Choosing $c = 1$ gives

$$\int_0^a \Phi^\sim(a/x)x\,dx = \int_0^a \frac{a^2\,dx}{x(3+\log a/x)^2} = a^2/3.$$

On the other hand, for any δ, $\varepsilon > 0$ we have

$$\int_0^a \Phi^\sim(\varepsilon/x)^{1+\delta} x\,dx = \infty.$$

The characterization of A_Φ given by Kerman and Torchinsky [4] shows that $A_\Phi = A_2$; in this case $w \in A_p$ for all $p > 2$ but w is not in A_2.

Let us now apply (4.1) with this choice of Φ. Take $\Psi(t) = t^2 - 1$ for $t > 1$ and 0 otherwise. Then L^Ψ consists of the functions whose restriction to every set of finite measure is in L^2. Since

$$\int_1^t (2s)(t/s)^2(1+\log t/s)^2\,ds = (2/3)t^2(1+\log t)^3,$$

we see that the Hardy–Littlewood maximal function is bounded from $L^2(\log L)^3(w\,dx)$ to $L^2_{\mathrm{loc}}(w\,dx)$ for all $w \in B_\Phi$.

For the case $w(x) = |x|$ on E^1, slightly better weighted bounds for Mf can be obtained by using the fact that w is in the weight class $A(2, 1)$ of Chung, Hunt, and Kurtz [2]. However, if we modify w by redefining $w(x) = |x|/(\log 2/x)^\varepsilon$ for $|x| < 1$, then $w \notin A(2, 1)$ for $\varepsilon > 0$ but $w \in B_\Phi$ for $\varepsilon > 1$.

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H^p spaces over open subsets of R^n

by

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Abstract. Part of the theory of H^p spaces over R^n, originated by C. Fefferman and E. M. Stein [4], is generalized to the case of arbitrary open subsets of R^n. The following subjects are treated: (1) Definition of $H^p(\Omega)$, where Ω is an open subset of R^n, by means of maximal functions; (2) Atomic decomposition for $H^p(\Omega)$; (3) Identification of the duals of $H^p(\Omega)$ with certain function spaces over Ω; (4) The complex method of interpolation for $H^p(\Omega)$ and $L^p(\Omega)$; (5) Extension of a distribution in $H^p(\Omega)$ to a distribution in $H^p(R^n)$. All the results are given in the situation that R^n has a parabolic metric.

1. Introduction. In this paper, we introduce H^p spaces over arbitrary open subsets of R^n by means of certain maximal functions and show that they have some properties similar to the H^p spaces over R^n (for the H^p spaces over R^n, see Calderón–Torchinsky [1], [2] or Torchinsky's book [10; Chapt. XIV]).

We briefly review our results.

Let φ be a function in $C_0^\infty(R^n)$ such that $\operatorname{supp}\varphi \subset \{x \in R^n \mid |x| < 1\}$ (if $x \in R^n$, then $|x|$ denotes the usual Euclidean norm of x) and $\int \varphi(x)\,dx = 1$. For $t > 0$, we define $(\varphi)_t$ by $(\varphi)_t(x) = t^{-n}\varphi(t^{-1}x)$ (we shall modify this definition afterwards; see the next to the last paragraph in this section). Let Ω be an open subset of R^n. For $f \in \mathscr{D}'(\Omega)$, we define the *radial maximal function* $f^+_{\varphi,\Omega}(x)$, $x \in \Omega$, by

$$f^+_{\varphi,\Omega}(x) = \sup\{|\langle f, (\varphi)_t(x-\cdot)\rangle| \mid 0 < t < \operatorname{dis}(x, \Omega^c)\},$$

where Ω^c denotes the complement of Ω (throughout this paper, $\mathscr{D}'(\Omega)$ denotes the set of distributions on Ω and $\langle f, \psi\rangle$, where $f \in \mathscr{D}'(\Omega)$ and $\psi \in C_0^\infty(\Omega)$, means $f(\psi)$; we use the same notation $\langle f, \psi\rangle$ if f is a distribution with compact support and ψ is a smooth function on R^n). For p with $0 < p \leqslant 1$, we define $H^p(\Omega)$ as the set of those $f \in \mathscr{D}'(\Omega)$ for which $f^+_{\varphi,\Omega}$ belongs to $L^p(\Omega)$. We consider $H^p(\Omega)$ a quasinormed linear space by defining the quasinorm of $f \in H^p(\Omega)$ to be equal to the $L^p(\Omega)$-norm of $f^+_{\varphi,\Omega}$. (By a *quasinorm* we mean a function σ on a linear space X which has the following properties: (i) $\sigma(x) > 0$ if $x \neq 0$ and $\sigma(0) = 0$; (ii) $\sigma(\lambda x) = |\lambda|\sigma(x)$ for all scalars λ and all $x \in X$; (iii) there exists a positive constant k such that $\sigma(x+y) \leqslant k(\sigma(x)+\sigma(y))$ for all $x, y \in X$.) Then the maximal inequality given by the author [8] shows that the above definition

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of $H^p(\Omega)$ does not depend on the choice of φ (up to equivalence of quasinorms) and that certain grand maximal functions associated with $f \in H^p(\Omega)$ belong to $L^p(\Omega)$. These results will be given in Section 2.

Let Ω be a proper open subset of $\boldsymbol{R}^n$. Using the characterization of $H^p(\Omega)$ in terms of the grand maximal function, we can prove that elements of $H^p(\Omega)$, $0<p\leqslant 1$, admit a certain atomic decomposition. We define p-atoms in the usual way (see [6] or [7] or Section 3.1 of the present paper). We call f a (p, Ω)-*atom*, where $0<p\leqslant 1$, if $f\in L^\infty(\Omega)$ and if there exists a ball $I=I_f$ such that $2I\subset\Omega$, $5I\cap\Omega^c\neq\emptyset$, $\operatorname{supp} f\subset I$ and $\|f\|_\infty\leqslant|I|^{-1/p}$ (where sI denotes the ball with the same center as I and with radius s times as large, and $|I|$ denotes the Lebesgue measure of I). Then every element of $H^p(\Omega)$, $0<p\leqslant 1$, can be written as a certain linear combination of p-atoms and (p, Ω)-atoms. This result, together with a similar result for $L^p(\Omega)$, $1<p\leqslant\infty$, will be given in Section 3.

Using the atomic decomposition, we obtain the following results. First, the dual of $H^p(\Omega)$, $0<p\leqslant 1$, can be identified with a Lipschitz space (when $0<p<1$) or the BMO space (when $p=1$) if the latter spaces are appropriately defined over Ω. Secondly, the complex method of interpolation can be applied to $H^p(\Omega)$ $(0<p\leqslant 1)$ and $L^p(\Omega)$ $(1<p\leqslant\infty)$ in the same way as in the case $\Omega=\boldsymbol{R}^n$. Thirdly, we can prove that if Ω satisfies a certain condition then every element of $H^p(\Omega)$, $0<p\leqslant 1$, can be extended to an element of $H^p(\boldsymbol{R}^n)$ and, moreover, there exists a bounded linear extension operator from $H^p(\Omega)$ to $H^p(\boldsymbol{R}^n)$. The result on the dual of $H^p(\Omega)$ will be given in Section 4, the complex method of interpolation in Section 5, and the extension in Section 6.

In the sections to follow, we shall give all our results in the situation that $\boldsymbol{R}^n$ is endowed with a group, $\{A(t)\,|\,t>0\}$, of linear transformations. The transformation $A(t)$ is a generalization of the scalar multiplication $x\mapsto tx$. The group $\{A(t)\}$ and the associated metric on $\boldsymbol{R}^n$ were given by A. P. Calderón and A. Torchinsky [1; Section 1]; their results will be summarized at the beginning of the next section. Except for the definition of $(\varphi)_t$, the assertions in the above paragraphs hold true in the case of general $\{A(t)\}$ if one regards $\operatorname{dis}(\cdot,\cdot)$ and ball as those with respect to the metric associated with the group $\{A(t)\}$. The definition of $(\varphi)_t$ in the general case will be given in the next section.

Notation. We use the letter C to denote a positive constant which need not be the same at different occurrences. For the use of C, see also the paragraph marked N.B. in the next section. If E is a measurable subset of $\boldsymbol{R}^n$, $0<p\leqslant\infty$ and if f is a measurable function on E, then $\|f\|_{p,E}$ denotes the $L^p(E)$-norm of f, i.e.,

$$\|f\|_{p,E}=\Big(\int_E|f(x)|^p\,dx\Big)^{1/p}\qquad\text{when } 0<p<\infty$$

and $\|f\|_{\infty,E}=\operatorname{ess\,sup}\{|f(x)|\,|\,x\in E\}$. If $E=\boldsymbol{R}^n$, then we shall abbreviate $\|f\|_{p,E}$ to $\|f\|_p$. If M is a positive number, $\mathscr{A}$ is a family of subsets of $\boldsymbol{R}^n$ and if $\sum_{E\in\mathscr{A}}\chi_E(x)\leqslant M$ for all $x\in\boldsymbol{R}^n$, where χ_E denotes the characteristic function of E, then we say that *the overlap of $\mathscr{A}$ does not exceed M.*

2. Maximal functions and $H^p(\Omega)$. First, we recall some preliminary results about the parabolic metric on $\boldsymbol{R}^n$. For details, see Calderón–Torchinsky [1; Section 1]. Let $\{A(t)\,|\,t>0\}$ be a set of linear transformations for $\boldsymbol{R}^n$ with the following properties: $t\mapsto A(t)x$ is continuous for every $x\in\boldsymbol{R}^n$, $A(t)A(s)=A(ts)$, $A(1)=$ the identity operator, and $|A(t)x|\geqslant t|x|$ if $t\geqslant 1$. Then it follows that there exist real numbers α and β such that $1\leqslant\alpha\leqslant\beta$,

$$t^\alpha|x|\leqslant|A(t)x|\leqslant t^\beta|x|\qquad\text{if } t\geqslant 1,$$
$$t^\beta|x|\leqslant|A(t)x|\leqslant t^\alpha|x|\qquad\text{if } t\leqslant 1.$$

Let γ be the positive number for which $\det A(t)=t^\gamma$ for all $t>0$. We denote by ϱ the unique function on $\boldsymbol{R}^n$ such that $\varrho(x)=1$ if and only if $|x|=1$, and that $\varrho(A(t)x)=t\varrho(x)$ for all $t>0$ and all $x\in\boldsymbol{R}^n$. It has the following properties:

$$\varrho(0)=0,\qquad \varrho(x)>0\quad\text{if } x\neq 0,$$
$$\varrho(x+y)\leqslant\varrho(x)+\varrho(y),\qquad \varrho(-x)=\varrho(x),$$
$$\varrho(x)^\beta\leqslant|x|\leqslant\varrho(x)^\alpha\quad\text{if } |x| \text{ or } \varrho(x)\leqslant 1,$$
$$\varrho(x)^\alpha\leqslant|x|\leqslant\varrho(x)^\beta\quad\text{if } |x| \text{ or } \varrho(x)\geqslant 1.$$

For x and y in $\boldsymbol{R}^n$, we set $\operatorname{dis}(x, y)=\varrho(x-y)$. This is in fact a distance function on $\boldsymbol{R}^n$, and it determines the same uniform topology as the usual Euclidean distance. For $x\in\boldsymbol{R}^n$ and $E\subset\boldsymbol{R}^n$, we define $\operatorname{dis}(x, E)=\inf\{\operatorname{dis}(x, y)\,|\,y\in E\}$ if $E\neq\emptyset$ and $\operatorname{dis}(x, E)=\infty$ if $E=\emptyset$. For a function f on $\boldsymbol{R}^n$ and for $t>0$, we define the function $(f)_t$ by

$$(f)_t(x)=t^{-\gamma}f(A(t^{-1})x).$$

For $x\in\boldsymbol{R}^n$ and $t>0$, we define the ball $B(x, t)$ by

$$B(x, t)=\{y\in\boldsymbol{R}^n\,|\,\operatorname{dis}(x, y)<t\}.$$

If $I=B(x, t)$ is a ball, we call x the center of I and t the radius of I, and we write $x=x_I$ and $t=r_I$. For a ball I and for $s>0$, we denote by sI the ball with the came center as I and with radius sr_I. We have $|B(x, t)|=|B(0, 1)|t^\gamma$.

N.B. Hereafter we fix the Euclidean space $\boldsymbol{R}^n$ and the group $\{A(t)\}$. Every constant denoted by the letter C depends only on the dimension n, the group $\{A(t)\}$ and other explicitly indicated parameters.

