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On a class of Markov type semigroups in spaces of uniformly continuous and bounded functions

by

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Abstract. We study a new class of Markov type semigroups (not strongly continuous in general) in the space of all real, uniformly continuous and bounded functions on a separable metric space E. Our results allow us to characterize the generators of Markov transition semigroups in infinite dimensions such as the heat and the Ornstein–Uhlenbeck semigroups.

1. Introduction. In this paper we study a new class of semigroups of bounded linear operators on $\mathcal{C}_{\mathrm{b}}(E)$, the Banach space of all real, uniformly continuous and bounded functions on a separable metric space E, endowed with the supremum norm $\|\cdot\|_0$. We call these semigroups π-semigroups. A π-semigroup P_t is characterized by the following assumptions:

(i) for any $f \in \mathcal{C}_{\mathrm{b}}(E)$ and $x \in E$, the map $[0,\infty[\to \mathbb{R}$, $t \mapsto P_t f(x)$, is continuous;

(ii) for any bounded sequence $(f_n) \subset \mathcal{C}_{\mathrm{b}}(E)$ such that f_n converges pointwise to $f \in \mathcal{C}_{\mathrm{b}}(E)$ (we briefly write $f_n \xrightarrow{\pi} f$), we have $P_t f_n \xrightarrow{\pi} P_t f$, $t \geq 0$;

(iii) there exist $M \geq 1$ and $\omega \geq 0$ such that $\|P_t f\|_0 \leq M e^{\omega t}\|f\|_0$, $f \in \mathcal{C}_{\mathrm{b}}(E)$, $t \geq 0$.

The main motivation of the paper is the study of semigroups of kernels in infinite dimensions. They arise as transition semigroups of Markov processes (see Definition 3.5) corresponding to solutions of stochastic differential equations and representing solutions of PDE's with infinitely many variables (we refer to [3], [4], [6], [8], [13], [18], [21], [22], [24]). These semigroups, when considered as a family of operators acting on $\mathcal{C}_{\mathrm{b}}(\Omega)$, where Ω is an open set of a separable Hilbert space H, turn out to be π-semigroups. On the other hand, in several cases the strong continuity fails to hold in $\mathcal{C}_{\mathrm{b}}(\Omega)$. This happens for instance for the Ornstein–Uhlenbeck semigroup or

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for the semigroup associated with a Dirichlet problem in a half space of H (see §4).

We define a generator $\mathcal{A}$ for a π-semigroup P_t as follows (see §3 for details):

$$\begin{cases} D(\mathcal{A}) = \{f \in \mathcal{C}_b(E) : \exists g \in \mathcal{C}_b(E),\ \exists \delta > 0 \text{ such that } \sup_{h\in]0,\delta]} \|\Delta_h f\|_0 < \infty \\ \qquad\qquad\qquad \text{and } \lim_{h\to 0^+} \Delta_h f(x) = g(x),\ x \in E\}, \\ \mathcal{A}f(x) = \lim_{h\to 0^+} \Delta_h f(x), \quad f \in D(\mathcal{A}),\ x \in E, \end{cases}$$

where $\Delta_h = h^{-1}(P_h - I)$. Notice that $\mathcal{A}$ does not have dense domain in general. We show that the resolvent operator of $\mathcal{A}$ can be obtained by a Laplace transform of P_t, which is pointwise defined in $\mathcal{C}_b(E)$.

In this paper there are three main results that we briefly present here. Let P_t be a π-semigroup and let $\mathcal{S}$ be any covering of E. We consider the following operator:

$$\begin{cases} D(\mathcal{A}_{\mathcal{S}}) = \{f \in D(\mathcal{A}) : \lim_{h\to 0^+} \sup_{x\in S} |\Delta_h f(x) - \mathcal{A}f(x)| = 0,\ S \in \mathcal{S}\}, \\ \mathcal{A}_{\mathcal{S}} f(x) = \mathcal{A}f(x), \quad x \in E. \end{cases}$$

Our first main theorem (see Theorem 3.7) is a kind of generalization of a well known result that states that for a $\mathcal{C}_0$-semigroup, the "weak" and "strong" generators coincide (see for instance [20, Theorem 1.3 of §2]).

