_1^0|. \tag{19}$$

For if (19) fails, then by (18)

$$\lim_{n\to\infty}\int \max_{1\leqslant i\leqslant 2^n} |SR_{[0,1]\setminus F_1^0}(TA_i^n)| > \frac{\Delta^2}{2}|A_1^0|. \tag{20}$$

Define $V\colon L^1\to L^1$ by

$$Vf = SR_{[0,1]\setminus F_1^0}(Tf).$$

Statement (20) asserts that V is an E-operator of constant larger that $\Delta^2/2$, so by Theorem 4.2 there is a non-atomic σ-ring $\mathscr{B}\subset\sigma\mathscr{A}\big((A_i^n)\big)$ such that

$$\|Vf\|\geqslant\frac{\Delta^2}{2}\|f\| \quad \text{for all } f\in L^1(\mathscr{B}).$$

Since $\|S\|=1$,

$$\|R_{[0,1]\setminus F_1^0}(Tf)\|\geqslant\frac{\Delta^2}{2}\|f\|$$

for all $f\in L^1(\mathscr{B})$.

By Dor's Theorem (Proposition 5.1), for any finite disjoint collection $(B_i)\subset\mathscr{B}$,

$$\int\limits_{[0,1]\setminus F_1^0}\max_i|TB_i|\geqslant\Big(\frac{\Delta^2}{2}\Big)^2\Big|\bigcup_i B_i\Big|.$$

By property (d) in Theorem 4.3,

$$\int\limits_{F_1^0}\max_i|TB_i|\geqslant(1-\varepsilon)\,\delta\Big|\bigcup_i B_i\Big|.$$

Addition of the last two inequalities would give that T is an E-operator of constant

$$(1-\varepsilon)\,\delta+\Big(\frac{\Delta^2}{2}\Big)^2>\delta+\frac{1}{2}\Big(\frac{\Delta^2}{2}\Big)^2$$

which is a contradiction. Hence (19) is true.

Next we use Lemma 6.1 to show that

$$\lim_{n\to\infty}\int \max_{\leqslant i\leqslant 2^n} |SR_{F^n}(TA_1^0)| \geqslant \frac{\Delta^2}{4}|A_1^0|. \tag{21}$$

For each n and i,

$$\begin{aligned}
&\|R_{F_i^n}(TA_1^0)-R_{F_1^0}(TA_i^n)\|\\
&\qquad=\Big\|R_{F_i^n}(TA_i^n)+\sum_{j\neq}R_{F_i^n}(TA_j^n)-R_{F_i^n}(TA_i^n)-R_{F_1^0\setminus F_i^n}(TA_i^n)\Big\|\\
&\qquad\leqslant\sum_{j\neq i}\|R_{F_i^n}(TA_j^n)\|+\|R_{F_1^0\setminus F_i^n}(TA_i^n)\|\\
&\qquad\leqslant\varepsilon\|TA_i^n\|+\varepsilon\int\limits_{F_i^n}|TA_i^n| \qquad \text{(by Theorem 4.3 (c), (b))}\\
&\qquad\leqslant 2\,\varepsilon\|TA_i^n\|\leqslant\frac{\Delta^2}{4}\|TA_i^n\|\leqslant\frac{\Delta^2}{4}|A_i^n|.
\end{aligned}$$

Summation over i and application of Lemma 6.1 and (19) give (21).

Lemma 6.2 and (21) complete the proof that S is an E-operator. ■

Corollary 6.3. *Let $T\colon L^1\to L^1$ be a bounded linear operator. If there exists a subspace Y isomorphic to L^1 with $T|_Y$ an isomorphism, then there exists a subspace Z isometric to L^1 with $T|_Z$ an isomorphism and with TZ complemented.*

Proof. Combine Theorems 4.1 and 4.2. ■

7. Further propositions and open problems. Our next proposition deals with the absolute value of an operator on L^1. If $T\colon L^1\to L^1$ is a bounded linear operator, its absolute value $|T|$ is the operator on L^1 defined for $f\geqslant 0$, $f\in L^1$, by

$$(|T|f)(t) = \sup\Big\{\sum_{i=1}^m|Tf_i(t)|\colon \sum_{i=1}^m f_i=f,\ f_i\geqslant 0\Big\}$$

for all $t\in[0,1]$, and defined for general $f\in L^1$ by linearity, writing f as the difference of two positive functions. It is a fact that $|T|$ is bounded with norm $\|T\|$. (See Chapter IV of [16] for a general discussion of $|T|$.)

Proposition 7.1. *Let $T\colon L^1\to L^1$ be a bounded linear operator. T is an E-operator if and only if $|T|$ is an E-operator.*

Proof. Suppose T is an E-operator. Since for any measurable set E, χ_E is non-negative, we have $|T|E\geqslant|TE|$. Hence any bush which shows T is an E-operator shows also that $|T|$ is an E-operator.