Next, we recall the definitions of the Lipschitz classes. For details, see [8; Section III]. For a nonnegative integer m, we denote by $\mathscr{P}_m$ the set of polynomial functions on $\boldsymbol{R}^n$ of degree not exceeding m. For $s>0$ and for $f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$, we set

$$\|f\|_{A(s)}=\sup_{I:\,\text{ball}}\Big[\inf\Big\{r_I^{-\gamma-s}\int_I|f(x)-P(x)|\,dx\,\Big|\,P\in\mathscr{P}_{[s]}\Big\}\Big],$$

where $[s]$ denotes the integer part of s. Let $s > 0$, Ω a proper open subset of $\mathbf{R}^n$ and f a function on Ω. We define $\tilde{f}$ by $\tilde{f}(x) = f(x)$ if $x \in \Omega$ and $\tilde{f}(x) = 0$ if $x \in \mathbf{R}^n \setminus \Omega$. If $\tilde{f} \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, then we set

$$\|f\|_{\Lambda(s;\Omega)} = \|\tilde{f}\|_{\Lambda(s)} + \sup_{x\in\Omega} \{|f(x)|(\operatorname{dis}(x, \Omega^c))^{-s}\}.$$

We denote by $\Lambda(s; \Omega)$ the set of those functions f on Ω such that $\tilde{f} \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $\|f\|_{\Lambda(s;\Omega)} < \infty$. (In the Remark in Section 4, we shall give a characterization of functions f in $\Lambda(s; \Omega)$ which does not use $\tilde{f}$ and which refers exclusively to the behavior of f in Ω.)

Let Φ be a function defined on $\mathbf{R}^n \times \mathbf{R}^n \times (0, \infty)$. We shall say that Φ is a *good kernel* if it satisfies the conditions (i)–(viii) below with some function K: $(0, \infty) \ni s \mapsto K_s \in (0, \infty)$.

(i) $\Phi(x, \cdot, t) \in C_0^\infty(B(x, t))$ for all $x \in \mathbf{R}^n$ and all $t > 0$.

(ii) $\|\Phi(x, \cdot, t)\|_{\Lambda(s)} \leqslant K_s t^{-\gamma-s}$ for all $x \in \mathbf{R}^n$, all $t > 0$ and all $s > 0$.

(iii) $\Phi(\cdot, y, t) \in L^1(\mathbf{R}^n)$ and $\int \Phi(x, y, t)\, dx = 1$ for all $y \in \mathbf{R}^n$ and all $t > 0$.

(iv) For $g \in L^\infty(\mathbf{R}^n)$ with compact support and for $t > 0$, we define the function $g \# \Phi(t)$ by

$$(g \# \Phi(t))(y) = \int g(x)\Phi(x, y, t)\, dx;$$

then

$$\|g \# \Phi(t)\|_{\Lambda(s)} \leqslant K_s \|g\|_{\Lambda(s)} \quad \text{for all } s > 0.$$

(v) If $t > 0$, $g \in L^\infty(\mathbf{R}^n)$ and g has compact support, then $g \# \Phi(t)$ belongs to $C_0^\infty(\mathbf{R}^n)$.

(vi) For every $f \in \mathscr{D}'(\mathbf{R}^n)$ and for every $t > 0$, the function $x \mapsto \langle f, \Phi(x, \cdot, t)\rangle$ is locally integrable on $\mathbf{R}^n$.

(vii) If $f \in \mathscr{D}'(\mathbf{R}^n)$, $t > 0$, $g \in L^\infty(\mathbf{R}^n)$ and g has compact support, then

$$\langle f, g \# \Phi(t)\rangle = \int g(x)\langle f, \Phi(x, \cdot, t)\rangle\, dx.$$

(viii) For every open subset Ω of $\mathbf{R}^n$ and for every $f \in \mathscr{D}'(\Omega)$, the *radial maximal function* $M^+_{\Phi,\Omega}(f)(x)$, $x \in \Omega$, which is defined by

$$M^+_{\Phi,\Omega}(f)(x) = \sup\{|\langle f, \Phi(x, \cdot, t)\rangle| \mid 0 < t < \operatorname{dis}(x, \Omega^c)\},$$

is measurable.

The following are typical examples of good kernels. Take $\varphi \in C_0^\infty(B(0, 1))$ such that $\int \varphi(x)\, dx = 1$. Set $\Phi_1(x, y, t) = (\varphi)_t(x-y)$ and

$$\Phi_2(x, y, t) = (\varphi)_{2^k}(x-y) \quad \text{if } 2^k \leqslant t < 2^{k+1},\ k \in \mathbf{Z}.$$

Then Φ_1 and Φ_2 are good kernels. The radial maximal function associated with Φ_1 coincides with $f^+_{\varphi,\Omega}$ of Section 1.

We shall use the following notation:

$$\|f\|_{p,\Phi,\Omega} = \|M^+_{\Phi,\Omega}(f)\|_{p,\Omega},$$

where Ω is an open subset of $\mathbf{R}^n$, $f \in \mathscr{D}'(\Omega)$, $0 < p \leqslant \infty$ and Φ is a good kernel.

Let Ω be an open subset of $\mathbf{R}^n$, $f \in \mathscr{D}'(\Omega)$ and $s > 0$. We define the *grand maximal function* $f^*_{s,\Omega}$ by

$$f^*_{s,\Omega}(x) = \sup_{\psi} |\langle f, \psi\rangle|, \quad x \in \mathbf{R}^n,$$

where the supremum is taken over all those ψ for which there exists a $t = t_\psi > 0$ such that $\psi \in C_0^\infty(B(x, t) \cap \Omega)$ and $\|\psi\|_{\Lambda(s;B(x,t)\cap\Omega)} \leqslant t^{-\gamma-s}$.

The following theorem is given in [8].

THEOREM A. *If Φ is a good kernel, $s > 0$ and $\gamma/(\gamma+s) < p \leqslant \infty$, then*

$$\|f^*_{s,\Omega}\|_p \leqslant C_{s,p,\Phi} \|f\|_{p,\Phi,\Omega}$$

for all open subsets Ω of $\mathbf{R}^n$ and for all $f \in \mathscr{D}'(\Omega)$.

COROLLARY. *If Φ and Φ' are good kernels and $0 < p \leqslant \infty$, then*

$$\|f\|_{p,\Phi',\Omega} \leqslant C_{p,\Phi,\Phi'} \|f\|_{p,\Phi,\Omega}$$

for all open subsets Ω of $\mathbf{R}^n$ and for all $f \in \mathscr{D}'(\Omega)$.

In fact, Theorem A is not given verbatim in [8]. It is, however, very similar to Corollary 1 of [8] and can be proved in the same way. The Corollary readily follows from Theorem A since $M^+_{\Phi',\Omega}(f)(x) \leqslant C_{s,\Phi'} f^*_{s,\Omega}(x)$ for all $x \in \Omega$.

The Corollary implies that some kind of statement containing an inequality for $\|f\|_{p,\Phi,\Omega}$ (for example, the statement of the form "$\|f\|_{p,\Phi,\Omega} \leqslant C_{p,\Phi}$ if $f \ldots$") holds for all good kernels Φ if it holds for one such kernel. In such statements, we shall omit "for all Φ" or "for some Φ".

Recall also the following fact: If $1 < p \leqslant \infty$, then

$$C_{s,p}^{-1} \|f\|_{p,\Omega} \leqslant \|f^*_{s,\Omega}\|_{p,\Omega} \leqslant C_{s,p} \|f\|_{p,\Omega}$$

(see e.g. [1; Section 1.6]).

Now we shall introduce the H^p spaces over open subsets of $\mathbf{R}^n$.

DEFINITION. Let Ω be an open subset of $\mathbf{R}^n$ and $0 < p \leqslant 1$. Take a good kernel Φ. Then we denote by $H^p(\Omega)$ the set of those $f \in \mathscr{D}'(\Omega)$ such that $\|f\|_{p,\Phi,\Omega} < \infty$. We consider $H^p(\Omega)$ a quasinormed linear space with the quasinorm $\|\cdot\|_{p,\Phi,\Omega}$.

The Corollary shows that the set $H^p(\Omega)$ does not depend on the choice of the good kernel Φ and that the quasinorms $\|\cdot\|_{p,\Phi,\Omega}$ for various Φ are mutually equivalent. If $\Omega = \mathbf{R}^n$, then $H^p(\Omega)$ defined above coincides with $H^p(\mathbf{R}^n)$ given by Calderón and Torchinsky [2].

The following lemma will be used in the next section.

LEMMA 1. *Suppose Ω is an open subset of $\boldsymbol{R}^n$, $f \in \mathscr{D}'(\Omega)$, I is a ball with $3I \subset \Omega$, $\psi \in C_0^\infty(3I)$, $t > 0$ and $s > 0$. Then*

$$|\langle f, \psi\rangle| \leqslant C_{s,t}|I| \sum_{|\alpha| \leqslant [s]+1} \sup_y |\partial_y^\alpha \psi(x_I + A(r_I)y)| \inf_{x \in tI} f_{s,\Omega}^*(x),$$

where α denotes a multi-index and $|\alpha|$ its order.

Proof. Suppose $x \in tI$. Then $3I \subset B(x, (t+3)r_I) \cap \Omega$. Set $J = B(x, (t+3)r_I)$. From the definition of $f_{s,\Omega}^*(x)$, it follows that

$$|\langle f, \psi\rangle| \leqslant ((t+3)r_I)^{\gamma+s} \|\psi\|_{A(s;J\cap\Omega)} f_{s,\Omega}^*(x).$$

On the other hand, we have

$$\begin{aligned}\|\psi\|_{A(s;J\cap\Omega)} &\leqslant \|\psi\|_{A(s;3I)} \leqslant C_s \|\psi\|_{A(s)} = C_s r_I^{-s} \big\|\psi(x_I + A(r_I)\cdot)\big\|_{A(s)} \\ &\leqslant C_s r_I^{-s} \sum_{|\alpha| \leqslant [s]+1} \sup_y |\partial_y^\alpha \psi(x_I + A(r_I)y)|\end{aligned}$$

(for the second and the last inequalities, see [8; Section III, 307 and 306.1]). Combining the above inequalities, we obtain the desired result.

3. Atomic decomposition

3.1. *Atomic decomposition for $H^p(\Omega)$.* Let $0 < p \leqslant 1$ and let Ω be a proper open subset of $\boldsymbol{R}^n$.

A function $f \in L^\infty(\boldsymbol{R}^n)$ is called a *p-atom* if there exists a ball $I = I_f$ satisfying $\operatorname{supp} f \subset I$ and $\|f\|_\infty \leqslant |I|^{-1/p}$ and if $\langle f, P\rangle = 0$ for all $P \in \mathscr{P}_{[\gamma/p-\gamma]}$. We call f a *(p, Ω)-atom* if $f \in L^\infty(\Omega)$ and if there exists a ball $I = I_f$ such that $2I \subset \Omega$, $5I \cap \Omega^c \neq \emptyset$, $\operatorname{supp} f \subset I$ and $\|f\|_{\infty,\Omega} \leqslant |I|^{-1/p}$. If f and I satisfy the above conditions, we shall say f is a p-atom or (p, Ω)-atom *supported on I*.

If f is a p-atom, then

$$\|f|\Omega\|_{p,\Phi,\Omega} \leqslant \|f\|_{p,\Phi,\boldsymbol{R}^n} \leqslant C_{p,\Phi},$$

where $f|\Omega$ denotes the restriction of f to Ω (cf. e.g. Latter [6]). If f is a (p, Ω)-atom, then

$$\|f\|_{p,\Phi,\Omega} \leqslant C_{p,\Phi}.$$

(This can be proved as follows. Suppose f is a (p, Ω)-atom supported on I. Take a good kernel Φ such that $\operatorname{supp}\Phi(x, \cdot, t) \subset B(x, t/2)$ for all $x \in \boldsymbol{R}^n$ and all $t > 0$. Then $M_{\Phi,\Omega}^+(f)(x) \leqslant C_\Phi |I|^{-1/p}$ and $M_{\Phi,\Omega}^+(f)(x) = 0$ if $x \notin 7I$. Hence $\|f\|_{p,\Phi,\Omega} \leqslant C_{p,\Phi}$. By the Corollary in Section 2, the conclusion holds for every good kernel Φ.) Hence, if $\{f_i\}$ is a sequence of p-atoms and $\{g_j\}$ is a sequence of (p, Ω)-atoms, and if $\{\lambda_i\}$ and $\{\mu_j\}$ are sequences of nonnegative numbers such that $\sum_i \lambda_i^p < \infty$ and $\sum_j \mu_j^p < \infty$, then the series $\sum_i \lambda_i f_i|\Omega$ and $\sum_j \mu_j g_j$ both commutatively converge in $H^p(\Omega)$ and

$$\Big\|\sum_i \lambda_i f_i|\Omega + \sum_j \mu_j g_j\Big\|_{p,\Phi,\Omega} \leqslant C_{p,\Phi}\Big(\sum_i \lambda_i^p + \sum_j \mu_j^p\Big)^{1/p}.$$

The following theorem means that the converse also holds.

