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A criterion for subharmonicity of a function of the spectrum

by

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Abstract. The following is a special case of the general result proven in the paper. Let $\chi\colon F_c(\boldsymbol{C})\to[-\infty, +\infty)$, where $F_c(\boldsymbol{C})$ denotes the collection of all non-empty compact subsets of the complex plane $\boldsymbol{C}$. Assume that $\chi(K)\leqslant\chi(L)$, if $K\subset L$, and $\chi(\bigcap K_n) = \lim\chi(K_n)$, whenever $K_{n+1}\subset K_n$ for $n = 1, 2, \ldots$. Then conditions (a) and (b) are equivalent: (a) for every analytic function a from $G\subset\boldsymbol{C}$ into a Banach algebra A the function $\lambda\to\chi\big(\sigma(a(\lambda))\big)$ is subharmonic; (b) the same for A commutative.

An application to uniform algebras is given.

1. Introduction. Consider a typical situation: we are given a Banach algebra A (the case A = the algebra of all bounded operators on a Banach space X being the most interesting) and an analytic function $a\colon G\to A$, where $G\subset\boldsymbol{C}$ is open; suppose that we are interested in studying the behaviour of the set-valued function $K(\lambda) = \sigma\big(a(\lambda)\big)$ (= the spectrum). One way of doing it is to consider some characteristics χ of compact sets, and to analyse the functions $\lambda\to\chi\big(K(\lambda)\big)$. In many instances $\chi\big(K(\lambda)\big)$ was found to be subharmonic (e.g. for $\chi(K) = \log\max\{|z|\colon z\in K\}$, cf. Vesentini [14], and $\chi(K) = \log\operatorname{diam}(K)$, cf. Aupetit [1], and $\chi(K) = n$th diameter of K or the logarithmic capacity of K, cf. Słodkowski [10]).

In the realm of uniform algebras J. Wermer [16] began to study the multifunction $K(\lambda) = \hat{g}\big(\hat{f}^{-1}(\lambda)\big)$, where $f, g\in A$, a uniform algebra on a compact space X, and $\lambda\in\sigma(f)\setminus f(X)$. Here, too, $\chi\big(K(\lambda)\big)$ is subharmonic for the same characteristics χ as above, cf. [3], [5], [8], [10], [17], [18].

Since this approach has resulted already in many interesting applications to uniform algebras and operator theory (see [1], [2], [3], [8], [12], [16], [18]), it seems worthwhile to find out some general and easily applicable conditions on χ, that would guarantee subharmonicity of $\chi\big(K(\lambda)\big)$ for $K(\lambda) = \sigma\big(a(\lambda)\big)$ or $g\big(f^{-1}(\lambda)\big)$. (Cf. [1], Ch. 3, § 1, Remarque.)

Incidentally, each concrete χ mentioned above fulfils trivially the following condition. (This observation was made by B. Aupetit.)

(*) If $a\colon G\to A$ is analytic, and A is a commutative Banach algebra then $\lambda\to\chi\big(\sigma(a(\lambda))\big)$ is subharmonic.

In this respect it is suprising to observe that, in general, (*) is "almost" all what is needed (up to mild topological assumptions). More specifically we prove:

THEOREM 1. *Let* $\chi: F_c(C) \to [-\infty, +\infty)$ *satisfy* (*) ($F_c(C)$ *denotes the family of all compact non-empty subsets of* C). *Assume moreover*:

(i) $\chi(K) \leqslant \chi(L)$ *whenever* $K \subset L$;

(ii) $\chi\left(\bigcap K_n\right) = \lim \chi(K_n)$ *if* $K_{n+1} \subset K_n$, $K_n \in F_c(C)$, $n = 1, 2, \ldots$

Then for every analytic function $a(\lambda)$, *into a Banach algebra* A, *the function* $\lambda \to \chi\big(\sigma(a(\lambda))\big)$ *is subharmonic. Similarly, for every* $f, g \in A$, *a uniform algebra on a compact set* X, *the function* $\lambda \to \chi\big(g(f^{-1}(\lambda))\big)$ *is subharmonic in* $\sigma(f) \setminus f(X)$.

In Section 2 we give a more general result (Theorem 2), for χ defined on subsets of $C \times F_c(C)$, and we show an application. Section 3 contains the proofs.

2. Results. In order to formulate Theorem 2 we need some auxiliary notions.

It is convenient to consider $F_c(C)$ with the upper semi-finite topology (shortly $\varkappa$-topology, cf. [6], § 18.II). It has the property that every u.s.c. map (= upper semi-continuous; cf. [6], § 18.I) is $\varkappa$-continuous. The topology of the Cartesian product of C and $(F_c(C), \varkappa)$ is also called $\varkappa$-topology. The upper limit of a sequence (K_n) of closed sets (cf. [6], § 29, III, Definition) is denoted by $\mathrm{Ls}\, K_n$.

DEFINITION 1. A function $\chi: H \to [-\infty, +\infty)$, where $H \subset C \times F_c(C)$ is $\varkappa$-open, is said to be *subharmonic* if it fulfils conditions (i) and (ii):

(i) For every sequence $(\lambda_n, K_n) \in H$, such that $\lambda_n \to \lambda_0$, and $\mathrm{Ls}\, K_n \subset K_0$, and $(\lambda_0, K_0) \in H$, we have

$$\limsup \chi(\lambda_n, K_n) \leqslant \chi(\lambda_0, K_0).$$

(ii) If m is an integer, $\varepsilon > 0$ and $(a_0, b_0, \ldots, a_m, b_m) \in C^{2m+2}$ is such that $\left(a_0, \bigcup_{i=1}^m D^-(a_i, \varepsilon)\right) \in H$, then the function

$$\lambda \to \chi\Big(a_0 + \lambda b_0, \bigcup_{i=1}^{m} D^-(a_i + \lambda b_i, \varepsilon)\Big)$$

is subharmonic in a neighbourhood of 0 (here D^- denotes the closed disc of given centre and radius).