THEOREM 1.1. *Let P_t be a π-semigroup and let $\mathcal{S}$ be any covering of E such that*

$$\lim_{t\to 0^+} \sup_{x\in S} |P_t f(x) - f(x)| = 0, \qquad f \in \mathcal{C}_b(E),\ S \in \mathcal{S}. \tag{1.1}$$

Then $\mathcal{A}_{\mathcal{S}} = \mathcal{A}$.

As a corollary, by taking $\mathcal{S} = \{H\}$, we deduce that if a π-semigroup P_t is also a $\mathcal{C}_0$-semigroup on $\mathcal{C}_b(E)$, then the generators of P_t as a $\mathcal{C}_0$-semigroup and as a π-semigroup coincide. We apply this fact to the heat semigroup in $\mathcal{C}_b(H)$, which is a Markov transition $\mathcal{C}_0$-semigroup (see Definition 3.5) and hence a π-semigroup as well. This way we obtain the second main result (see Theorem 4.1) that provides a new characterization for the generator of the heat semigroup, extending a classical theorem due to Gross (see Theorem 3 and Corollary 3.2 of [13]). We also investigate a "natural" locally convex topology on $\mathcal{C}_b(E)$, considered in [11], which induces the π-convergence for sequences (see Theorem 2.2).

The theory of π-semigroups is a development of Cerrai's theory of weakly continuous semigroups (see [5], [6] and Remark 2.4). They were introduced to study the Ornstein–Uhlenbeck semigroup on $\mathcal{C}_b(H)$, whose generator was defined through the pointwise Laplace transform of the semigroup. The same approach has been used to define a generator for other semigroups such as

the Mehler semigroups (see [12]) and the semigroup arising from an infinite-dimensional Dirichlet problem (see [21]). In Section 4 we show that all these semigroups are in fact π-semigroups and that their generators can also be defined through a pointwise limit of $\Delta_h f$ or equivalently through a uniform limit of $\Delta_h f$ on each compact set.

Our theory can be used to study Markov transition semigroups associated with solutions of stochastic differential equations of more general type (some results in this direction are contained in [24]).

Since π-semigroups are not strongly continuous in general, a comparison with other types of semigroups seems to be in order. A way to treat the lack of strong continuity for a semigroup is to find a suitable linear locally convex topology weaker than the norm topology of the underlying Banach space but more appropriate for the semigroup. Let us remark that the classical Yosida approach (see §IX.3 of [23]) in the treatment of semigroups of linear operators on locally convex spaces does not work in our case. Indeed, it requires that the locally convex topologies are sequentially complete (see Claim 4 in the proof of Theorem 2.2).

Several papers about semigroups on general locally convex spaces are available in the literature (see [16], [17] and the references of [16]). On this subject we show that π-semigroups are weakly integrable semigroups in the Jefferies sense (see [16], [17] and Remark 3.10). However our approach is different and simpler. In order to treat π-semigroups, we do not utilize weak Pettis type integration and do not have to consider the properties of a particular locally convex topology on $\mathcal{C}_b(E)$, which is difficult to characterize (see Theorem 2.2). We will only work with the norm topology in $\mathcal{C}_b(E)$.

We consider connections with the class of integrated semigroups, which has been intensively studied (see for instance [1], [15]). We point out that any generator of a π-semigroup is the generator of an integrated semigroup on $\mathcal{C}_b(E)$ as well (see Remark 3.9). However our results do not follow from the general theory of integrated semigroups.

Finally, one can consider analytic semigroups T_t on a Banach space X (i.e. the map $t \mapsto T_t$ is analytic in $]0,\infty[$ with values in $\mathcal{L}(X)$) without requiring the strong continuity at $t = 0$. This theory is developed in the book [19] by Lunardi to systematically treat parabolic PDE's in finite dimensions. Unfortunately the Ornstein–Uhlenbeck semigroup is not analytic in $\mathcal{C}_b(\mathbb{R}^n)$ (see [7]). In infinite dimensions the situation is worse, even the heat semigroup is not analytic in $\mathcal{C}_b(H)$ (see [14]).