THEOREM 1. *Let Ω be a proper open subset of $\boldsymbol{R}^n$, $0 < p \leqslant 1$ and $f \in H^p(\Omega)$. Then there exist sequences of nonnegative numbers $\{\lambda_i\}$ and $\{\mu_j\}$, a sequence of p-atoms $\{f_i\}$ and a sequence of (p, Ω)-atoms $\{g_j\}$ such that each f_i is supported on a ball I_i satisfying $2I_i \subset \Omega$ and*

$$\Big(\sum_i \lambda_i^p + \sum_j \mu_j^p\Big)^{1/p} \leqslant C_{p,\Phi}\|f\|_{p,\Phi,\Omega},$$

$$f = \sum_i \lambda_i f_i|\Omega + \sum_j \mu_j g_j.$$

We shall prove this theorem in Sections 3.2–3.4 (Section 3.2 will be devoted to a preliminary argument). There, in fact, we shall give a more detailed result; we give an explicit way to obtain the decomposition, which contains much information on the p-atoms and (p, Ω)-atoms arising in the decomposition, and moreover we show that a similar decomposition also holds for functions in $L^p(\Omega)$ if $1 < p \leqslant \infty$. In Sections 4–6, we need the detailed results of Sections 3.2–3.4.

Remark. The basic idea of our proof of Theorem 1 is the same as that of Latter [6] or Latter–Uchiyama [7] (this idea goes back to Herz [5] or Coifman [3]). The point peculiar to our situation (general $\Omega \subset \boldsymbol{R}^n$) is the claim (3.27) in Section 3.3.

3.2. *Whitney decomposition.* In this section, we give a certain decompositon of proper open subsets of $\boldsymbol{R}^n$ and the associated partition of unity. The result to be given below is a modification of the similar result known as the Whitney decomposition in the case of $\boldsymbol{R}^n$ with the usual metric (for the Whitney decomposition, see e.g. Stein's book [9; Chapt. VI, § 1] or Torchinsky's book [10; Chapt. XIII, § 4.6]).

Let $\mathscr{G}_0$ be the set of balls with radius 1 and with center of the form $(m_1/\sqrt{n}, \ldots, m_n/\sqrt{n})$ with $m_j \in \boldsymbol{Z}$. For each $k \in \boldsymbol{Z}$, let $\mathscr{G}_k$ be the set of balls of the form $A(2^k)(I)$ with $I \in \mathscr{G}_0$. Let $\mathscr{G} = \bigcup_{k\in\boldsymbol{Z}} \mathscr{G}_k$.

Then the following hold. If $I \in \mathscr{G}_k$, then $r_I = 2^k$. For each $k \in \boldsymbol{Z}$, the balls in $\mathscr{G}_k$ cover $\boldsymbol{R}^n$. For each $t > 0$, there exists $C_t > 0$ such that for every $k \in \boldsymbol{Z}$ the overlap of the set $\{tI \mid I \in \mathscr{G}_k\}$ does not exceed C_t. If $k \in \boldsymbol{Z}$, $t > 0$ and E is a compact subset of $\boldsymbol{R}^n$, then there are only a finite number of balls I such that $I \in \mathscr{G}_k$ and $tI \cap E \neq \emptyset$.

If U is a proper open subset of $\boldsymbol{R}^n$, then we set

$$\mathscr{W}(U) = \{I \in \mathscr{G} \mid 20r_I < \operatorname{dis}(x_I, U^c) \leqslant 43r_I\}.$$

If U and Ω are proper open subsets of $\boldsymbol{R}^n$ such that $U \subset \Omega$ and if $b > 0$, then we set

$$\mathscr{W}^b(U, \Omega) = \{I \in \mathscr{W}(U) \mid \operatorname{dis}(x_I, \Omega\backslash U) \geqslant br_I\}.$$

LEMMA 2. *Let U be a proper open subset of $\boldsymbol{R}^n$.*

(i) *For every compact set $E \subset U$, there exist only a finite number of balls I such that $I \in \mathscr{W}(U)$ and $19I \cap E \neq \emptyset$.*

(ii) $\bigcup_{I \in \mathscr{W}(U)} I = U$.

(iii) *The overlap of $\{19I \mid I \in \mathscr{W}(U)\}$ does not exceed C.*

If in addition V is a proper open subset of $\boldsymbol{R}^n$ and $U \subset V$, then:

(iv) *If $I \in \mathscr{W}(U)$, $J \in \mathscr{W}(V)$ and $2I \cap 2J \neq \emptyset$, then $r_I \leqslant 2r_J$.*

(v) *For each fixed $I \in \mathscr{W}(U)$, the number of balls J satisfying $J \in \mathscr{W}(V)$ and $2I \cap 2J \neq \emptyset$ does not exceed C.*

Proof. We prove (ii). Set $t = 1/42$. Suppose $x \in U$. Take $I \in \mathscr{G}$ such that $t\,\mathrm{dis}(x, U^c) < r_I \leqslant 2t\,\mathrm{dis}(x, U^c)$ and $I \ni x$. Then

$$\mathrm{dis}(x_I, U^c) \leqslant \mathrm{dis}(x_I, x) + \mathrm{dis}(x, U^c) < (1+1/t) r_I = 43 r_I,$$

$$\mathrm{dis}(x_I, U^c) \geqslant \mathrm{dis}(x, U^c) - \mathrm{dis}(x_I, x) > (1/2t - 1) r_I = 20 r_I,$$

and, hence, $I \in \mathscr{W}(U)$. This proves (ii). Proofs of the other claims are done by the well known argument concerning the Whitney decomposition (cf. [9; Chapt. VI, § 1]); they are left to the reader.

LEMMA 3. *Let U and Ω be proper open subsets of $\boldsymbol{R}^n$ such that $U \subset \Omega$.*

(i) *If $b > 43$, then*

$$\mathscr{W}^b(U, \Omega) = \{I \in \mathscr{W}(\Omega) \mid \mathrm{dis}(x_I, \Omega \setminus U) \geqslant b r_I\}$$

and, in particular, $\mathscr{W}^b(U, \Omega) \subset \mathscr{W}(\Omega)$.

(ii) *If $b > 6$, $I \in \mathscr{W}^b(U, \Omega)$, $J \in \mathscr{W}(U)$ and $2I \cap 2J \neq \emptyset$, then $J \in \mathscr{W}^{b/2-3}(U, \Omega)$.*

(iii) *If $b > 92$ and $I \in \mathscr{W}^b(U, \Omega)$, then*

$$\{J \in \mathscr{W}(U) \mid 2I \cap 2J \neq \emptyset\} = \{J \in \mathscr{W}(\Omega) \mid 2I \cap 2J \neq \emptyset\}.$$

(iv) *If V is a proper open subset of $\boldsymbol{R}^n$ such that $U \subset V \subset \Omega$ and if $b > 43$, then $\mathscr{W}^b(U, \Omega) \subset \mathscr{W}^b(V, \Omega)$.*

Proof. (i) Since $U^c = (\Omega \setminus U) \cup \Omega^c$, the equality

$$\mathrm{dis}(x, U^c) = \min\{\mathrm{dis}(x, \Omega \setminus U), \mathrm{dis}(x, \Omega^c)\} \tag{3.1}$$

holds for every $x \in \boldsymbol{R}^n$. Hence if $\mathrm{dis}(x_I, \Omega \setminus U) \geqslant b r_I$ with $b > 43$ and if $I \in \mathscr{W}(U)$ or $I \in \mathscr{W}(\Omega)$, then $\mathrm{dis}(x_I, U^c) = \mathrm{dis}(x_I, \Omega^c)$. From this (i) follows easily.

(ii) If b, I and J satisfy the assumptions of (ii), then $r_I \geqslant r_J/2$ (Lemma 2(iv)) and, hence,

$$\mathrm{dis}(x_J, \Omega \setminus U) \geqslant \mathrm{dis}(x_I, \Omega \setminus U) - \mathrm{dis}(x_I, x_J) > b r_I - 2r_I - 2r_J \geqslant (b/2 - 3) r_J.$$

(iii) Let $b > 92$ and $I \in \mathscr{W}^b(U, \Omega)$. If $J \in \mathscr{W}(U)$ and $2I \cap 2J \neq \emptyset$, then $J \in \mathscr{W}^{b/2-3}(U, \Omega) \subset \mathscr{W}(\Omega)$ (by (ii) and (i)). Conversely, suppose $J \in \mathscr{W}(\Omega)$ and $2I \cap 2J \neq \emptyset$. Since $I \in \mathscr{W}(\Omega)$ (by (i)), we obtain $r_I \geqslant r_J/2$ (Lemma 2(iv)). Hence, in the same way as in the proof of (ii), we have $\mathrm{dis}(x_J, \Omega \setminus U) > (b/2 - 3) r_J$. Hence, by (i), $J \in \mathscr{W}^{b/2-3}(U, \Omega) \subset \mathscr{W}(U)$. This proves (iii).

(iv) This follows from (i) since $\mathrm{dis}(x, \Omega \setminus U) \leqslant \mathrm{dis}(x, \Omega \setminus V)$ for all x. This completes the proof of Lemma 3.

LEMMA 4. *Let U, V and Ω be proper open subsets of $\boldsymbol{R}^n$ such that $U \subset V \subset \Omega$.*

(i) *If $b > 43$, $I \in \mathscr{W}(U)$ and if there exists a J such that $J \in \mathscr{W}(V) \setminus \mathscr{W}^b(U, \Omega)$ and $2I \cap 2J \neq \emptyset$, then $\mathrm{dis}(x_I, \Omega \setminus U) < (2b+6) r_I$.*

(ii) *If $b > 0$ and $J \in \mathscr{W}(V) \setminus \mathscr{W}^b(U, \Omega)$, then $\mathrm{dis}(x_J, \Omega \setminus U) < \max\{21, b\} r_J$.*

Proof. (i) Suppose b, I and J satisfy the assumptions of (i). If $\mathrm{dis}(x_I, \Omega \setminus U) \leqslant \mathrm{dis}(x_I, \Omega^c)$, then, using (3.1) with $x = x_I$, we see that $\mathrm{dis}(x_I, \Omega \setminus U) = \mathrm{dis}(x_I, U^c) \leqslant 43 r_I < (2b+6) r_I$. Thus assume $\mathrm{dis}(x_I, \Omega \setminus U) > \mathrm{dis}(x_I, \Omega^c)$. Using (3.1) again, we see that $I \in \mathscr{W}(\Omega)$. Hence, by Lemma 2(iv), we have $r_J \leqslant 2r_I$. On the other hand, using (3.1) with $x = x_J$ and with U replaced by V, we see that either of the following holds:

$$20 r_J < \mathrm{dis}(x_J, \Omega \setminus V) \leqslant 43 r_J, \tag{3.2}$$

$$20 r_J < \mathrm{dis}(x_J, \Omega^c) \leqslant 43 r_J. \tag{3.3}$$

If (3.2) holds, then

$$\begin{aligned}\mathrm{dis}(x_I, \Omega \setminus U) &\leqslant \mathrm{dis}(x_J, \Omega \setminus U) + \mathrm{dis}(x_J, x_I)\\ &\leqslant \mathrm{dis}(x_J, \Omega \setminus V) + \mathrm{dis}(x_J, x_I) < 43 r_J + 2r_J + 2r_I\\ &\leqslant 92 r_I < (2b+6) r_I.\end{aligned}$$

Suppose (3.3) holds. Then $J \in \mathscr{W}(\Omega)$. From this and the assumption that $J \notin \mathscr{W}^b(U, \Omega)$, it follows that $\mathrm{dis}(x_J, \Omega \setminus U) < b r_J$ (Lemma 2(i)). Hence

$$\mathrm{dis}(x_I, \Omega \setminus U) \leqslant \mathrm{dis}(x_J, \Omega \setminus U) + \mathrm{dis}(x_J, x_I) < b r_J + 2r_J + 2r_I \leqslant (2b+6) r_I.$$

Thus (i) has been proved.