In case χ does not depend on λ, Def. 1 agrees with assumptions on χ in Theorem 1. This theorem follows from Theorem 2 (cf. Sec. 3), due to the fact that the multifunctions $\sigma(a(\lambda))$ and $\hat{g}(\hat{f}^{-1}(\lambda))$ are analytic in the sense of the following definition.

DEFINITION 2 ([7], [11]). An u.s.c. multifunction $K: G \to F_c(C)$, where $G \subset C$ is open, is said to be *analytic* if the set

$$U = \{(\lambda, z):\ \lambda \in G, z \notin K(\lambda)\}$$

is a domain of holomorphy.

THEOREM 2. *Let* $G \subset C$ *be open and* $H \subset C \times F_c(C)$ *be* $\varkappa$*-open. Assume that* $K: G \to F_c(C)$ *is an analytic set-valued function*, $\chi: H \to [-\infty, +\infty)$ *is subharmonic and* $\{(\lambda, K(\lambda)):\ \lambda \in G\} \subset H$.

Then the function $\lambda \to \chi(\lambda, K(\lambda))$ *is subharmonic in* G.

The following Corollary is an immediate application of Theorem 2. (It was announced in [9]; it may be also obtained by applying [11], Th. 3.2, (v); Cor. 5.1 (iii) and Def. 4.1.)

COROLLARY. *Let* $G \subset C$ *be open and let* $K: G \to F_c(C)$ *be an analytic set-valued function. Assume that* $\psi(\lambda, z_1, \ldots, z_n)$ *is a plurisubharmonic function defined in a neighbourhood* W *of the set* $\{(\lambda, z_1, \ldots, z_n):\ \lambda \in G,\ z_i \in K(\lambda), i = 1, \ldots, n\}$.

Then the function $\varphi(\lambda) = \max\{\psi(\lambda, z_1, \ldots, z_n):\ z_i \in K(\lambda),\ i = 1, \ldots, n\}$ *is subharmonic in* G.

Observe that if we specialize $\psi(\lambda, z_1, \ldots, z_n) = \big(2/n(n-1)\big) \sum_{i<j} \log|z_i - z_j|$, then we obtain subharmonicity of nth diameter of $K(\lambda)$, mentioned in the introduction (cf. [10] for another proof for $K(\lambda) = \sigma(a(\lambda))$ and [8], [10] in case of $g(f^{-1}(\lambda))$).

Proof. Put $H = \{(\lambda, K) \in C \times F_c(C):\ \{\lambda\} \times K^n \subset W\}$; by [6], § 18.II this set is $\varkappa$-open. We define

$$\chi(\lambda, K) = \max\{\psi(\lambda, z_1, \ldots, z_n):\ z_i \in K,\ i = 1, \ldots, n\} \tag{1}$$

for $(\lambda, K) \in H$. Since $\varphi(\lambda) = \psi(\lambda, K(\lambda))$, we have to show that $\chi: H \to [-\infty, +\infty)$ is subharmonic in sense of Definition 1. Condition (i) of Definition 1 is satisfied, for ψ is u.s.c. (details omitted). Assume that a_i, b_i, ε are the same as in Condition (ii), Def. 1.

Observe that

$$\begin{aligned}\varphi(\lambda) &= \chi\Big(a_0 + \lambda b_0, \bigcup_{k=1}^{m} D^-(a_k + \lambda b_k, \varepsilon)\Big)\\ &= \max\{\psi(a_0 + \lambda b_0, a_{k(1)} + \lambda b_{k(1)} + z_{k(1)}, \ldots, a_{k(n)} + \lambda b_{k(n)} + z_{k(n)}:\\ &\qquad 1 \leqslant k(i) \leqslant m,\ |z_{k(i)}| \leqslant \varepsilon,\ i = 1, \ldots, n\}.\end{aligned}$$

Thus $\varphi(\lambda)$ can be represented as the supremum of a family of subharmonic functions (defined in a common neighbourhood of zero) and since φ is u.s.c., therefore it is subharmonic by [15], Sec. 9.6 (in some neighbourhood of zero). ∎

The detailed proof of Theorem 2 is given in Section 3. Now we describe its basic steps. The following approximation lemma helps to reduce the general problem to simpler cases.

LEMMA 1. *Let* $G \subset C$ *be open and* $K: G \to F_c(C)$ *and* $K_n: G \to F_c(C)$ *be u.s.c. multifunctions, and let* $\chi: H \to [-\infty, +\infty)$, *where* $H \subset C \times F_c(C)$ *is* $\varkappa$*-open, satisfy Condition* (i) *of Definition* 1.

Assume that the sets $\{(\lambda, K(\lambda)) \colon \lambda \in G\}$ *and* $\{(\lambda, K_n(\lambda)) \colon \lambda \in G\}$ *are contained in* H, *and for every compact subset* $Z \subset G$ *it holds*

$$\sup\{|z| \colon z \in K_n(\lambda),\ \lambda \in Z,\ n = 1, 2, \ldots\} < \infty. \tag{2}$$

If, moreover, the relations $K(\lambda) \subset \bigcap_n K_n(\lambda)$ *and* $\operatorname{Ls} K_n(\lambda) = K(\lambda)$ *hold, and all functions*

$$\varphi_n(\lambda) = \chi(\lambda, K_n(\lambda))$$

are subharmonic in G, *then the function*

$$\varphi(\lambda) = \chi(\lambda, K(\lambda))$$

is subharmonic as well.