Suppose now that $|T|$ is an E-operator. From Theorem 4.3 (c) (using $\varepsilon=\frac{1}{4}$) there exist trees (A_i^n) and (F_i^n) with

$$\int\limits_{F_j^n}\sum_{\substack{j=1\\ i\neq i}}^{2^n}|T|A_j^n\leqslant\frac{1}{4}\int\limits_{F_i^n}|T|A_i^n\leqslant\frac{1}{4}\int\limits_{F_i^n}|T|A_1^0.$$

Summation over i and reversal of the order of summation gives

$$\sum_{j=1}^{2^n} \int_{F_1^0 \setminus F_j^n} |T| A_j^n \leqslant \frac{1}{4} \int_{F_1^0} |T| A_1^0. \tag{22}$$

From the definition of $|T|$, we can find functions $f_k \geqslant 0$, $1 \leqslant k \leqslant m$, with $\sum_{k=1}^{m} f_k = \chi_{A_1^0}$ and with

$$\int_{F_1^0} \sum_{k=1}^{m} |Tf_k| \geqslant \frac{1}{2} \int_{F_1^0} |T| A_1^0. \tag{23}$$

We shall show that there is a constant $C > 0$ such that

$$\sum_{k=1}^{m} \int_{F_1^0} \max_{1 \leqslant i \leqslant 2^n} |TR_{A_i^n} f_k| \geqslant C \tag{24}$$

for all n. This is enough to complete the proof, for if T were not an E-operator, then for each k,

$$\int \max |TR_{A_i^n} f_k| \to 0$$

as $n \to \infty$, by Lemma 6.2.

We now show (24). $\sum_{j=1}^{2^n} |TR_{A_j^n} f_k| \geqslant |Tf_k|$. Hence

$$\sum_{k=1}^{m} \sum_{j=1}^{2^n} \int_{F_1^0} |TR_{A_j^n} f_k| \geqslant \sum_{k=1}^{m} \int_{F_1^0} |Tf_k| \geqslant \frac{1}{2} \int_{F_1^0} |T| A_1^0 \tag{25}$$

by (23). Notice that by the definition of $|T|$,

$$\sum_{k=1}^{m} |TR_{A_j^n} f_k| \leqslant |T| A_j^n.$$

So by (22),

$$\sum_{j=1}^{2^n} \sum_{k=1}^{m} \int_{F_1^0 \setminus F_j^n} |TR_{A_j^n} f_k| \leqslant \sum_{j=1}^{2^n} \int_{F_1^0 \setminus F_j^n} |T| A_j^n \leqslant \frac{1}{4} \int_{F_1^0} |T| A_1^0. \tag{26}$$

Subtraction of (26) from (25) yields

$$\sum_{k=1}^{m} \sum_{j=1}^{2^n} \int_{F_j^n} |TR_{A_j^n} f_k| \geqslant \frac{1}{4} \int_{F_1^0} |T| A_1^0 = C > 0.$$

This implies (24). ■

For our last proposition we present an application suggested by a question of A. Pełczyński. Let G be the circle group and let μ be normalized Haar measure on G. Let T_g be the translation operator on $L^1(G, \mu)$ defined by

$$T_g(f)(a) = f(a-g)$$

for each $a, g \in G$.

Suppose $S\colon L^1(G) \to L^1(G)$ is a bounded linear operator. Consider the operator

$$\tilde{S} = \int_G T_g S T_{-g} \, d\mu(g).$$

PROPOSITION 7.2. *If $\tilde{S}$ is an E-operator, then S is an E-operator.*

Proof. Suppose S is not an E-operator. By Theorem 4.1 $T_g S T_{-g}$ is not an E-operator. Then for any fixed bush (E_i^n),

$$\lambda_g^n = \int \max_i |T_g S T_{-g}(E_i^n)| \to 0$$

as $n \to \infty$. For each n, λ_g^n is a continuous function of g, and $\lambda_g^n \leqslant \|S\|$. The bounded convergence theorem implies that

$$\int_G \lambda_g^n \, d\mu(g) \to 0$$

as $n \to \infty$.

For any partition (A_j) of G into arcs, and any $g_j \in A_j$, we have for each n

$$\int_G \max_i \Big| \sum_j T_{g_j} S T_{-g_j}(E_i^n)(t) \cdot \mu(A_j) \Big| dt \leqslant \int_G \sum_j \max_i |T_{g_j} S T_{-g_j}(E_i^n)(t) \cdot \mu(A_j)| \, dt = \sum_j \lambda_{g_j}^n \mu(A_j).$$

By taking the limit over a sequence of refining partitions, we conclude that

$$\int_G \max_i |\tilde{S} E_i^n| \leqslant \int_G \lambda_g^n \, d\mu(g).$$

Since the right-hand side tends to 0 as $n \to \infty$, and since this is true for any bush (E_i^n), the operator $\tilde{S}$ cannot be an E-operator. ■

We end by posing two problems. The first is part of the larger question whether any complemented infinite dimensional subspace of L^1 is isomorphic either to L^1 or to l_1. Recall that an operator T is said to have the Dunford–Pettis property if the image under T of a weakly compact set has compact closure.

PROBLEM 1. Suppose a projection $P: L^1 \to L^1$ fails to have the Dunford–Pettis property. Must P be an E-operator?

H. P. Rosenthal [13] has constructed an example of an operator on L^1 which, when restricted to the span of the Rademacher functions, is the identity operator (and hence fails to have the Dunford–Pettis property), but which is not an E-operator.

The second problem concerns a local version of the property which defines an E-operator.

PROBLEM 2. Let T be an operator on L^1. Suppose there exists a constant $\delta > 0$ such that for each n,

$$\sup \int \max_{1\leqslant i\leqslant n} |TE_i| \geqslant \delta$$

where the supremum is taken over all partitions (E_i), $i = 1, \ldots, n$, of $[0, 1]$ with $|E_i| = 1/n$. Must T be an E-operator?

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