(ii) Suppose b and J satisfy the assumptions of (ii). Since $J \in \mathscr{W}(V)$, we have

$$20 r_J < \mathrm{dis}(x_J, V^c) \leqslant 43 r_J. \tag{3.4}$$

Since $J \notin \mathscr{W}^b(U, \Omega)$, one of the following four inequalities holds:

$$\mathrm{dis}(x_J, \Omega \setminus U) < b r_J, \tag{3.5}$$

$$\mathrm{dis}(x_J, U^c) > 43 r_J, \tag{3.6}$$

$$\mathrm{dis}(x_J, \Omega \setminus U) \leqslant 20 r_J, \tag{3.7}$$

$$\mathrm{dis}(x_J, \Omega^c) \leqslant 20 r_J. \tag{3.8}$$

If (3.5) or (3.7) holds, then the conclusion of (ii) holds. The inequalities (3.6) or (3.8) cannot hold simultaneously with (3.4) since $\mathrm{dis}(x_J, U^c) \leqslant \mathrm{dis}(x_J, V^c) \leqslant \mathrm{dis}(x_J, \Omega^c)$. This proves (ii) and completes the proof of Lemma 4.

Now we shall give a partition of unity on a proper open subset of $\boldsymbol{R}^n$. Take a function $\theta \in C_0^\infty(B(0, 2))$ such that $0 \leqslant \theta(x) \leqslant 1$ for all $x \in \boldsymbol{R}^n$, and $\theta(x) = 1$ for $x \in B(0, 1)$. For each $I \in \mathscr{G}$, set

$$\theta_I(x) = \theta\big(A(r_I^{-1})(x-x_I)\big).$$

Then $\theta_I \in C_0^\infty(2I)$, $0 \leqslant \theta_I(x) \leqslant 1$ for all x, and $\theta_I(x) = 1$ for $x \in I$. For each proper open subset U of $\boldsymbol{R}^n$ and for each $I \in \mathscr{G}$, we define the function φ_I^U on $\boldsymbol{R}^n$ as follows:

$$\varphi_I^U(x) = \begin{cases} \theta_I(x)/\sum_{J \in \mathscr{W}(U)} \theta_J(x) & \text{if } I \in \mathscr{W}(U) \text{ and } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 5. *Let U be a proper open subset of $\boldsymbol{R}^n$.*

(i) *$\varphi_I^U \in C_0^\infty(2I)$ for all $I \in \mathscr{G}$.*

(ii) *$0 \leqslant \varphi_I^U(x) \leqslant 1$ for all $I \in \mathscr{G}$ and all $x \in \boldsymbol{R}^n$.*

(iii) *$\sum_{I \in \mathscr{G}} \varphi_I^U(x) = 1$ if $x \in U$.*

(iv) *If $I \in \mathscr{G}$ and $0 < t \leqslant 10 r_I$, then*

$$\left|\partial_x^\alpha \varphi_I^U\big(A(t)x\big)\right| \leqslant C_{\theta,\alpha}(t/r_I)^{|\alpha|}$$

for all $x \in \boldsymbol{R}^n$ and all multi-indices α.

(v) *If Ω is a proper open subset of $\boldsymbol{R}^n$, $\Omega \supset U$, $b > 92$ and $I \in \mathscr{W}^b(U, \Omega)$, then $\varphi_I^U = \varphi_I^\Omega$.*

Proof. The claims (i)–(iii) follow easily from Lemma 2 and the definition of φ_I^U. The claim (iv) can be proved by the use of Lemma 2 and the fact that the operator norm of the linear transformation $A(t)$ does not exceed Ct if $0 < t \leqslant 10$. The claim (v) can be proved by the use of Lemma 3(i), (iii).

3.3. Proof of Theorem 1. We shall prove Theorem 1 and, at the same time, prove that a similar decomposition also holds for functions in $L^p(\Omega)$ if $1 < p \leqslant \infty$. Throughout Sections 3.3 and 3.4, we asume that Ω is a proper open subset of $\boldsymbol{R}^n$ and either $f \in H^p(\Omega)$ and $0 < p \leqslant 1$, or $f \in L^p(\Omega)$ and $1 < p \leqslant \infty$.

Take a nonnegative integer m such that $m \geqslant [\gamma/p - \gamma]$, and take $s > 0$ such that $\gamma/(\gamma+s) < p$. For each $k \in \boldsymbol{Z}$, set

$$U(k) = \{x \in \Omega \mid f_{s,\Omega}^*(x) > 2^k\}.$$

This is an open subset of Ω since $f_{s,\Omega}^*$ is lower-semicontinuous. We have $U(k) \subset U(k-1)$. For each $k \in \boldsymbol{Z}$ and each $I, J \in \mathscr{G}$, we define $P_{I,J}^k$ and $P_{I,J}^\Omega$ as follows. If $I \in \mathscr{W}\big(U(k)\big)$, $J \in \mathscr{W}\big(U(k-1)\big)$ and $2I \cap 2J \neq \emptyset$, then $P_{I,J}^k$ is the unique element of $\mathscr{P}_m$ such that

$$\langle f\varphi_I^{U(k)}\varphi_J^{U(k-1)} - P_{I,J}^k \chi_{8I}\chi_{8J}, Q\rangle = 0 \quad \text{for all } Q \in \mathscr{P}_m;$$

for other (k, I, J), we set $P_{I,J}^k = 0$. If $I, J \in \mathscr{W}(\Omega)$ and $2I \cap 2J \neq \emptyset$, then $P_{I,J}^\Omega$ is the unique element of $\mathscr{P}_m$ such that

$$\langle f\varphi_I^\Omega\varphi_J^\Omega - P_{I,J}^\Omega \chi_{8I}\chi_{8J}, Q\rangle = 0 \quad \text{for all } Q \in \mathscr{P}_m;$$

for other (I, J), we set $P_{I,J}^\Omega = 0$.

Then for every $\psi \in C_0^\infty(\Omega)$ and every $k \in \boldsymbol{Z}$,

$$\sum_{I,J} |\langle f\varphi_I^{U(k)}\varphi_J^{U(k-1)}, \psi\rangle| < \infty, \tag{3.9}$$

$$\sum_{I,J} |\langle P_{I,J}^k \chi_{8I}\chi_{8J}, \psi\rangle| < \infty, \tag{3.10}$$

where the sums are taken over all $I \in \mathscr{G}$ and all $J \in \mathscr{G}$ (this convention will be used hereafter; a sum over balls should be taken over all balls in $\mathscr{G}$ if no restriction is indicated). This will be proved in Section 3.4. Hence the series $\sum_{I,J} f\varphi_I^{U(k)}\varphi_J^{U(k-1)}$ and $\sum_{I,J} P_{I,J}^k \chi_{8I}\chi_{8J}$ both commutatively converge in $\mathscr{D}'(\Omega)$ and the iterated series formula $\sum_{I,J} = \sum_I(\sum_J) = \sum_J(\sum_I)$ holds. By Lemma 2(i), the same holds for the series $\sum_{I,J} P_{I,J}^\Omega \chi_{8I}\chi_{8J}$. Since $\{\varphi_J^{U(k-1)}\}_J$ is a partition of unity on $U(k-1)$ and $\operatorname{supp}\varphi_I^{U(k)}$ is a compact subset of $U(k) \subset U(k-1)$, we have

$$\sum_J f\varphi_I^{U(k)}\varphi_J^{U(k-1)} = f\varphi_I^{U(k)}. \tag{3.11}$$

For each $k \in \boldsymbol{Z}$, we set

$$g^k = f - \sum_I f\varphi_I^{U(k)} + \sum_{I,J} P_{I,J}^k \chi_{8I}\chi_{8J}.$$

In Section 3.4, we shall prove the following:

$$g^k \to \sum_{I,J} P_{I,J}^\Omega \chi_{8I}\chi_{8J} \quad \text{in } \mathscr{D}'(\Omega) \text{ as } k \to -\infty, \tag{3.12}$$

$$g^k \to f \quad \text{in } \mathscr{D}'(\Omega) \text{ as } k \to +\infty, \tag{3.13}$$

$$\Big\|f - \sum_I f\varphi_I^{U(k)}\Big\|_{\infty,\Omega} \leqslant C_s 2^k \quad \text{for every } k \in \boldsymbol{Z}. \tag{3.14}$$

(By (3.14) we mean, in particular, that $f - \sum_I f\varphi_I^{U(k)}$ belongs to $L^\infty(\Omega)$.) For the moment we assume these results and continue the proof.

Using (3.11), we see that $g^k - g^{k-1}$ can be written as

$$g^k - g^{k-1} = \sum_J h_J^k \tag{3.15}$$

with

$$h_J^k = -\sum_I f\varphi_I^{U(k)}\varphi_J^{U(k-1)} + \sum_I P_{I,J}^k \chi_{8I}\chi_{8J} + f\varphi_J^{U(k-1)} - \sum_K P_{J,K}^{k-1}\chi_{8J}\chi_{8K}.$$

We set

$$h_I^\Omega = \sum_J P_{I,J}^\Omega \chi_{8I}\chi_{8J}. \tag{3.16}$$

Combining (3.12), (3.13), (3.15) and (3.16), we obtain

$$f = \lim_{M \to \infty} \sum_{k=-M}^{M} \sum_J h_J^k + \sum_I h_I^\Omega. \tag{3.17}$$

We shall prove the following:

(3.18) $$h_J^k = 0 \quad \text{if } J \notin \mathscr{W}(U(k-1)),$$

(3.19) $$h_J^k \in L^\infty(\Omega) \quad \text{and} \quad \|h_J^k\|_{\infty,\Omega} \leqslant C_{\theta,m,s} 2^k,$$

(3.20) $$\operatorname{supp} h_J^k \subset (\text{the closure of } 8J) \subset 9J,$$

(3.21) $$\langle h_J^k, Q\rangle = 0 \quad \text{for all } Q \in \mathscr{P}_m,$$

(3.22) $$h_I^\Omega = 0 \quad \text{if } I \notin \mathscr{W}(\Omega),$$

(3.23) $$h_I^\Omega \in L^\infty(\Omega) \quad \text{and} \quad \|h_I^\Omega\|_{\infty,\Omega} \leqslant C_{\theta,m,s} \inf_{x \in 9I} f_{s,\Omega}^*(x),$$

(3.24) $$\operatorname{supp} h_I^\Omega \subset (\text{the closure of } 8I) \subset 9I.$$

Before we prove the above claims, we shall see some of their consequences. First, if $0 < p \leqslant 1$, then Theorem 1 follows from the above claims and (3.17). This can be seen from the following observations: If we set $\lambda_J^k = \|h_J^k\|_{\infty,\Omega} |9J|^{1/p}$, then $(\lambda_J^k)^{-1}(h_J^k)^\sim$ is a p-atom and

$$\sum_{k,J} (\lambda_J^k)^p \leqslant C_{\theta,m,s,p} \sum_{k\in \mathbf{Z}} \sum_{J \in \mathscr{W}(U(k-1))} 2^{kp}|J| \leqslant C_{\theta,m,s,p} \sum_{k \in \mathbf{Z}} 2^{kp}|U(k-1)|$$
$$\leqslant C_{\theta,m,s,p} \int_\Omega f_{s,\Omega}^*(x)^p\, dx \leqslant C_{\theta,m,s,p,\Phi} \|f\|_{p,\Phi,\Omega}^p$$

(the second inequality follows from Lemma 2 (ii) and (iii)); if we set $\mu_I = \|h_I^\Omega\|_{\infty,\Omega}|9I|^{1/p}$, then $\mu_I^{-1} h_I^\Omega$ is a (p, Ω)-atom and

$$\sum_I \mu_I^p \leqslant C_{\theta,m,s,p} \sum_{I \in \mathscr{W}(\Omega)} \big(\inf_{x\in I} f_{s,\Omega}^*(x)\big)^p |I| \leqslant C_{\theta,m,s,p} \sum_{I \in \mathscr{W}(\Omega)} \int_I f_{s,\Omega}^*(x)^p\, dx$$
$$\leqslant C_{\theta,m,s,p} \int_\Omega f_{s,\Omega}^*(x)^p\, dx \leqslant C_{\theta,m,s,p,\Phi} \|f\|_{p,\Phi,\Omega}^p$$

(the third inequality follows from Lemma 2 (ii) and (iii)). Secondly, the claims (3.18)–(3.24) imply the following inequalities:

$$\sum_{k,J} |h_J^k(x)| \leqslant \sum_{k \in \mathbf{Z}} \Big(\sum_{J \in \mathscr{W}(U(k-1))} C_{\theta,m,s} 2^k \chi_{9J}(x)\Big)$$
$$\leqslant C_{\theta,m,s} \sum_{k \in \mathbf{Z}} 2^k \chi_{U(k-1)}(x) \leqslant C_{\theta,m,s} f_{s,\Omega}^*(x),$$
$$\sum_I |h_I^\Omega(x)| \leqslant C_{\theta,m,s} f_{s,\Omega}^*(x).$$

Finally, from the above results, we see that the series $\sum_{k,J} h_J^k$ commutatively converges in $\mathscr{D}'(\Omega)$, to say the least. Hence, the formula (3.17) can be rewritten, without ambiguity, as

(3.17′) $$f = \sum_{k,J} h_J^k + \sum_I h_I^\Omega,$$

where the sum in k is taken over $\mathbf{Z}$.