The next two lemmas cover the non-trivial part of the proof of Theorem 2; the first one may be considered as an extension of the usual maximum principle for plurisubharmonic functions (consider e.g. χ defined by (1)).

Lemma 2. (i) *Let* $H \subset F_c(C)$ *be* $\varkappa$*-open, and* $\chi \colon H \to [-\infty, +\infty)$ *be subharmonic. Then for every* $Z \in H$ *it holds* $\chi(Z) = \chi(\partial Z)$. *In particular, if* $X, Z \in H$ *and* $\partial Z \subset X \subset Z$, *then* $\chi(X) = \chi(Z)$.

(ii) *If, moreover,* $H = F_c(C)$, *then* $\chi(Z) = \chi(Z\hat{\ })$ *for every* $Z \in F_c(C)$ (*where* $Z\hat{\ }$ *denotes the polynomial hull of* Z).

Lemma 3. *Theorem* 2 *holds under the additional assumption that there is a neighbourhood* V *of* $(G \times \boldsymbol{C}) \cap \partial U$, *where*

$$U = \{(\lambda, z) \in G \times \boldsymbol{C} \colon z \notin K(\lambda)\},$$

and a strictly pulrisubharmonic function $\varrho \colon V \to \boldsymbol{R}$, *such that*

$$U \cap V = \{(\lambda, z) \in V \colon \varrho(\lambda, z) < 0\}.$$

The proof is an application of a result of E. E. Levi (in that being similar to [11], Sec. 2 and [19]), of Lemma 2, and of the following criterion for subharmonicity (cf. [10], Proposition which is a direct consequence of [15], Sec. 9.4, p. 58).

Proposition 1. *Let* $G \subset C$ *be open and* $\varphi \colon G \to [-\infty, +\infty)$ *be* u.s.c. *Assume that for every* $\lambda_0 \in G$ *there are* $\varepsilon > 0$ *and a subharmonic function* $\psi \colon D(\lambda_0, \varepsilon) \to [-\infty, +\infty)$, *such that* $\psi(\lambda_0) = \varphi(\lambda_0)$ *and* $\psi(\lambda) \leqslant \varphi(\lambda)$ *for* $\lambda \in D(\lambda_0, \varepsilon)$. *Then* φ *is subharmonic.*

Once we have Lemma 3 we prove Theorem 2 approximating (in sense of Lemma 1) a given multifunction $K(\lambda)$ by multifunctions $K_n(\lambda)$ satisfying assumptions of Lemma 3.

3. Proofs.

Proof of Lemma 1. By Definition 1, (i), if $K \subset K'$, then $\chi(\lambda, K) \leqslant \chi(\lambda, K')$, and, if $\bigcup_n K_n$ is relatively compact, $\limsup \chi(\lambda, K_n) \leqslant \chi(\lambda, \operatorname{Ls} K_n)$. Therefore $\varphi(\lambda) = \limsup \varphi_n(\lambda)$. Moreover, $\varphi(\lambda)$ is u.s.c. by Definition 1, (i); the sequence (φ_n) is locally uniformly bounded (from the above by (2), and φ_n are subharmonic: therefore by [15], § 9.6, $\varphi(\lambda) = \limsup \varphi_n(\lambda)$ is subharmonic, too). ∎

Proposition 2. *Let* $H \subset C \times F_c(C)$ *be* $\varkappa$*-open and let* $\chi \colon H \to [-\infty, +\infty)$ *satisfy condition* (i) *of Definition* 1. *Then each of the following conditions is equivalent to subharmonicity of* χ.

(i) *For every* $\varepsilon > 0$ *and for every integer* m *the function*

$$\psi(z_0, z_1, \ldots, z_m) = \chi\Big(z_0, \bigcup_{i=1}^{m} D^-(z_i, \varepsilon)\Big)$$

is plurisubharmonic on the open set on which it is defined.

(ii) *For every integer* m, $\varepsilon_1 > 0, \ldots, \varepsilon_m > 0$, *and functions* $a_0, a_1, \ldots, a_m$, *analytic in a neighbourhood of* 0, *the function* $\lambda \to \chi(a_0(\lambda), K(\lambda))$ *is subharmonic in a neighbourhood of* 0, *where*

$$K(\lambda) = \bigcup_{i=1}^{m} D^-(a_i(\lambda), \varepsilon_i). \tag{3}$$

(iii) *Let* $G \subset \boldsymbol{C}$ *be open, and* $f \colon G \times Y \to \boldsymbol{C}$ *be continuous, where* Y *is a compact space. Assume that* $f(\cdot, y)$ *is analytic for every* y *and* $\{(\lambda, f(\{\lambda\} \times Y)) \colon \lambda \in G\} \subset H$. *Then the function* $\lambda \to \chi(\lambda, f(\{\lambda\} \times Y))$ *is subharmonic.*

Proof. Condition (i) is equivalent to subharmonicity of χ. Indeed, by our assumptions on χ, function ψ is u.s.c., and so it is plurisubharmonic if and only if its restriction to every complex line is subharmonic (where it is defined). But the latter condition is the same as (ii) of Definition 1. For convenience we consider a modification of condition (ii). (ii)′ = (ii) with all $\varepsilon_i = \varepsilon > 0$.

(i) ⇒ (ii)′. The function (3) is, in this case, the composite of the plurisubharmonic function ψ with the analytic function $\lambda \to (a_0(\lambda), a_1(\lambda), \ldots, a_m(\lambda))$, and so it is subharmonic.