In the last section we show that the theory of π-semigroups can also be developed on $\mathcal{BC}(E)$, the Banach space of all real continuous and bounded functions on E, endowed with the supremum norm. Notice that many Markov transition semigroups, such as the heat semigroup, are not strongly continuous on $\mathcal{BC}(H)$.

In forthcoming papers we will consider the Cauchy problem for π-semigroups and provide a Hille–Yosida type theorem. Some results in this direction as well as various remarks and details about π-semigroups are contained in [22].

2. Basic concepts. Let (E,d) be a separable metric space with metric d. We denote by $\mathcal{C}_{\rm b}(E)$ the set of all real, uniformly continuous and bounded functions on E. We consider $\mathcal{C}_{\rm b}(E)$ as a Banach space endowed with the sup norm: $\|f\|_0=\sup_{x\in E}|f(x)|$, $f\in\mathcal{C}_{\rm b}(E)$. A sequence $(f_n)\subset\mathcal{C}_{\rm b}(E)$ is said to be *π-convergent* to a map f and we write $f_n\xrightarrow{\pi}f$ as $n\to\infty$ if the following conditions hold:

$$\text{(2.1)}\qquad \begin{aligned} &\text{(a)}\quad f\in\mathcal{C}_{\rm b}(E),\qquad \sup_{n\ge1}\|f_n\|_0<\infty;\\ &\text{(b)}\quad \lim_{n\to\infty}f_n(x)=f(x),\qquad x\in E.\end{aligned}$$

Similarly, let J be a real interval and $\widehat{t}\in J$. Let $F:J\setminus\{\widehat{t}\}\to\mathcal{C}_{\rm b}(E)$. We say that $F(t)\xrightarrow{\pi}f$ as $t\to\widehat{t}$ if for any sequence $(t_n)\subset J\setminus\{\widehat{t}\}$ that converges to $\widehat{t}$ we have $F(t_n)\xrightarrow{\pi}f$ as $n\to\infty$. This implies that there exists a neighborhood U of $\widehat{t}$ such that $\sup_{t\in U\setminus\{\widehat{t}\}}\|F(t)\|_0<\infty$.

Let us remark that π-convergence for sequences of real bounded Borel functions is considered in the theory of Markov processes; for instance in the book [11] by Ethier and Kurtz it occurs as *bpc* (bounded pointwise convergence). Now we introduce π-semigroups on $\mathcal{C}_{\rm b}(E)$.

DEFINITION 2.1. Let P_t, $t\ge0$, be a semigroup of bounded linear operators on $\mathcal{C}_{\rm b}(E)$, that is, $P_0=I$ and $P_{t+s}=P_tP_s$ for $t,s\ge0$. We say that P_t is a *π-semigroup* on $\mathcal{C}_{\rm b}(E)$ if the following conditions hold ([1]):

(2.2) (i) there exist $M\ge1$ and $\omega\ge0$ such that $\|P_t\|_{\mathcal{L}(\mathcal{C}_{\rm b}(E))}\le Me^{\omega t}$, $t\ge0$;
(ii) for any $x\in E$ and $f\in\mathcal{C}_{\rm b}(E)$, the map $[0,\infty[\to\mathbb{R}$, $t\mapsto P_tf(x)$, is continuous;
(iii) for any $(f_n)\subset\mathcal{C}_{\rm b}(E)$, $f_n\xrightarrow{\pi}f$ implies that $P_tf_n\xrightarrow{\pi}P_tf$ as $n\to\infty$, for all $t\ge0$.

Let us remark that condition (i) is equivalent to requiring that the semigroup P_t is locally bounded (i.e. for any $T>0$, there exists a constant C_T such that $\|P_t\|_{\mathcal{L}(\mathcal{C}_{\rm b}(E))}\le C_T$, $t\in[0,T]$). The proof is standard.