Now we shall prove (3.18)–(3.24). The claims (3.18), (3.20), (3.22) and (3.24) are clear. (3.21) follows from the definition of $P_{I,J}^k$ and (3.11). In order to prove the other claims, we need some more observations.

The following estimates hold for all $t > 0$:

(3.25) $$\|P_{I,J}^k \chi_{8I}\chi_{8J}\|_\infty \leqslant C_{\theta,m,s,t} \inf_{x \in tI} f_{s,\Omega}^*(x),$$

(3.26) $$\|P_{I,J}^\Omega \chi_{8I}\chi_{8J}\|_\infty \leqslant C_{\theta,m,s,t} \inf_{x \in tI} f_{s,\Omega}^*(x).$$

We can prove these estimates by using Lemma 1 and Lemma 5 (iv) and arguing in the same way as in Latter–Uchiyama [7; pp. 393–394].

(3.23) follows from (3.26) and Lemma 2 (iii) (or (v)).

In order to prove (3.19), we use the following fact:

(3.27) $$h_J^k = 0 \quad \text{if } J \in \mathscr{W}^{200}(U(k), \Omega).$$

This can be seen as follows. Let $J \in \mathscr{W}^{200}(U(k), \Omega)$. Then, by Lemma 3(i) and (iv), we also have

$$J \in \mathscr{W}(\Omega) \cap \mathscr{W}^{200}(U(k-1), \Omega) \cap \mathscr{W}^{200}(U(k-2), \Omega).$$

Hence, by Lemma 3(iii), it follows that

(3.28) $$\{I \in \mathscr{W}(U(k)) \mid 2I \cap 2J \neq \varnothing\} = \{I \in \mathscr{W}(\Omega) \mid 2I \cap 2J \neq \varnothing\},$$

(3.29) $$\{K \in \mathscr{W}(U(k-2)) \mid 2J \cap 2K \neq \varnothing\} = \{K \in \mathscr{W}(\Omega) \mid 2J \cap 2K \neq \varnothing\}.$$

By Lemma 3 (ii), we see that $I \in \mathscr{W}^{97}(U(k), \Omega)$ for every I in the set of (3.28) and that $K \in \mathscr{W}^{97}(U(k-2), \Omega)$ for every K in the set of (3.29). Hence, by Lemma 5(v), we see that $\varphi_J^{U(k-1)} = \varphi_J^\Omega$, $\varphi_I^{U(k)} = \varphi_I^\Omega$ for all I in the set of (3.28), and $\varphi_K^{U(k-2)} = \varphi_K^\Omega$ for all K in the set of (3.29). Using these results, we compare the definitions of $P_{I,J}^k$ and $P_{J,K}^{k-1}$ with those of $P_{I,J}^\Omega$ and $P_{J,K}^\Omega$ to find that $P_{I,J}^k = P_{I,J}^\Omega$ for all $I \in \mathscr{G}$ and that $P_{J,K}^{k-1} = P_{J,K}^\Omega$ for all $K \in \mathscr{G}$. Hence

(3.30) $$\sum_I P_{I,J}^k \chi_{8I}\chi_{8J} - \sum_K P_{J,K}^{k-1}\chi_{8J}\chi_{8K} = \sum_I P_{I,J}^\Omega \chi_{8I}\chi_{8J} - \sum_K P_{J,K}^\Omega \chi_{8J}\chi_{8K} = 0$$

(since $P_{I,J}^\Omega = P_{J,I}^\Omega$). On the other hand, since $\{\varphi_I^{U(k)}\}_I$ is a partition of unity on $U(k)$ and $\operatorname{supp}\varphi_J^{U(k-1)} \subset 2J \subset U(k)$, we have

(3.31) $$-\sum_I f\varphi_I^{U(k)}\varphi_J^{U(k-1)} + f\varphi_J^{U(k-1)} = 0.$$

Now (3.27) follows from (3.30) and (3.31).

We now prove (3.19). By (3.18) and (3.27), we may and shall assume $J \in \mathscr{W}(U(k-1)) \setminus \mathscr{W}^{200}(U(k), \Omega)$. We write

$$h_J^k = \Big(f - \sum_I f\varphi_I^{U(k)}\Big)\varphi_J^{U(k-1)} + \sum_I P_{I,J}^k \chi_{8I}\chi_{8J} - \sum_K P_{J,K}^{k-1}\chi_{8J}\chi_{8K}$$
$$= a_J^k + b_J^k - c_J^k, \quad \text{say}.$$

For a_J^k, we use (3.14) to see that $a_J^k \in L^\infty(\Omega)$ and $\|a_J^k\|_{\infty,\Omega} \leqslant C_s 2^k$. Using (3.25) with $t = 406$ and Lemma 4(i) with $U = U(k)$, $V = U(k-1)$ and $b = 200$, we see that

$$\|P_{I,J}^k \chi_{8I} \chi_{8J}\|_\infty \leqslant C_{\theta,m,s} 2^k \quad \text{for all } I$$

(recall that $P_{I,J}^k \neq 0$ only if $I \in \mathscr{W}(U(k))$ and $2I \cap 2J \neq \emptyset$). This, together with Lemma 2(iii), implies that $b_J^k \in L^\infty(\Omega)$ and $\|b_J^k\|_{\infty,\Omega} \leqslant C_{\theta,m,s} 2^k$. Similarly, using (3.25) with $t = 200$ and with k, I, J replaced by $k-1$, J, K respectively and Lemma 4(ii) with U, V and b as above, we see that $c_J^k \in L^\infty(\Omega)$ and $\|c_J^k\|_{\infty,\Omega} \leqslant C_{\theta,m,s} 2^k$. Combining these results, we obtain (3.19).

We have now proved (3.18)–(3.24).

3.4. Proof of Theorem 1 (continued). We now prove (3.9), (3.10), (3.12), (3.13) and (3.14).

Proof of (3.9). Fix $k \in \mathbf{Z}$ and $\psi \in C_0^\infty(\Omega)$. The term in the sum of (3.9) does not vanish only if $I \in \mathscr{W}(U(k))$, $J \in \mathscr{W}(U(k-1))$ and $2I \cap 2J \cap \operatorname{supp}\psi \neq \emptyset$. For such I and J, we have $r_I \leqslant 2r_J$(Lemma 2(iv)) and

$$20 r_I < \operatorname{dis}(x_I, U(k)^c) \leqslant \operatorname{dis}(x_I, \Omega^c) < 2r_I + \max_{x \in \operatorname{supp}\psi} \operatorname{dis}(x, \Omega^c)$$

and, hence, $r_I \leqslant C_{\psi,\Omega}$. Hence, using Lemma 1 and Lemma 5(iv), we have

$$|\langle f\varphi_I^{U(k)} \varphi_J^{U(k-1)}, \psi\rangle| \leqslant C_{s,\theta,\psi,\Omega} |I| \inf_{x \in 50I} f_{s,\Omega}^*(x).$$

Since for each $I \in \mathscr{W}(U(k))$ the number of balls J satisfying $J \in \mathscr{W}(U(k-1))$ and $2I \cap 2J \neq \emptyset$ does not exceed C (Lemma 2(v)), we sum up the above inequalities over J to obtain

$$\sum_J |\langle f\varphi_I^{U(k)} \varphi_J^{U(k-1)}, \psi\rangle| \leqslant C_{s,\theta,\psi,\Omega} |I| \inf_{x \in 50I} f_{s,\Omega}^*(x). \tag{3.32}$$

To sum up this over I, we consider the two cases $I \in \mathscr{W}^{50}(U(k), \Omega)$ and $I \in \mathscr{W}(U(k)) \setminus \mathscr{W}^{50}(U(k), \Omega)$ separately. There are only a finite number of balls I such that $I \in \mathscr{W}^{50}(U(k), \Omega)$ and $2I \cap \operatorname{supp}\psi \neq \emptyset$ (this can be seen from Lemma 3(i) and Lemma 2(i)). Hence the sum of the left-hand side of (3.32) over $I \in \mathscr{W}^{50}(U(k), \Omega)$ is finite. If $I \in \mathscr{W}(U(k)) \setminus \mathscr{W}^{50}(U(k), \Omega)$, then $\inf\{f_{s,\Omega}^*(x) \mid x \in 50I\} \leqslant 2^k$. Hence the sum of the left-hand side of (3.32) over $I \in \mathscr{W}(U(k)) \setminus \mathscr{W}^{50}(U(k), \Omega)$ does not exceed

$$\sum_{\substack{I \in \mathscr{W}(U(k)) \\ 2I \cap \operatorname{supp}\psi \neq \emptyset}} C_{s,\theta,\psi,\Omega} |I| 2^k < \infty.$$

This proves (3.9).

Proof of (3.10). The term in the sum of (3.10) does not vanish only if $I \in \mathscr{W}(U(k))$, $J \in \mathscr{W}(U(k-1))$, $2I \cap 2J \neq \emptyset$ and $8I \cap \operatorname{supp}\psi \neq \emptyset$. Using (3.25) and Lemma 2(iii) (or (v)), we obtain

$$\sum_J |\langle P_{I,J}^k \chi_{8I} \chi_{8J}, \psi\rangle| \leqslant C_{\theta,m,s} \inf_{x \in 50I} f_{s,\Omega}^*(x) \|\psi\|_{1,8I}.$$

Thus we can prove (3.10) in the same way as (3.9).

Proof of (3.12). This is trivial if $f = 0$. Assume $f \neq 0$. Then $f_{s,\Omega}^*(x) > 0$ for all x and, hence, $U(k) \uparrow \Omega$ as $k \downarrow -\infty$. Thus the convergence

$$f - \sum_I f\varphi_I^{U(k)} \to 0 \quad \text{in } \mathscr{D}'(\Omega) \text{ as } k \to -\infty$$

is obvious. We shall prove the convergence

$$\sum_{I,J} P_{I,J}^k \chi_{8I} \chi_{8J} \to \sum_{I,J} P_{I,J}^\Omega \chi_{8I} \chi_{8J} \quad \text{in } \mathscr{D}'(\Omega) \text{ as } k \to -\infty.$$

For each x, we have $\operatorname{dis}(x, U(k)^c) \uparrow \operatorname{dis}(x, \Omega^c)$ as $k \downarrow -\infty$. This implies that for each $I \in \mathscr{G}$, there exists an integer k_I such that

$$\{I\} \cap \mathscr{W}(U(k)) = \{I\} \cap \mathscr{W}(\Omega) \quad \text{if } k \leqslant k_I. \tag{3.33}$$

Fix a $\psi \in C_0^\infty(\Omega)$. There exists an integer k_1 such that $U(k_1) \supset \operatorname{supp}\psi$. We set

$$\mathscr{A} = \{I \in \mathscr{W}(\Omega) \cup \bigcup_{k \leqslant k_1} \mathscr{W}(U(k)) \mid 8I \cap \operatorname{supp}\psi \neq \emptyset\}.$$

If $I \in \mathscr{A}$, then

$$20 r_I < \operatorname{dis}(x_I, \Omega^c) < 8r_I + \max_{x \in \operatorname{supp}\psi} \operatorname{dis}(x, \Omega^c),$$

$$43 r_I \geqslant \operatorname{dis}(x_I, U(k_1)^c) > \min_{x \in \operatorname{supp}\psi} \operatorname{dis}(x, U(k_1)^c) - 8r_I.$$

Since the r_I are integral powers of 2, the above inequalities imply that $\{r_I \mid I \in \mathscr{A}\}$ is a finite set. This fact and the fact that each $\mathscr{G}_k$ contains only a finite number of balls I for which $8I$ intersects a given compact set imply that $\mathscr{A}$ is a finite set. Hence $\bigcup_{I \in \mathscr{A}} 2I$ is a relatively compact subset of Ω. Thus, by the same argument as above, there exists an integer k_2 for which the set

$$\mathscr{B} = \{J \in \mathscr{W}(\Omega) \cup \bigcup_{k \leqslant k_2} \mathscr{W}(U(k)) \mid 2J \cap \bigcup_{I \in \mathscr{A}} 2I \neq \emptyset\}$$

is finite. We set $\mathscr{C} = \mathscr{A} \cup \mathscr{B}$. This is also a finite set. Hence we use (3.33) to see that there exists an integer k_3 such that

$$\mathscr{C} \cap \mathscr{W}(U(k)) = \mathscr{C} \cap \mathscr{W}(\Omega) \quad \text{if } k \leqslant k_3. \tag{3.34}$$