(ii)′ ⇒ (iii). Take any G_0 relatively compact in G; we will show that $\varphi(\lambda) = \chi(\lambda, f(\{\lambda\} \times Y))$ is subharmonic in G_0. Put $K(\lambda) = f(\{\lambda\} \times Y)$ for $\lambda \in G_0$; we apply Lemma 1 to show that the function $\varphi(\lambda) = \chi(\lambda, K(\lambda))$ is subharmonic in G_0. Take a sequence $(\varepsilon_n)_{n=1}^{\infty}$ with $\lim \varepsilon_n = 0$ and $\varepsilon_n > 0$, $n = 1, 2, \ldots$ The set

$$\{(\lambda, z) \in G_0^- \times \boldsymbol{C} \colon z \in K(\lambda)\}$$

is compact and covered by open sets

$$V(y) = \{(\lambda, z) \in G \times \boldsymbol{C} \colon |a(\lambda, y) - z| < \varepsilon_n\},$$

where $y \in Y$; we take a finite subcovering $V(y_{n,1}), \ldots, V(y_{n,j(n)})$ and define

$$K_n(\lambda) = \bigcup_{j=1}^{j(n)} D^-\big(a_i(\lambda, y_{n,i}), \varepsilon_n\big).$$

One checks easily (we omit this) that the multifunctions

$$K_n\colon G_0 \to F_c(C); \qquad K\colon G_0 \to F_c(C)$$

satisfy all assumptions of Lemma 1. In particular, functions $\varphi_n(\lambda) = \chi\big(\lambda, K_n(\lambda)\big)$ are subharmonic by condition (ii)' and so is φ, by Lemma 1.

(iii) $\Rightarrow$(ii). Just take $Y = \bigcup_{i=1}^m D^-(d_i, \varepsilon_i)$, so that all discs are mutually disjoint and put $f(\lambda, z) = a_i(\lambda) + z - d_i$, if $z \in D^-(d_i, \varepsilon_i)$. The rest follows. Finally (ii) implies subharmonicity of χ immediately. ∎

Proof of Theorem 1. We will apply Theorem 2 to $K(\lambda) = \sigma\big(a(\lambda)\big)$ and $g\big(f^{-1}(\lambda)\big)$; this can be done because these two multifunctions are analytic in sense of Definition 2 by [11], Corollaries 3.3 and 3.4. Next, in case χ does not depend on λ, conditions (i) and (ii) of Theorem 1 imply condition (i) of Definition 1. Indeed, if $\mathrm{cl}\big(\bigcup_{n=1}^\infty K_n\big)$ is compact, then $\mathrm{Ls}K_n = \bigcap_n A_n$, where $A_n = \bigcup_{i \geqslant n} K_i$ (cf. [6], § 29.IV.8). Since $K_n \subset A_n$ for every n, and since (A_n) is a decreasing sequence of compact sets, we have $\chi(\mathrm{Ls}K_n) = \chi(\bigcap_n A_n) = \lim \chi(A_n) \geqslant \limsup \chi(K_n)$, as desired. Finally, $(*)$ implies condition (iii) of Proposition 2 and so χ is subharmonic in sense of Definition 1 (indeed: put $A = C(Y)$ and define $a(\lambda) \in A$ by $a(\lambda)(y) = f(\lambda, y)$; the details follow easily). ∎

Proof of Lemma 2. Observe first that (i) $\Rightarrow$(ii); clearly $\partial(Z\hat{\ }) \subset Z \subset Z\hat{\ }$; since χ is monotonous we have $\chi\big(\partial(Z\hat{\ })\big) \leqslant \chi(Z) \leqslant \chi(Z\hat{\ })$. The extreme terms being equal, $\chi(Z) = \chi(Z\hat{\ })$.

Concerning part (i), take a sequence (ε_n) with $\varepsilon_n > 0$ and $\lim \varepsilon_n = 0$, and cover ∂Z by open discs $D(z_{n,i}, \varepsilon_n)$, where $z_{n,1}, \ldots, z_{n,m(n)} \in \partial Z$. Put $Z_n = Z \cup A_n$, where $A_n = \bigcup_{i=1}^{m(n)} D^-(z_{n,i}, \varepsilon_n)$. Then $\partial Z \subset A_n$, for $n = 1, 2, \ldots$ $\mathrm{Ls}A_n = \partial Z$, and $Z \subset Z_n$, $\mathrm{Ls}Z_n = Z$. By these relations and properties of χ it holds $\chi(Z) = \limsup \chi(Z_n)$ and $\chi(\partial Z) = \limsup \chi(A_n)$. Thus to get $\chi(Z) = \chi(\partial Z)$ it is enough to prove that $\chi(Z_n) = \chi(A_n)$, which follows from the following fact.

Assertion. Let χ, H and Z satisfy assumptions of Lemma 2. Let $\varepsilon > 0$ and $z_1, \ldots, z_m \in \partial Z$ be such that $\partial Z \subset \bigcup_{i=1}^m D(z_i, \varepsilon)$ and $Z \cup \bigcup_{i=1}^m D^-(z_i, \varepsilon) \in H$. Define $K\colon C \to H$ setting $K(0) = Z \cup \bigcup_{i=1}^m D^-(z_i, \varepsilon)$ and, for $t \neq 0$, $K(t) = \bigcup_{i=1}^m D^-(z_i, \varepsilon)$. Then $t \to \chi\big(K(t)\big)$ is subharmonic.

This subharmonic function is constant on $C \setminus \{0\}$, therefore by [13], II.4, Remark, it must be constant on C. Applying this to above situation we get $\chi(Z_n) = \chi(A_n)$, as desired.