Let P_t be a π-semigroup. We define the *type* of P_t as the real number

$$\omega=\inf\{\alpha\ge0:\text{there exists }M_\alpha\ge1\text{ with }\|P_t\|_{\mathcal{L}(\mathcal{C}_{\rm b}(E))}\le M_\alpha e^{\alpha t},\ t\ge0\}.$$

([1]) Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be two real Banach spaces. $\mathcal{L}(X,Y)$ stands for the Banach space of all bounded linear operators from X into Y, endowed with the norm $\|T\|_{\mathcal{L}(X,Y)}=\sup_{\|x\|_X\le1}\|Tx\|_Y$, $T\in\mathcal{L}(X,Y)$. We also set $\mathcal{L}(X)=\mathcal{L}(X,X)$, $X'=\mathcal{L}(X,\mathbb{R})$ and $\|\cdot\|'=\|\cdot\|_{X'}$.

Let E be infinite and $\mathcal{S}=\{S_i\}_{i\in I}$ be a *nontrivial covering* of E, i.e. $S_i\subset E$, $i\in I$, $E=\bigcup_{i\in I}S_i$ and there exists $S_{i_0}\in\mathcal{S}$ that is infinite. In the sequel we also consider π-semigroups P_t that satisfy the following additional condition:

$$\text{(2.3)}\qquad \lim_{t\to0^+}\sup_{x\in S}|P_tf(x)-f(x)|=0,\qquad f\in\mathcal{C}_{\rm b}(E),\ S\in\mathcal{S}.$$

Notice that in case P_t satisfies (2.3) with $\mathcal{S}=\{E\}$, it is also a strongly continuous semigroup on $\mathcal{C}_{\rm b}(E)$. ∎

Clearly π-convergence of sequences of functions does not define a unique topology on $\mathcal{C}_{\rm b}(E)$. However, there exists a "natural" locally convex topology τ_0 on $\mathcal{C}_{\rm b}(E)$ that induces the π-convergence for sequences (see [11], pp. 495–496, and Remark 2.3). In the next theorem we show that τ_0 can be defined in a different way, based on an embedding result concerning $\mathcal{C}_{\rm b}(E)'$ of independent interest (see Claim 1 in the proof of Theorem 2.2). Moreover we prove that τ_0 is neither sequentially complete nor metrizable.

We fix some preliminary notations. We consider $\mathcal{M}(E)$, the linear space of all finite Borel signed measures on E. Let $\mu\in\mathcal{M}(E)$. By the Hahn–Jordan Decomposition Theorem, $\mu=\mu_+-\mu_-$, where μ_+ and μ_- are positive measures. Moreover the variation of μ is $|\mu|=\mu_++\mu_-$. It is straightforward to verify that $\mathcal{M}(E)$ is a Banach space endowed with the norm $\|\mu\|_{\mathcal{M}}=|\mu|(E)$, $\mu\in\mathcal{M}(E)$.

Let X be a Banach space and Y be a subspace of X'. The locally convex topology $\sigma(X,Y)$ is the weakest topology on X making each $\eta\in Y$ continuous.

THEOREM 2.2. *The space $\mathcal{M}(E)$ can be considered as a closed subset of $\mathcal{C}_{\rm b}(E)'$. This way the Hausdorff locally convex topology $\tau_0=\sigma(\mathcal{C}_{\rm b}(E),\mathcal{M}(E))$ on $\mathcal{C}_{\rm b}(E)$ satisfies*:

(∗) *for any $(f_n)\subset\mathcal{C}_{\rm b}(E)$, f_n converges to $f\in\mathcal{C}_{\rm b}(E)$ with respect to $\tau_0\Leftrightarrow f_n\xrightarrow{\pi}f$.*

The topology τ_0 is not metrizable and not sequentially complete.

Proof. The proof is split up into several parts.

CLAIM 1. *$(\mathcal{M}(E),\|\cdot\|_{\mathcal{M}})$ is isometrically embedded in $(\mathcal{C}_{\rm b}(E)',\|\cdot\|')$.*

Consider the map $F:\mathcal{M}(E)\to\mathcal{C}_{\rm b}(E)'$ defined by

$$\text{(2.4)}\qquad \langle F_\mu,f\rangle=\int_E f(y)\,\mu(dy),\qquad \mu\in\mathcal{M}(E),\ f\in\mathcal{C}_{\rm b}(E).$$

We assert that F is an isometry. It is evident that $\|F_\mu\|'\le\|\mu\|_{\mathcal{M}}$ for any $\mu\in\mathcal{M}(E)$; let us prove the converse inequality.