Set $k_4 = \min\{k_1, k_2, k_3\}$. We shall prove that

$$\sum_{I,J} \langle P_{I,J}^k \chi_{8I} \chi_{8J}, \psi\rangle = \sum_{I,J} \langle P_{I,J}^\Omega \chi_{8I} \chi_{8J}, \psi\rangle$$

for $k \leqslant k_4$. It is sufficient to prove that the following equalities hold for $k \leqslant k_4$:

$$\{I \in \mathscr{W}(U(k)) \mid 8I \cap \operatorname{supp}\psi \neq \emptyset\} = \{I \in \mathscr{W}(\Omega) \mid 8I \cap \operatorname{supp}\psi \neq \emptyset\}, \tag{3.35}$$

$$\{J \in \mathscr{W}(U(k-1)) \mid 8J \cap \operatorname{supp}\psi \neq \emptyset\} = \{J \in \mathscr{W}(\Omega) \mid 8J \cap \operatorname{supp}\psi \neq \emptyset\}, \tag{3.36}$$

$$P_{I,J}^k = P_{I,J}^\Omega \quad \text{if } I \in \text{the set of (3.35) and } J \in \text{the set of (3.36)}. \tag{3.37}$$

The claims (3.35) and (3.36) easily follow from (3.34). The claim (3.37) is proved as follows. Suppose $k \leqslant k_4$, I belongs to the set of (3.35) and J belongs to the set of (3.36). Using (3.34), we see that

$$\{I' \in \mathscr{W}(U(k)) \mid 2I' \cap 2I \neq \emptyset\} = \{I' \in \mathscr{W}(\Omega) \mid 2I' \cap 2I \neq \emptyset\},$$
$$\{J' \in \mathscr{W}(U(k-1)) \mid 2J' \cap 2J \neq \emptyset\} = \{J' \in \mathscr{W}(\Omega) \mid 2J' \cap 2J \neq \emptyset\}.$$

Hence, from the definition of φ_I^U it follows that $\varphi_I^{U(k)} = \varphi_I^\Omega$ and $\varphi_J^{U(k-1)} = \varphi_J^\Omega$. Now comparing the definitions of $P_{I,J}^k$ and $P_{I,J}^\Omega$, we see that $P_{I,J}^k = P_{I,J}^\Omega$. This proves (3.37) and completes the proof of (3.12).

Proof of (3.13). If $p = \infty$, then $U(k) = \emptyset$ for all sufficiently large k and (3.13) is trivial. So assume $0 < p < \infty$. Fix a $\psi \in C_0^\infty(\Omega)$. What we have to show is that

$$(3.38) \qquad \sum_{I,J} \langle f\varphi_I^{U(k)} \varphi_J^{U(k-1)} - P_{I,J}^k \chi_{8I} \chi_{8J}, \psi \rangle \to 0$$

as $k \to +\infty$ (recall (3.11)). Set

$$f_{I,J}^k = f\varphi_I^{U(k)} \varphi_J^{U(k-1)} - P_{I,J}^k \chi_{8I} \chi_{8J}.$$

Notice that $\langle f_{I,J}^k, \psi \rangle \neq 0$ only if

$$(3.39) \qquad \begin{aligned} &I \in \mathscr{W}(U(k)), \quad 8I \cap \operatorname{supp}\psi \neq \emptyset, \\ &J \in \mathscr{W}(U(k-1)), \quad 2I \cap 2J \neq \emptyset. \end{aligned}$$

Let Q_I be the polynomial in $\mathscr{P}_m$ which is the sum of the terms of degree $\leqslant m$ in Taylor's series of $\psi(x)$ at x_I. Then $\langle f_{I,J}^k, \psi \rangle = \langle f_{I,J}^k, \psi - Q_I \rangle$. Using this equality, Lemma 1, Lemma 5 (iv) and (3.25), we obtain

$$(3.40) \quad |\langle f_{I,J}^k, \psi \rangle| \leqslant C_{s,\theta,m} |I| \inf_{x \in 50I} f_{s,\Omega}^*(x) \sum_{|\alpha| \leqslant [s]+1} \sup_{y \in B(0,8)} \left|\partial_y^\alpha (\psi - Q_I)(x_I + A(r_I)y)\right|.$$

Since $|U(k)| \leqslant 2^{-kp} \|f_{s,\Omega}^*\|_{p,\Omega}^p$ (Chebyshev's inequality), we have $\sup\{r_I \mid I \in \mathscr{W}(U(k))\} \to 0$ as $k \to \infty$.

From now on we assume k is sufficiently large. Suppose I satisfies the first and second conditions of (3.39). Then

$$\operatorname{dis}(x_I, \Omega^c) > \min_{x \in \operatorname{supp}\psi} \operatorname{dis}(x, \Omega^c) - 8r_I > 43r_I$$

(the latter inequality holds because r_I is small) and, hence, $\operatorname{dis}(x_I, \Omega \setminus U(k)) \leqslant 43r_I$ (we have used (3.1)). Thus

$$\inf\{f_{s,\Omega}^*(x) \mid x \in 50I\} \leqslant 2^k.$$

Moreover, the operator norm of $A(r_I)$ is majorized by r_I and $|A(r_I)y| \leqslant r_I \varrho(y) < 8r_I$ for all $y \in B(0, 8)$. Thus

$$\sum_{|\alpha| \leqslant [s]+1} \sup_{y \in B(0,8)} \left|\partial_y^\alpha (\psi - Q_I)(x_I + A(r_I)y)\right| \leqslant C_{\psi,m,s} r_I^{m+1}.$$

Combining these estimates with (3.40), we obtain

$$|\langle f_{I,J}^k, \psi \rangle| \leqslant C_{s,\theta,\psi,m} 2^k |I|^{(\gamma+m+1)/\gamma}.$$

Summing up this over J (it is sufficient to consider those J which satisfy the third and fourth conditions of (3.39)), we obtain

$$\sum_J |\langle f_{I,J}^k, \psi \rangle| \leqslant C_{s,\theta,\psi,m} 2^k |I|^{(\gamma+m+1)/\gamma}$$

(we have used Lemma 2 (v)). Summing up this over I, we obtain

$$\begin{aligned} \sum_{I,J} |\langle f_{I,J}^k, \psi \rangle| &\leqslant C_{s,\theta,\psi,m} 2^k \Big(\sum_{I \in \mathscr{W}(U(k))} |I| \Big)^{(\gamma+m+1)/\gamma} \leqslant C_{s,\theta,\psi,m} 2^k |U(k)|^{(\gamma+m+1)/\gamma} \\ &\leqslant C_{s,\theta,\psi,m} 2^k |U(k)|^{1/p} \leqslant C_{s,\theta,\psi,m} \Big(\int_{U(k)} f_{s,\Omega}^*(x)^p \, dx \Big)^{1/p} \end{aligned}$$

(the second inequality follows from Lemma 2 (ii) and (iii); the third holds because $|U(k)|$ is small and $(\gamma+m+1)/\gamma > 1/p$). Since $f_{s,\Omega}^* \in L^p(\Omega)$ and $|U(k)| \to 0$ as $k \to \infty$, the last expression in the above series of inequalities tends to zero as $k \to \infty$. This proves (3.38) and completes the proof of (3.13).

Proof of (3.14). Let $k \in \mathbf{Z}$. Take a function η such that $\eta \in C_0^\infty(B(0, 1))$ and $\int \eta(x) \, dx = 1$. For $\varepsilon > 0$, set

$$\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dis}(x, \Omega^c) > \varepsilon\}$$

and define $f_\varepsilon(x)$, $x \in \Omega_\varepsilon$, by

$$f_\varepsilon(x) = \langle f, (\eta)_\varepsilon(x - \cdot) \rangle - \sum_{I: r_I > \varepsilon} \langle f, (\eta)_\varepsilon(x - \cdot) \rangle \varphi_I^{U(k)}(x).$$

It is easy to see that f_ε is a smooth function on Ω_ε. We shall prove that

$$(3.41) \qquad |f_\varepsilon(x)| \leqslant C_{s,\eta} 2^k \quad \text{if } 45\varepsilon < \varepsilon' \text{ and } x \in \Omega_{\varepsilon'}$$

and that, for each fixed $\psi \in C_0^\infty(\Omega)$,

$$(3.42) \qquad \int f_\varepsilon(x) \psi(x) \, dx \to \langle f - \sum_I f\varphi_I^{U(k)}, \psi \rangle \quad \text{as } \varepsilon \to 0.$$

The assertion (3.14) readily follows from these results.

We now prove (3.41). Suppose $45\varepsilon < \varepsilon'$ and $x \in \Omega_{\varepsilon'}$. First, we consider the case $\operatorname{dis}(x, U(k)^c) > 45\varepsilon$. Then obviously $x \in U(k)$. Moreover, for each $I \in \mathscr{W}(U(k))$ with $2I \ni x$,

$$43r_I \geqslant \operatorname{dis}(x_I, U(k)^c) \geqslant \operatorname{dis}(x, U(k)^c) - \operatorname{dis}(x_I, x) > 45\varepsilon - 2r_I$$

and, hence, $r_I > \varepsilon$. Hence, $f_\varepsilon(x) = 0$ in this case. Next, consider the case $\operatorname{dis}(x, U(k)^c) \leqslant 45\varepsilon$. Then $\operatorname{dis}(x, \Omega \setminus U(k)) \leqslant 45\varepsilon$ (we have used (3.1) and the assumption $\operatorname{dis}(x, \Omega^c) > \varepsilon' > 45\varepsilon$). Hence, using Lemma 1 with $\psi = (\eta)_\varepsilon(x - \cdot)$, $I = B(x, \varepsilon)$ and $t = 46$, we have

$$|\langle f, (\eta)_\varepsilon(x - \cdot) \rangle| \leqslant C_{s,\eta} \inf_{y \in B(x, 46\varepsilon)} f_{s,\Omega}^*(y) \leqslant C_{s,\eta} 2^k.$$

Hence, $|f_\varepsilon(x)| \leqslant C_{s,\eta} 2^k$ in this case. This proves (3.41).

Next we prove (3.42). Fix a $\psi \in C_0^\infty(\Omega)$. Take $\varepsilon' > 0$ such that $\operatorname{supp}\psi \subset \Omega_{\varepsilon'}$. Suppose $\varepsilon < \varepsilon'$, $I \in \mathscr{W}(U(k))$ and $r_I > \varepsilon$. Then the function $(\check{\eta})_\varepsilon * (\varphi_I^{U(k)}\psi)$, where $\check{\eta}(x) = \eta(-x)$ and $*$ denotes the convolution, belongs to $C_0^\infty(3I)$ and

$$\int \langle f, (\eta)_\varepsilon(x-\cdot)\rangle \varphi_I^{U(k)}(x)\psi(x)\,dx = \langle f, (\check{\eta})_\varepsilon * (\varphi_I^{U(k)}\psi)\rangle.$$

Using Lemma 1 and Lemma 5 (iv) and arguing in the same way as in the proof of (3.9), we see that the absolute value of the right-hand side of the above equality is majorized by

$$C_{s,\eta,\theta,\psi,\Omega}|I| \inf_{x\in 50I} f_{s,\Omega}^*(x). \tag{3.43}$$

(Notice that the above constant C does not depend on ε.) By the same argument as in the proof of (3.9), the sum of (3.43) over the balls I in $\mathscr{W}(U(k))$ satisfying $2I \cap \operatorname{supp}\psi \neq \emptyset$ is finite. Hence we can change the order of $\lim_{\varepsilon\to 0}$ and $\sum_I$ below:

$$\lim_{\varepsilon\to 0} \sum_{I:r_I>\varepsilon} \int \langle f, (\eta)_\varepsilon(x-\cdot)\rangle \varphi_I^{U(k)}(x)\psi(x)\,dx$$
$$= \sum_I \lim_{\varepsilon\to 0} \int \langle f, (\eta)_\varepsilon(x-\cdot)\rangle \varphi_I^{U(k)}(x)\psi(x)\,dx = \sum_I \langle f, \varphi_I^{U(k)}\psi\rangle.$$

We also have

$$\lim_{\varepsilon\to 0} \int \langle f, (\eta)_\varepsilon(x-\cdot)\rangle \psi(x)\,dx = \langle f, \psi\rangle.$$

Combining the above two results, we obtain (3.42). This completes the proof of (3.14).

We have thus proved Theorem 1 as well as all the assertions in Section 3.3.