It remains to prove the Assertion. We will apply Lemma 1; first we approximate the multifunction K by a suitable sequence K_n. Fix $\eta > 0$ such

that $B^-(\partial Z, 4\eta) \subset \bigcup_{i=1}^m D(z_i, \varepsilon)$, where $B^-(\partial Z, 4\eta)$ denotes $\{z \in C\colon \mathrm{dist}(z, \partial Z) \leqslant 4\eta\}$. Fix a point z_0 outside $K(0)$ and consider multifunctions of the form $L_a(t) = z_0 + at + D^-(0, \eta)$; we construct K_n with their help. Denote $\mathrm{Int}\,L_a = \{(t, z)\colon |z - z_0 - at| < \eta\}$ and by $\mathrm{Gr}(L_a)$ the graph of L_a, that is $\{(\lambda, z)\colon z \in L_a(\lambda)\}$. For every n there is some $\delta_n > 0$, such that whenever $\mathrm{Gr}(L_a)$ intersects $D^-(0, \delta_n) \times B^-(Z, \eta)$ and $L_a(t) \subset B^-(Z, \eta)$, then $|t| \leqslant 1/n$. (We omit an easy proof.) Observe that the sets $\mathrm{Int}(L_a)$, $a \in C$, cover $\partial D(0, \delta_n) \times Z$, and choose a finite subcovering, say $\mathrm{Int}(L_{a_i})$, $= 1, 2, \ldots, j(n)$; further on we write L_i instead of L_{a_i}. Put

$$F_i = \{t \in C\colon L_i(t) \subset B^-(Z, \eta)\};$$

we define K_n as the multifunction whose graph is equal to the set

$$\bigcup_{i=1}^{j(n)} \mathrm{Gr}(L_i | F_i) \cup \mathrm{Gr}(K) \cup D^-(0, \delta_n) \times Z, \tag{4}$$

where $L_i | F_i$ denotes the restriction of L_i to F_i. Now it is clear that K_n is an u.s.c. multifunction mapping C into $H \subset F_c(C)$. Moreover, by definition of δ_n and by (4) it holds

$$\mathrm{Gr}(K) \subset \mathrm{Gr}(K_n) \subset \mathrm{Gr}(K) \cup D^-(0, 1/n);$$

therefore for every $t \in C$ we have $K(t) \subset K_n(t)$, $n = 1, 2, \ldots$ and $\mathrm{Ls}K_n(t) = K(t)$.

Observe now that if we show that $\varphi_n(t) = \chi\big(K_n(t)\big)$ is subharmonic, then $(K_n)_{n=1}^\infty$ and K will satisfy all assumptions of Lemma 1 and so subharmonicity of $t \to \chi\big(K(t)\big)$ will follow. By Proposition 2, (ii) it is enough to show that each multifunction K_n is locally of the form (3). We consider three cases. If $t_0 \in D(0, \delta_n)$, let $b_1, \ldots, b_s$ be an η-net of Z, then, in a neighbourhood of t_0, it holds

$$K_n(t) = \bigcup_{i=1}^m D^-(z_i, \varepsilon) \cup \bigcup_{i=1}^s D^-(b_i, \eta),$$

as desired. If $t_0 \in \partial D(0, \delta_n)$, then the open sets $\mathrm{Int}(L_i)$, $i = 1, 2, \ldots, j(n)$, cover $\{t_0\} \times Z$, and so they also cover $D^-(t_0, \mu) \times Z$ for $\mu > 0$ small enough. Consequently the representation

$$K_n(t) = \bigcup_{i=1}^m D^-(z_i, \varepsilon) \cup \bigcup_{i=1}^s D^-(b_i, \eta)$$

holds in $D(t_0, \mu)$, too. The last case, $t_0 \notin D^-(0, \delta_n)$, is more delicate. Put

$$G_i = \{t \in C\colon L_i(t) \subset B(Z, \eta)\},$$

where

$$B(Z, \eta) = \{z \in C\colon \mathrm{dist}(z, Z) < \eta\};$$

clearly G_i is open and $G_i^- \subset F_i$. Divide indices $\{1, 2, \ldots, j(n)\}$ into three sets

$$I_1 = \{i:\ t_0 \in G_i\}; \quad I_2 = \{i:\ t_0 \in F_i \setminus G_i\} \quad \text{and} \quad I_3 = \{i:\ t_0 \notin F_i\}.$$

Only in case $i \in I_2$ the behaviour of $L_i|F_i$ near t_0 needs some comments. If $t_0 \in F_i \setminus G_i$, then $L_i(t_0) \subset B^-(Z, \eta)$ but $L_i(t_0) \not\subset B(Z, \eta)$, hence there is an $x \in L_i(t_0) \cap \partial B(Z, \eta)$ and so for some $x_1 \in Z$ we have $|x - x_1| = \eta$. Clearly $x_1 \in \partial Z$ and so

$$L_i(t_0) \subset B^-(\partial Z, 3\eta) \qquad (\operatorname{diam} L_i(t_0) = 2\eta),$$

and

$$L_i(t) \subset B(\partial Z, 4\eta) \subset \bigcup_{i=1}^{m} D^-(z_i, \varepsilon)$$

if t is near t_0. Consequently, in a neighbourhood of t_0, we have

$$K(t) = \bigcup_{i=1}^{m} D^-(z_i, \varepsilon) \cup \bigcup_{i \in I_1} L_i(t).$$

This ends the proof.

Proof of Lemma 3. We will apply Proposition 1 to show that $\varphi(\lambda) = \chi(\lambda, K(\lambda))$ is subharmonic. For each $\lambda_0 \in G$ we find $r = r(\lambda_0) > 0$ and an u.s.c. multifunction $L = L_{\lambda_0}$: $D(\lambda_0, r) \to F_c(C)$, such that $\psi(\lambda) = \chi(\lambda, L(\lambda))$ is subharmonic, $L(\lambda) \subset K(\lambda)$ for $\lambda \in D(\lambda_0, r)$ and $\partial K(\lambda_0) \subset L(\lambda_0)$. Observe that $\psi(\lambda) = \chi(\lambda, L(\lambda)) \leqslant \varphi(\lambda)$ (since $\chi(\lambda, \cdot)$ is monotonous) and $\psi(\lambda_0) = \varphi(\lambda_0)$ (by Lemma 2); thus the assumptions of Proposition 1 are fulfilled, and so $\varphi(\lambda) = \chi(\lambda, K(\lambda))$ is subharmonic. It remains to construct L with the desired properties.