For this purpose fix a $\mu \in \mathcal{M}(E)$, $\mu = \mu_+ - \mu_-$. There exist two Borel sets A_+ and A_- such that $A_+ \cap A_- = \emptyset$, $A_+ \cup A_- = E$ and $\mu_+(A_+) = \mu_+(E)$, $\mu_-(A_-) = \mu_-(E)$.

Fix $\varepsilon > 0$. By a property of finite Borel measures (see for instance Theorem 4.3.7 of [2]), we can choose a closed subset of E, $C_- \subset A_-$, such that $\mu_-(A_- \setminus C_-) < \varepsilon$. Now the crucial point is to show that there exists a closed subset of E, $C_+ \subset A_+$, such that

(2.5) $\qquad \mu_+(A_+ \setminus C_+) < \varepsilon$ and C_+ and C_- are separated

(i.e. $d(C_+, C_-) = \inf_{x \in C_+, y \in C_-} d(x,y) > 0$). We first take a closed set $C \subset A_+$ such that $\mu_+(A_+ \setminus C) < \varepsilon/2$. Then we consider a sequence of closed sets defined as follows: $C_n = \{x \in C : d(x, C_-) \geq n^{-1}\}$, $n \geq 1$.

Now we prove that $\mu_+(B_n) \to 0$ as $n \to \infty$, where $B_n = C \setminus C_n$. We have $B_n \downarrow B_0$ (i.e. $B_{n+1} \subset B_n$ and $\bigcap_{n \geq 1} B_n = B_0$). Since $C \cap C_- = \emptyset$ we get $B_0 = \emptyset$ and so $\mu_+(C \setminus C_n) \to 0$ as $n \to \infty$. Hence there exists n_0 such that $\mu_+(C \setminus C_{n_0}) < \varepsilon/2$. Thus (2.5) is proved by setting $C_+ = C_{n_0}$. Indeed, $d(C_{n_0}, C_-) \geq n_0^{-1}$ and

$$\mu_+(A_+ \setminus C_{n_0}) \leq \mu_+(A_+ \setminus C) + \mu_+(C \setminus C_{n_0}) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Consider the Borel map $g = I_{A_+} - I_{A_-}$ (2). It is clear that $\int_E g(y)\,\mu(dy) = |\mu|(E)$. Since C_+ and C_- are separated closed sets we can take a map $f \in \mathcal{C}_b(E)$ such that $\|f\|_0 = 1$, $f(x) = 1$ if $x \in C_+$ and $f(x) = -1$ if $x \in C_-$ (for instance set $f(x) = [d(x, C_-) + d(x, C_+)]^{-1}[d(x, C_-) - d(x, C_+)]$, $x \in E$). We can verify that

$$\Big|\int_E f(y)\,\mu(dy) - \int_E g(y)\,\mu(dy)\Big| \leq \int_E |f(y) - g(y)|\,|\mu|(dy)$$
$$= \int_{A_+ \setminus C_+} |f(y) - g(y)|\,\mu_+(dy) + \int_{A_- \setminus C_-} |f(y) - g(y)|\,\mu_-(dy) \leq 4\varepsilon.$$

Therefore $\|F_\mu\|' \geq \langle F_\mu, f\rangle \geq \int_E g(y)\,\mu(dy) - 4\varepsilon = |\mu|(E) - 4\varepsilon$. By the arbitrariness of ε we conclude that $\|F_\mu\|' \geq \|\mu\|_{\mathcal{M}}$. Thus F is an isometry.

CLAIM 2. *$\tau_0 = \sigma(\mathcal{C}_b(E), \mathcal{M}(E))$ satisfies condition* $(*)$.

For any $x \in E$ we denote by δ_x the Dirac measure with support $\{x\}$. Notice that τ_0 is a Hausdorff topology, since Dirac measures separate the elements of $\mathcal{C}_b(E)$.

We prove property $(*)$:

$\Leftarrow$ It is clear, by the Dominated Convergence Theorem.

$\Rightarrow$ If $f_n \to f$ with respect to τ_0, then using the Dirac measures we immediately conclude that $\lim_{n\to\infty} f_n(x) = f(x)$, $x \in E$.