4. Dual of $H^p(\Omega)$. Let Ω be a proper open subset of $\boldsymbol{R}^n$. We denote by $\mathrm{BMO}(\Omega)$ the set of those functions f in $L^1_{\mathrm{loc}}(\Omega)$ such that

$$\|f\|_{\mathrm{BMO}(\Omega)} = \sup_I \Big[\inf_{c\in C}\{|I|^{-1}\int_I |f(x)-c|\,dx\}\Big] + \sup_J \{|J|^{-1}\int_J |f(x)|\,dx\} < \infty,$$

where the first sup is taken over those balls I such that $2I \subset \Omega$ and the second over those balls J such that $2J \subset \Omega$ and $5J \cap \Omega^c \neq \emptyset$. We consider $\mathrm{BMO}(\Omega)$ a Banach space with the above norm. If $f \in \mathrm{BMO}(\Omega)$, then f is integrable over every bounded measurable subset of Ω (we can prove this by covering the bounded set with $9I$, $I \in \mathscr{W}(\Omega)$). We denote by $X(\Omega)$ the set of those $f \in L^\infty(\Omega)$ which vanish outside some bounded subset of Ω.

Before we give the theorem about the dual of $H^p(\Omega)$, we give two lemmas.

LEMMA 6. *Let Ω be a proper open subset of $\boldsymbol{R}^n$ and E a bounded closed subset of Ω. Suppose $f \in \mathscr{D}'(\Omega)$, $\operatorname{supp} f \subset E$ and $0 < p < q \leqslant \infty$. Then*

$$\|f\|_{p,\Phi,\Omega} \leqslant C_{p,q,E,\Omega,\Phi}\|f\|_{q,\Phi,\Omega}.$$

Proof. It is sufficient to prove the inequality for one good kernel Φ. We take a good kernel Φ such that $\operatorname{supp}\Phi(x,\cdot,t) \subset B(x, t/2)$ for all $x \in \boldsymbol{R}^n$ and all $t > 0$. Then we easily see that $M^+_{\Phi,\Omega}(f)$ vanishes outside $\tilde{E}$, where

$$\tilde{E} = \{x \in \Omega \mid \operatorname{dis}(x, E) < \sup_{y\in E} \operatorname{dis}(y, \Omega^c)\}.$$

Hence Hölder's inequality gives

$$\|f\|_{p,\Phi,\Omega} \leqslant |\tilde{E}|^{1/p-1/q}\|f\|_{q,\Phi,\Omega}.$$

Since $\tilde{E}$ is also a bounded set, we have $|\tilde{E}| < \infty$. This proves the lemma.

LEMMA 7. *Let Ω be a proper open subset of $\boldsymbol{R}^n$ and $0 < p \leqslant 1$. Then both $C_0^\infty(\Omega)$ and $X(\Omega)$ are dense subsets of $H^p(\Omega)$.*

Proof. The inclusion $C_0^\infty(\Omega) \subset X(\Omega)$ is obvious. The inclusion $X(\Omega) \subset H^p(\Omega)$ follows from Lemma 6. Hence, in order to complete the proof, it is sufficient to prove that $C_0^\infty(\Omega)$ is dense in $H^p(\Omega)$. If f is a p-atom supported on a ball I satisfying $2I \subset \Omega$ or if f is a (p, Ω)-atom, then f can be approximated by functions in $C_0^\infty(\Omega)$ with respect to the quasinorm in $H^p(\Omega)$. In fact, if η is the same function as in the proof of (3.14) (Section 3.4), then $\tilde{f} * (\eta)_\varepsilon$ belongs to $C_0^\infty(\Omega)$ for sufficiently small $\varepsilon > 0$ and converges to f in $H^p(\Omega)$ as $\varepsilon \to 0$ (this latter conclusion is seen with the aid of Lemma 6). This fact and Theorem 1 imply that $C_0^\infty(\Omega)$ is dense in $H^p(\Omega)$. This completes the proof.

We denote by $(H^p(\Omega))'$ the dual of $H^p(\Omega)$, i.e., the space of all bounded linear functionals on $H^p(\Omega)$.

THEOREM 2. *Let Ω be a proper open subset of $\boldsymbol{R}^n$.*

(i) *For every $g \in \mathrm{BMO}(\Omega)$, there exists a unique $T \in (H^1(\Omega))'$ such that*

$$T(f) = \int_\Omega g(x)f(x)\,dx \quad \text{for all } f \in X(\Omega). \tag{4.1}$$

Conversely, for every $T \in (H^1(\Omega))'$, there exists a unique $g \in \mathrm{BMO}(\Omega)$ which satisfies (4.1). *The correspondence $T \leftrightarrow g$ determined through* (4.1) *is a Banach space isomorphism between $(H^1(\Omega))'$ and $\mathrm{BMO}(\Omega)$.*

(ii) *Let $0 < p < 1$. Then* (4.1) *determines a one-to-one correspondence between $T \in (H^p(\Omega))'$ and $g \in \Lambda(\gamma/p-\gamma;\,\Omega)$, and this correspondence is a Banach space isomorphism.*

Proof. We prove (ii). Suppose $g \in \Lambda(\gamma/p-\gamma;\,\Omega)$ and $f \in X(\Omega)$. Set $m = [\gamma/p-\gamma]$ and $s = m+1$ (thus $\gamma/(\gamma+s) < p$). With these m and s, decompose f as in Section 3.3; let (3.17′) be the decomposition. Then there exists a bounded set which depends only on $\operatorname{supp} f$ and Ω and which includes all the supports of h_J^k and h_I^Ω. Moreover,

$$\sup_x \{\sum_{k,J} |h_J^k(x)| + \sum_I |h_I^\Omega(x)|\} < \infty$$

(see Section 3.3). Hence

$$\int g(x)f(x)\,dx = \sum_{k,J}\int g(x)h_J^k(x)\,dx+\sum_{I}\int g(x)h_I^{\Omega}(x)\,dx$$

(we can change the order of $\int$ and $\sum$). Using the estimates (3.18)–(3.24), we easily see that the absolute value of the right-hand side of the above equality is majorized by

$$C_{\theta,p}\|g\|_{\Lambda(\gamma/p-\gamma;\Omega)}\|f^*_{s,\Omega}\|_{p,\Omega}.$$

Hence, by Theorem A, we have

$$\left|\int g(x)f(x)\,dx\right| \leqslant C_{p,\Phi}\|g\|_{\Lambda(\gamma/p-\gamma;\Omega)}\|f\|_{p,\Phi,\Omega}$$

(we can take the function θ depending only on the dimension n and the group $\{A(t)\}$). The above inequality, combined with Lemma 7, shows that the linear functional $X(\Omega)\ni f\mapsto\int g(x)f(x)\,dx$ can be uniquely extended to a bounded linear functional on $H^p(\Omega)$.

Conversely, suppose $T\in(H^p(\Omega))'$. Let E be a bounded subset of Ω. Then Lemma 6 implies that $L^2(E)\subset H^p(\Omega)$ and, hence, that the restriction of T to $L^2(E)$ is a bounded linear functional on $L^2(E)$. Hence there exists $g_E\in L^2(E)$ such that $T(f)=\int_E g_E(x)f(x)\,dx$ for all $f\in L^2(E)$. If E and E' are bounded subsets of Ω and $E\subset E'$, then $g_E(x)=g_{E'}(x)$ for a.e. $x\in E$. Hence, it is easy to see that there exists a measurable function g on Ω which is square integrable over every bounded measurable subset of Ω and satisfies (4.1). Such a g is obviously unique. We shall show that g belongs to $\Lambda(\gamma/p-\gamma;\ \Omega)$.

Let $\|T\|$ be the norm of T in $(H^p(\Omega))'$, i.e.,

$$\|T\| = \sup\{|T(f)|/\|f\|_{p,\Phi,\Omega}\mid f\in H^p(\Omega),\ f\neq 0\}.$$

Taking supremum of $\left|\int_\Omega g(x)f(x)\,dx\right| = |T(f|\Omega)|$ over all p-atoms f and over all $(p,\ \Omega)$-atoms f, we find that

$$\|\tilde{g}\|_{\Lambda(\gamma/p-\gamma)}+\sup_J\{|J|^{-\gamma/p}\int_J|g(x)|\,dx\}\leqslant C_{p,\Phi}\|T\|, \tag{4.2}$$

where the sup is taken over those balls J such that $2J\subset\Omega$ and $5J\cap\Omega^c\neq\emptyset$. Take a function $\eta\in C_0^\infty(B(0,\ 1))$ such that

$$\int\eta(x)x^\alpha\,dx = \begin{cases}1 & \text{if } \alpha=0,\\ 0 & \text{if } 0<|\alpha|\leqslant[\gamma/p-\gamma],\end{cases}$$

and set $G(x,\ t)=(\tilde{g}*(\eta)_t)(x)$. Then

$$|\partial_t G(x,\ t)|\leqslant C_\eta\|\tilde{g}\|_{\Lambda(\gamma/p-\gamma)}t^{\gamma/p-\gamma-1}\leqslant C_{\eta,p,\Phi}\|T\|\,t^{\gamma/p-\gamma-1}$$

(for the first inequality, see [8; Section III, 305.1]). If $x\in\Omega$ and $u=\operatorname{dis}(x,\ \Omega^c)$, then from (4.2) we can easily deduce the estimate $|G(x,\ u/2)|\leqslant C_{\eta,p,\Phi}\|T\|u^{\gamma/p-\gamma}$. Hence,

$$|g(x)| = \left|G(x,\ u/2)-\int_0^{u/2}\partial_t G(x,\ t)\,dt\right|\leqslant C_{\eta,p,\Phi}\|T\|u^{\gamma/p-\gamma}.$$

This and (4.2) imply $\|g\|_{\Lambda(\gamma/p-\gamma;\Omega)}\leqslant C_{p,\Phi}\|T\|$ (we can take η depending only on n, $\{A(t)\}$ and p). This completes the proof of (ii).

The assertion (i) can be proved in a similar way; it is left to the reader.

Remark. Modifying the above argument, we can prove the following results.

(i) If $g\in\mathrm{BMO}(\Omega)$, then $\|\tilde{g}\|_{\mathrm{BMO}(\boldsymbol{R}^n)}\leqslant C\|g\|_{\mathrm{BMO}(\Omega)}$, where $\|\cdot\|_{\mathrm{BMO}(\boldsymbol{R}^n)}$ is defined by

$$\|f\|_{\mathrm{BMO}(\boldsymbol{R}^n)} = \sup_{I:\,\text{ball}}\Big[\inf_{c\in\boldsymbol{C}}\{|I|^{-1}\int_I|f(x)-c|\,dx\}\Big].$$

(ii) A function g on Ω belongs to $\Lambda(s;\ \Omega)$, $s>0$, if and only if $g\in L^1_{\mathrm{loc}}(\Omega)$ and

$$\sup_I\Big[\inf_{P\in\mathscr{P}_{[s]}}\{r_I^{-\gamma-s}\int_I|g(x)-P(x)|\,dx\}\Big]+\sup_J\{r_J^{-\gamma-s}\int_J|g(x)|\,dx\}<\infty,$$

where the first sup is taken over those balls I such that $2I\subset\Omega$ and the second over those balls J such that $2J\subset\Omega$ and $5J\cap\Omega^c\neq\emptyset$.

5. Complex interpolation. In this section, we use the following notations: $S=\{z\in\boldsymbol{C}\mid 0<\operatorname{Re}z<1\}$, $\bar{S}=$ the closure of S in $\boldsymbol{C}$ and $\Omega_\varepsilon=$ the same as in Section 3.4 (proof of (3.14)).

The following theorem shows the complex method of interpolation as applied to $H^p(\Omega)$ and $L^p(\Omega)$.

THEOREM 3. *Let Ω be a proper open subset of $\boldsymbol{R}^n$ and let $0<p<q<r<\infty$, $0<t<1$ and $1/q=(1-t)/p+t/r$.*

(i) *Let K_0 and K_1 be nonnegative numbers. Suppose $\{f_z\mid z\in\bar{S}\}$ is a family of distributions on Ω which satisfies the conditions* (A.1)–(A.4) *below.*

(A.1) *For each $\psi\in C_0^\infty(\Omega)$, the function $z\mapsto\langle f_z,\ \psi\rangle$ is analytic in S.*

(A.2) *For each $\varepsilon>0$ and for each $\psi\in C_0^\infty(B(0,\ \varepsilon))$, the function $(z,\ x)\mapsto\langle f_z,\ \psi(x-\cdot)\rangle$ is continuous in $\bar{S}\times\Omega_\varepsilon$.*

(A.3) *For each $\varepsilon>0$, each $\psi\in C_0^\infty(B(0,\ \varepsilon))$ and each compact subset E of Ω_ε, the function $(z,\ x)\mapsto\langle f_z,\ \psi(x-\cdot)\rangle$ is bounded on $\bar{S}\times E$.*

(A.4) *$\|f_z\|_{p,\Phi,\Omega}\leqslant K_0$ if $\operatorname{Re}z=0$, and $\|f_z\|_{r,\Phi,\Omega}\leqslant K_1$ if $\operatorname{Re}z=1$.*

Then

$$\|f_t\|_{q,\Phi,\Omega}\leqslant C_{p,q,r,\Phi}K_0^{1-t}K_1^t;$$

in particular, $f_t\in H^q(\Omega)$ (when $0<q\leqslant 1$) or $f_t\in L^q(\Omega)$ (when $1<q<\infty$).