Fix $\lambda_0 \in G$ and denote, for $y = (\lambda_0, z_0) \in V$,

$$p^y(\lambda, z) = \varrho_\lambda(y)(\lambda - \lambda_0) + \varrho_z(y)(z - z_0) + \varrho_{\lambda\lambda}(y)(\lambda - \lambda_0)^2 + 2\varrho_{\lambda z}(y)(\lambda - \lambda_0)(z - z_0) + \varrho_{zz}(y)(z - z_0)^2, \tag{5}$$

(where $\varrho_z = \frac{1}{2}\left(\frac{\partial \varrho}{\partial z_1} - i\frac{\partial \varrho}{\partial z_2}\right)$, $z = z_1 + iz_2$, etc., cf. [4], I.A.2). By [4], IX.B.2, polynomials p^y have the following property: for every $y_0 = (\lambda_0, z_0) \in V$, such that $\varrho(y_0) = 0$, there is $r = r(y_0) > 0$ such that whenever $y' = (\lambda', z') \in D(\lambda_0, r) \times D(z_0, r)$ and $\varrho(y') = 0$, then

$$\{(\lambda, z) \in D(\lambda_0, r) \times D(z_0, r):\ \varrho(\lambda, z) \leqslant 0,\ p^{y'}(\lambda, z) = 0\} = \{y'\}. \tag{6}$$

Denote $Y = \{\lambda_0\} \times \partial K(\lambda_0)$, and let $\bigcup_{i=1}^{m} D(\lambda_0, r_i) \times D(z_0, r_i)$, where $r_i = r(\lambda_0, z_i)$, $z_i \in \partial K(\lambda_0)$, be a finite covering of the compact set Y; by [6], § 41.VI Cor. 4d there is $r_0 > 0$ such that for every $z_0 \in \partial K(\lambda_0)$ it holds

$$D(\lambda_0, r_0) \times D(z_0, r_0) \subset D(\lambda_0, r_i) \times D(z_i, r_i)$$

for some i. This and (6) implies that for every $y = (\lambda_0, z_0) \in Y$

$$\{p^y = 0\} \cap D(\lambda_0, r_0) \times D(z_0, r_0) \subset \operatorname{Gr}(K). \tag{7}$$

Denote by C_y^r, for $r > 0$ and $y \in Y$, the connected component of $\{p^y = 0\} \cap D^-(\lambda_0, r) \times C$ containing y. Construction of the multifunction L is based on the following assertion (to be shown later).

ASSERTION. *There is $r > 0$ such that for every $y = (\lambda_0, z_0) \in Y$*

$$C_y^r \subset D^-(\lambda_0, r) \times D(z_0, r_0), \tag{8}$$

$$\pi(C_y^r) = D^-(\lambda_0, r), \quad \textit{where} \quad \pi(\lambda, z) = \lambda. \tag{9}$$

Let L' be the multifunction whose graph is $\bigcup_{y \in Y} C_y^r$. By (9) $L'(\lambda) \neq \emptyset$ for every $\lambda \in D^-(\lambda_0, r)$. Since ϱ is C^2, one can show (cf. (5)) that the map $y \to C_y^r$: $Y \to F_c(C^2)$ is u.s.c., and so $L(\lambda) \in F_c(C)$ for $\lambda \in D^-(\lambda_0, r)$ and, moreover, L': $D^-(\lambda_0, r) \to F_c(C)$ is u.s.c., too. Put $L = L'|D(\lambda_0, r)$; by (8) and (7) we have $L(\lambda) \subset K(\lambda)$ for $\lambda \in D(\lambda_0, r)$. Finally $\partial K(\lambda_0) = Y \subset L(\lambda_0)$, by construction.

Thus we have all properties of L that we need, except for subharmonicity of $\psi(\lambda) = \chi(\lambda, L(\lambda))$. For this we apply Lemma 1 first. Take (ε_n) with $\lim \varepsilon_n = 0$ and $\varepsilon_n > 0$. Clearly for every $y \in Y$ the set $V(y) = C_y^r + \{0\} \times D(0, \varepsilon_n)$ is relatively open in $D^-(0, r) \times C$, and $\operatorname{Gr}(L') \subset \bigcup_{y \in Y} V(y)$. Choose a finite covering $\{V(y_{n,1}), \ldots, V(y_{n,y(n)})\}$ of $\operatorname{Gr}(L')$, and let L_n' be the multifunction such that

$$\operatorname{Gr}(L_n') = \bigcup_{i=1}^{j(n)} \left(C_{y_{n,i}}^r + \{0\} \times D^-(0, \varepsilon_n)\right).$$

Put $L_n = L_n'|D(\lambda_0, r)$. By the construction $L(\lambda) \subset L_n(\lambda)$ for $n = 1, 2, \ldots$ and $\lambda \in D(\lambda_0, r)$. Since the remaining assumptions of Lemma 1 are fulfilled, it is enough to prove that $\psi_n(\lambda) = \chi(\lambda, L_n(\lambda))$, $n = 1, 2, \ldots$ are subharmonic, in order to get that $\psi(\lambda)$ is subharmonic. Let $\lambda_1 \in D(\lambda_0, r)$ be arbitrary, and let $\delta > 0$ be small enough, so that for every $i = 1, \ldots, j(n)$ the set $C_{y_{n,i}}^r \cap (D(\lambda_1, \delta) \times C)$ is the graph of a single or two-valued analytic function, with at most one ramification point at λ_1. A closer look at (5) assures us that each of these functions has one of the following three forms: either $a_i(\lambda)$, or $b_j(\lambda) \pm \sqrt{(\lambda - \lambda_1)^2}\, c_j(\lambda)$, or $d_k(\lambda) \pm \sqrt{\lambda - \lambda_1}\, e_k(\lambda)$, where a_i, b_j, c_j, d_k, e_k are regular in $D(\lambda_1, \delta)$. Introduce a uniformizing variable t, so that $\lambda - \lambda_1 = t^2$. Put