(ii) *Conversely, suppose $f\in\mathscr{D}'(\Omega)$ and $\|f\|_{p,\Phi,\Omega}<\infty$ or, equivalently, suppose $f\in H^q(\Omega)$ (when $0<q\leqslant 1$) or $f\in L^q(\Omega)$ (when $1<q<\infty$). Then there exists a family $\{f_z\mid z\in\bar{S}\}\subset\mathscr{D}'(\Omega)$ which satisfies the equation $f_t=f$ and the conditions* (A.1)–(A.4) *with*

$$K_0=C_{p,q,r,\Phi}(\|f\|_{q,\Phi,\Omega})^{q/p},\quad K_1=C_{p,q,r,\Phi}(\|f\|_{q,\Phi,\Omega})^{q/r}.$$

Proof. (i) can be proved in the same way as in Calderón–Torchinsky [2; Theorem 3.1]. (ii) is proved as follows. Take a nonnegative integer m such that $m \leqslant [\gamma/p-\gamma]$, and take $s > 0$ such that $\gamma/(\gamma+s) < p$. With these m and s, decompose f as in Section 3.3 and obtain (3.17′). Set $c = q(1/p-1/r)$, $a_I = \inf\{f^*_{s,\Omega}(x) \mid x \in I\}$ $(I \in \mathscr{W}(\Omega))$ and

$$f_z = \sum_{k,J} 2^{kc(t-z)} h^k_J + \sum_{I \in \mathscr{W}(\Omega)} a_I^{c(t-z)} h^\Omega_I.$$

Then $\{f_z \mid z \in \bar{S}\}$ satisfies all the required conditions. Details are left to the reader.

6. Extension operator. We begin with the following lemma.

LEMMA 8. *Let Ω be a proper open subset of $\boldsymbol{R}^n$ and $0 < q < 1$. Then $H^p(\Omega) \subset H^q(\Omega)+L^\infty(\Omega)$ if $q < p \leqslant 1$, and $L^p(\Omega) \subset H^q(\Omega)+L^\infty(\Omega)$ if $1 < p < \infty$.*

Proof. Suppose that $q < p \leqslant 1$ and $f \in H^p(\Omega)$ or that $1 < p < \infty$ and $f \in L^p(\Omega)$. Set $m = [\gamma/q-\gamma]$ and $s = m+1$ (thus $\gamma/(\gamma+s) < q$). With these m and s, decompose f as in Section 3.3 and obtain (3.17′). For $I \in \mathscr{W}(\Omega)$, set $a_I = \inf\{f^*_{s,\Omega}(x) \mid x \in I\}$. Set

$$f_0 = \sum_{k>0,J} h^k_J + \sum_{I:a_I>1} h^\Omega_I, \quad f_1 = \sum_{k\leqslant 0,J} h^k_J + \sum_{I:a_I\leqslant 1} h^\Omega_I.$$

Then $f = f_0+f_1$, $f_0 \in H^q(\Omega)$ and $f_1 \in L^\infty(\Omega)$. This proves the lemma.

THEOREM 4. *Let Ω be a proper open subset of $\boldsymbol{R}^n$ and $A > 1$. Suppose that for every $x \in \Omega$, there exists an $x' \in \Omega^c$ such that $\operatorname{dis}(x, x') < A \operatorname{dis}(x, \Omega^c)$ and $\operatorname{dis}(x', \Omega) > A^{-1} \operatorname{dis}(x, \Omega^c)$. Then for each q with $0 < q < 1$ there exists a linear mapping T_q from $H^q(\Omega)+L^\infty(\Omega)$ to $\mathscr{D}'(\boldsymbol{R}^n)$ such that* (i) *$T_q(f)|\Omega = f$ for every $f \in H^q(\Omega)+L^\infty(\Omega)$ and* (ii) *if $q \leqslant p \leqslant \infty$, then*

$$\|T_q(f)\|_{p,\Phi,\boldsymbol{R}^n} \leqslant C_{A,q,p,\Phi}\|f\|_{p,\Phi,\Omega} \quad \textit{for all } f \in H^q(\Omega)+L^\infty(\Omega).$$

In particular, if Ω satisfies the assumption of this theorem with some $A > 1$ and if $0 < p \leqslant 1$, then every element of $H^p(\Omega)$ can be extended to an element of $H^p(\boldsymbol{R}^n)$.

Proof of Theorem 4. Let $0 < q < 1$. We define a map S from $\bigcup_{1<p\leqslant\infty} L^p(\Omega)$ to $\mathscr{D}'(\boldsymbol{R}^n)$ as follows. Let $f \in L^p(\Omega)$ and $1 < p \leqslant \infty$. Decompose f as in Section 3.3 with $m = [\gamma/q-\gamma]$ and $s = m+1$ (thus $\gamma/(\gamma+s) < q$) and obtain (3.17′). For each $I \in \mathscr{W}(\Omega)$, take $x'_I \in \Omega^c$ satisfying $\operatorname{dis}(x_I, x'_I) < A \operatorname{dis}(x_I, \Omega^c)$ and $\operatorname{dis}(x'_I, \Omega) > A^{-1} \operatorname{dis}(x_I, \Omega^c)$, set $I' = B(x'_I, 10A^{-1}r_I)$, and let Q_I be the unique element of $\mathscr{P}_m$ such that

$$\langle h^\Omega_I - Q_I\chi_{I'}, P\rangle = 0 \quad \text{for all } P \in \mathscr{P}_m.$$

We set

$$S(f) = \tilde{f} - \sum_{I \in \mathscr{W}(\Omega)} Q_I\chi_{I'}. \tag{6.1}$$

It is easy to see that $8I \subset C_A I'$, $I' \subset C_A I$, $2I' \cap \Omega = \varnothing$ and that the overlap of $\{I' \mid I \in \mathscr{W}(\Omega)\}$ does not exceed C_A. Moreover,

$$\|Q_I\chi_{I'}\|_\infty \leqslant C_{q,A}\|h^\Omega_I\|_{\infty,\Omega} \leqslant C_{q,A,\theta} \inf_{x\in I} f^*_{s,\Omega}(x). \tag{6.2}$$

From these facts, we easily see that

$$\|S(f)\|_p \leqslant C_{q,A,\theta,p}\|f\|_{p,\Omega} \quad \text{if } 1 < p \leqslant \infty. \tag{6.3}$$

It is also easy to see that $S(f)|\Omega = f$ for all f in the domain of S and that $S|L^p(\Omega)$, $1 < p \leqslant \infty$, is linear. Using (3.17′), we can rewrite (6.1) as

$$S(f) = \sum_{k,J} (h^k_J)^\sim + \sum_I (h^\Omega_I)', \tag{6.4}$$

where $(h^\Omega_I)' = (h^\Omega_I)^\sim - Q_I\chi_{I'}$.

Suppose $q \leqslant r \leqslant 1$. Then the above $(h^k_J)^\sim$ and $(h^\Omega_I)'$ are scalar multiples of r-atoms. Hence, using (6.4), (3.18)–(3.24) and (6.2), we see that

$$\|S(f)\|_{r,\Phi,\boldsymbol{R}^n} \leqslant C_{\theta,q,r,A,\Phi}\|f\|_{r,\Phi,\Omega} \tag{6.5}$$

for all f in the domain of S, in particular, for all $f \in C_0^\infty(\Omega)$. Hence, since $C_0^\infty(\Omega)$ is dense in $H^r(\Omega)$ (Lemma 7), the linear operator $S|C_0^\infty(\Omega)$ can be uniquely extended to a bounded linear operator from $H^r(\Omega)$ to $H^r(\boldsymbol{R}^n)$. We denote this extended operator by S_r. So $S_r(f)|\Omega = f$ for all $f \in H^r(\Omega)$.

If $q \leqslant r$, $r' \leqslant 1$, then $S_r(f) = S_{r'}(f)$ for all f in $H^r(\Omega) \cap H^{r'}(\Omega)$. This can be seen from the fact that if $f \in H^r(\Omega) \cap H^{r'}(\Omega)$ then there exists a sequence in $C_0^\infty(\Omega)$ which converges to f both in $H^r(\Omega)$ and in $H^{r'}(\Omega)$ (see the proof of Lemma 7). Similarly, if $q \leqslant r \leqslant 1 < p \leqslant \infty$, then $S_r(f) = S(f)$ for all f in $H^r(\Omega) \cap L^p(\Omega)$. (This can be proved by the same argument as above if $p < \infty$; the case $p = \infty$ can be reduced to the case $p < \infty$ since $H^r(\Omega) \cap L^\infty(\Omega) \subset L^p(\Omega)$ for all p with $1 < p < \infty$.)

We define the map T_q from $H^q(\Omega)+L^\infty(\Omega)$ to $\mathscr{D}'(\boldsymbol{R}^n)$ as follows: If $f = f_0+f_1$ with $f_0 \in H^q(\Omega)$ and $f_1 \in L^\infty(\Omega)$, then $T_q(f) = S_q(f_0)+S(f_1)$. The above paragraph shows that T_q is a well-defined linear mapping. If $q \leqslant p \leqslant 1$ and $f \in H^p(\Omega)$, then f can be written as $f = f_0+f_1$ with $f_0 \in H^q(\Omega) \cap H^p(\Omega)$ and $f_1 \in L^\infty(\Omega) \cap H^p(\Omega)$ (see the proof of Lemma 8). From this, we see that the restriction of T_q to $H^p(\Omega)$, $q \leqslant p \leqslant 1$, coincides with S_p. Similarly, the restrictions of T_q and S to $L^p(\Omega)$, $1 < p \leqslant \infty$, coincide. Hence, using (6.3) and (6.5), we see that T_q has the property (ii) of the theorem (notice that we can take θ depending only on n and $\{A(t)\}$). It is also easy to see that T_q has the property (i). This completes the proof of Theorem 4.

Remark. From Theorems 3 and 4 and their proofs, we can deduce the following result: If Ω and A satisfy the assumption of Theorem 4, then

$$C^{-1}\|\tilde{g}\|_{\mathrm{BMO}(\boldsymbol{R}^n)} \leqslant \|g\|_{\mathrm{BMO}(\Omega)} \leqslant C_A\|\tilde{g}\|_{\mathrm{BMO}(\boldsymbol{R}^n)}$$

and

$$\|\tilde{g}\|_{A(s)} \leqslant \|g\|_{A(s;\Omega)} \leqslant C_{s,A} \|\tilde{g}\|_{A(s)}$$

for all functions g on Ω with $\tilde{g} \in L^1_{\text{loc}}(\boldsymbol{R}^n)$ and for all $s > 0$.

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Weighted inequalities for the Hilbert transform and the adjoint operator in the continuous case

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Abstract. We prove two-weight norm inequalities in L^2 for the Hilbert transform in $\boldsymbol{R}$, of the Helson, Szegö and Sarason type.

I. Introduction. Arocena, Cotlar and Sadosky (see [3]) proved that the theory of generalized Toeplitz kernels can be used to obtain the theorems of Helson, Szegö and Sarason type (see [9, 10, 13]), with refinements.

Nevertheless in the case of two measures they do not obtain the Helson, Szegö and Sarason formula and in the case of $\boldsymbol{R}$ they consider, as Adams does (see [1]), functions with vanishing moments. In this paper, we consider two tempered measures, functions with vanishing Fourier transform in an interval, and use the theory of generalized Toeplitz kernels to give a constructive exponential characterization of Helson, Szegö and Sarason type for the Hilbert transform; and we do the same for finite measures, but with the adjoint operator.

The problems considered here arose in a natural way when we studied the following prediction theory problem proposed by Professor Ibragimov (private communication): characterize the continuous parameter weakly stationary completely linearly regular process such that the maximal correlation coefficient ϱ_t is $O(e^{-\lambda t})$ (see also [11]). In the previous papers (cf. [6–8]) this theory and an analogue of Theorem 1 were considered to obtain results about the rate of convergence of the maximal correlation coefficient in the continuous case including a solution to the problem stated by Professor Ibragimov.

An extension, to matrix-valued measures, of the results presented here is given in [5].

II. Basic problems

DEFINITION. A measure μ is *tempered of order* $\leqslant 2$ if $\mu/(x^2+1)$ is a finite measure.

Set

$M(\boldsymbol{R})$ = the positive finite Borel measures in $\boldsymbol{R}$,
$M^2(\boldsymbol{R})$ = the positive Borel tempered measures of order $\leqslant 2$ in $\boldsymbol{R}$.