$$R(t) = \bigcup_i D^-(a_i(\lambda), \varepsilon_n) \cup \bigcup_j D^-(b_j(\lambda) + t^2 c_j(\lambda), \varepsilon_n) \cup \bigcup_j D^-(b_j(\lambda) - t^2 c_j(\lambda), \varepsilon_n) \cup \bigcup_k D^-(d_k(\lambda) + t e_k(\lambda), \varepsilon_n) \cup \bigcup_k D^-(d_k(\lambda) - t e_k(\lambda), \varepsilon_n),$$

where $\lambda = \lambda_1+t^2$, and $|t| < \delta^{1/2}$. By Proposition 2 (ii) the function $\psi^{\sim}(t) = \chi\big(\lambda_1+t^2, R(t)\big)$ is subharmonic in $D(0, \delta^{1/2})$. On the other hand, $\psi_n(\lambda) = \psi_n(\lambda_1+t^2) = \psi^{\sim}(t)$. Since for $t \neq 0$ the mapping $t \to t^2+\lambda_1$ is locally biholomorphic, ψ_n is subharmonic in $D(\lambda_1, \delta)\setminus\{\lambda_1\}$. Moreover ψ_n is bounded from the above in $D(\lambda_1, \delta)$, and $\limsup_{s>0}\psi_n(\lambda_1+s) = \limsup_{s>0}\psi^{\sim}(s^{1/2}) = \psi^{\sim}(0) = \psi_n(\lambda_1)$ (by the Oka–Rothstein theorem [1], App. II, Th. 11); therefore by a result due to Brelot [13], Th. III. 30, ψ_n is subharmonic in $D(\lambda_1, \delta)$.

We close the proof of Lemma 3 with the proof of the Assertion. Observe first that

$$(10)\qquad \big(\varrho_z(y), \varrho_{zz}(y)\big) \neq (0, 0) \quad \text{for all } y \in Y.$$

Indeed, if not, then by (5) $\{\lambda = \lambda_0\} = \{\lambda_0\}\times C \subset \{p^y = 0\}$, and so, by (7), $D(z_0, r_0) \subset K(\lambda_0)$, which is false since $z_0 \in \partial K(\lambda_0)$. By (10) there are compact sets A and B, such that $A\cup B = Y$, and $\varrho_z(y) \neq 0$ for every $y \in A$, and $\varrho_{zz}(y) \neq 0$ for every $y \in B$.

We consider two cases.

Case A: there is $r > 0$ such that for every $y \in A$ relations (8) and (9) hold.

Rewrite polynomials (5) as

$$p^y(\lambda, z) = a^y(\lambda)(z-z_0)^2+b^y(\lambda)(z-z_0)+c^y(\lambda)$$

where $a^y(\lambda) = \varrho_{zz}(y)$;

$$b^y(\lambda) = \varrho_z(y)+2\varrho_{\lambda z}(y)(\lambda-\lambda_0),$$

and

$$c^y(\lambda) = \varrho_\lambda(y)(\lambda-\lambda_0)+\varrho_{\lambda\lambda}(y)(\lambda-\lambda_0)^2.$$

Let us remark that if $|ac|$ is small in comparison with $|b|$, then it is convenient to represent the roots of $a(z-z_0)^2+b(z-z_0)+c$ in the following form

$$(11)\qquad z_+-z_0 = -2c/\big(b\big(1+(1-4acb^{-2})^{1/2}\big)\big),$$

$$(12)\qquad z_--z_0 = -b\big(1+(1-4acb^{-2})^{1/2}\big)/(2a);$$

last formula valid for $a \neq 0$, otherwise z_- does not exist; $u^{1/2}$ denotes here the branch taking 1 at $u = 1$.

Fix M, $\alpha > 0$ such that $\alpha < \inf_{y\in Y}|\varrho_z(y)|$ and $\sup_{y\in Y}|\varrho_{zz}(y)| < M$. Then for every $\varepsilon > 0$ there is $r = r(\varepsilon) > 0$ such that for $y \in Y$ and $\lambda \in D^-(\lambda_0, r)$ it holds

$$|c| = |c^y(\lambda)| < \varepsilon; \qquad \alpha < |b| = |b^y(\lambda)| \quad \text{and} \quad |a| = |a^y(\lambda)| < M.$$

Using these estimations in (11) and (12) we obtain that if $\varepsilon > 0$ is small enough, then for every $\lambda \in D^-(\lambda_0, r)$, $r = r(\varepsilon)$ it holds $z_+^y(\lambda) \in D(z_0, \text{const}\cdot\varepsilon)$,

while $z_-^y(\lambda) \notin D(z_0, \alpha/4M)$; consequently

$$C_y^r = \{(\lambda, z_+^y(\lambda)):\ \lambda \in D^-(\lambda_0, r)\} \subset D^-(\lambda_0, r)\times D(z_0, r_0),$$

for every $y \in Y$ (for small ε), as desired. We get (9) by observing that (11) is well defined (with $a = a^y(\lambda)$, etc.) for every $\lambda \in D^-(\lambda_0, r)$, if $r = r(\varepsilon)$ and $\varepsilon M\alpha^{-2} < 1/8$.

Case B: there is $r > 0$ such that for all $y \in B$ relations (8), (9) hold.

Relation (9) holds for every $y \in B$ because $\varrho_{zz}(y) \neq 0$ (cf. (5)). Choose $\beta > 0$ such that $\beta \leqslant \inf_{y\in B}|\varrho_{zz}(y)|$. Suppose that (8) does not hold on B uniformly, i.e. there are sequences $y(n) = \big(\lambda_0, z(n)\big) \in B$ and $y'(n) = \big(\lambda'(n), z'(n)\big) \in C_{y(n)}^{1/n}$ such that $|z'(n)| \geqslant r_0$; observe that $\lim\lambda'(n) = \lambda_0$. Since $\big|\varrho_{zz}\big(y(n)\big)\big| \geqslant \beta$, the roots $z = z'(n)$ of $p^{y(n)}\big(\lambda'(n), z\big)$ form a bounded sequence; passing to subsequences, if necessary, we may assume that $y(n)\to y = (\lambda_0, z_0)$ and $y'(n)\to y' = (\lambda_0, z')$. In particular,

$$(13)\qquad p^y(y') = 0 = p^y(y) \quad \text{and} \quad |y-y'| \geqslant r_0.$$

Observe that there is no sequence $\lambda''\big(n(k)\big)$, with $n(k)\nearrow\infty$, such that $\lambda''\big(n(k)\big)\to\lambda_0$ and polynomials $p^{y(n(k))}\big(\lambda''\big(n(k)\big), z\big)$ have double roots. Suppose on the contrary that $z''(k)$ are such double roots. Then

$$p^{y(n(k))}\big(\lambda''\big(n(k)\big), z''(k)\big) = 0 = \frac{\partial}{\partial z}\,p^{y(n(k))}\big(\lambda''\big(n(k)\big), z''(k)\big).$$

By the latter equation and by the relations $\lim\lambda''\big(n(k)\big) = \lambda_0$, $\varrho_{zz}\big(y(n(k))\big) > \beta$, the sequence $z''(k)$ converges to some z''; by both equations $p^y(\lambda_0, z'') = 0$ and $\frac{\partial}{\partial z}p^y(\lambda_0, z'') = 0$. Consequently the quadratic polynomial $p^y(\lambda_0, z)$ has at least three roots ($z_0 \neq z'$, cf. (13), and a double one at z''), which is impossible (otherwise $p^y|\{\lambda_0\}\times C = 0$, contrary to (7)).

Thus there is $r' > 0$ such that for every $\lambda \in D^-(\lambda_0, r')$ and for $n = 1, 2, \ldots$, the equation $p^{y(n)}(\lambda, z) = 0$ has two different solutions. Thus the set $\{p^{y(n)} = 0\}\cap D^-(\lambda_0, r')\times C$ is the graph of a two-valued (algebraic) function, without ramification points, and, by the monodromy theorem, has two components (branches): $C_{y(n)}^{r'}$ is one of them and denote by S_n the other. We may also assume that $\{p^y = 0\}\cap D^-(\lambda_0, r')\times C$ has two components, $C_y^{r'}$ and S (use the fact that $p^y(\lambda_0, \cdot)$ does not have double roots and take smaller r', if needed). Since $y'(n)\to y' \in S$, and $\operatorname{Ls} C_{y(n)}^{r'} \subset C_y^{r'}$ and $\operatorname{Ls} S_n \subset S$, therefore $y'(n) \in S_n$ for large n, instead to $C_{y(n)}^{r'}$, as assumed. This settles case B.

The smaller of values of r obtained in cases A and B fulfils the Assertion. ■

Proof of Theorem 2. It suffices to prove that $\varphi(\lambda) = \chi\big(\lambda, K(\lambda)\big)$ is subharmonic on each relatively compact subregion G_0 of G. Fix such G_0;

since K is u.s.c., there is $R > 0$ such that

$$Z = G_0^- \times \{z: |z| = R\} \subset U = \{(\lambda, z) \in G \times C: z \notin K(\lambda)\}.$$

By Definition 2 the set U is a domain of holomorphy and by [4], IX.D.4, there is a sequence $(D_n)_{n=1}^{\infty}$ of strictly pseudoconvex domains (in the sense of Levi) such that $D_n \subset D_{n+1}$, $n = 1, 2, \ldots$, and $\bigcup_n D_n = U$; clearly we may assume that $Z \subset D_1$. Moreover, by [4], IX.A.4, there is a neighbourhood V_n' of ∂D_n, and a strictly plurisubharmonic function ϱ_n': $V_n' \to \boldsymbol{R}$ such that

$$D_n \cap V_n' = \{(\lambda, z) \in V_n': \varrho_n'(\lambda, z) < 0\}.$$

Put

$$U_n = (G_0 \times C) \cap (D_n \cup G_0 \times \{|z| > R\}),$$

and let K_n be the multifunction with the graph $G_0 \times C \setminus U_n$. Now it is clear (we omit the details) that K_n: $G_0 \to F_c(C)$ and is u.s.c.; moreover, if we put $V_n = V_n' \cap (G_0 \times \{|z| < R\})$ and $\varrho_n = \varrho_n' | V_n$, then K_n, ϱ_n and V_n satisfy all assumptions of Lemma 3 and so $\varphi_n(\lambda) = \chi(\lambda, K_n(\lambda))$ is subharmonic in G_0, $n = 1, 2, \ldots$ Observe that multifunctions $K|G_0$, K_n, $n = 1, 2, \ldots$ satisfy all assumptions of Lemma 1 (in particular $\bigcap_n K_n(\lambda) = K(\lambda)$, and $K_{n+1}(\lambda) \subset K_n(\lambda)$, $n = 1, 2, \ldots$ for $\lambda \in G_0$); therefore $\varphi(\lambda) = \chi(\lambda, K(\lambda))$ is subharmonic in G_0). ∎

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