(2) For any $B \subset E$, $I_B(x) = 0$ if $x \notin B$, $I_B(x) = 1$ if $x \in B$.

Assume by contradiction that $\sup_{n\geq 1} \|f_n\|_0 = \infty$. We can suppose that $\lim_{n\to\infty} f_n(x) = 0$, $x \in E$, and that $f_n \geq 0$.

Since (f_n) is not uniformly bounded, there exists a subsequence, denoted by (f_k), and a sequence $(x_k)_{k\geq 1} \subset E$ such that $f_k(x_k) > 2^k$, $k \geq 1$. Now consider the measure μ_0 defined by $\mu_0(B) = \sum_{k=1}^\infty 2^{-k}\delta_{x_k}(B)$ for any Borel set B in E. It is simple to verify that $\mu_0 \in \mathcal{M}(E)$. Moreover we have, for any $k \geq 1$,

$$\int_E f_k(y)\,\mu_0(dy) \geq 2^{-k}\int_E f_k(y)\,\delta_{x_k}(dy) = 2^{-k} f_k(x_k) \geq 1.$$

Thus f_k cannot converge to 0 with respect to τ_0 and we have obtained a contradiction.

CLAIM 3. *τ_0 is not metrizable.*

Actually it is possible to prove that τ_0 does not satisfy the first countable axiom, even if $E = \mathbb{R}$. We use the following theorem: if X is a Banach space and the topology $\sigma(X, X')$ satisfies the first countable axiom then X has finite dimension. For the proof we refer the reader to §II, p. 10 of [9].

Let us remark that the previous result also holds, with the same proof, if the topology $\sigma(X, X')$ is replaced by $\sigma(X, Y)$, where Y is a closed subspace of X'. Now to conclude one observes that, by Claim 1, using the isometry F, $\mathcal{M}(E)$ can be considered as a closed subset of $\mathcal{C}_b(E)'$.

CLAIM 4. *τ_0 is not sequentially complete.*

Actually we are able to prove a stronger statement: *any locally convex topology τ on $\mathcal{C}_b(E)$ satisfying condition $(*)$ (with τ_0 replaced by τ) is not sequentially complete.*

Denote by Γ the family of all seminorms on $\mathcal{C}_b(E)$ which are continuous with respect to τ. We say that $(f_n) \subset \mathcal{C}_b(E)$ is a *τ-Cauchy sequence* if

$$\lim_{n\to\infty} \sup_{k\geq 1} q(f_{n+k} - f_n) = 0 \quad \text{ for any } q \in \Gamma.$$

We fix $a \in E$ and introduce the functions $\widehat{f}_n(x) = \exp[-nd(x,a)]$, $x \in E$, $n \geq 1$. It is clear that $(\widehat{f}_n) \subset \mathcal{C}_b(E)$; we prove that it is a τ-Cauchy sequence.

Assume otherwise. Then there exist $\varepsilon > 0$, $\widehat{q} \in \Gamma$, and a sequence of integers (k_n) such that $\widehat{q}(\widehat{f}_{n+k_n} - \widehat{f}_n) \geq \varepsilon$ for any $n \geq 1$. Hence the sequence $\widehat{g}_n = \widehat{f}_{n+k_n} - \widehat{f}_n$ cannot converge to $0 \in \mathcal{C}_b(E)$ with respect to τ.

We obtain a contradiction by showing that $\widehat{g}_n$ does τ-converge to 0 as $n \to \infty$. Notice that by $(*)$, $\widehat{g}_n$ τ-converges to 0 if and only if $\widehat{g}_n \xrightarrow{\pi} 0$ as $n \to \infty$. Now since

$$\widehat{g}_n(x) = e^{-nd(x,a)}(e^{-k_n d(x,a)} - 1), \quad x \in E,\ n \geq 1,$$

we have $\|\widehat{g}_n\|_0 \le 1$, $n \ge 1$. Let us consider the pointwise convergence of $\widehat{g}_n$. If $x = a$, then $\widehat{g}(a) = 0$ for any $n \ge 1$. If $x \ne a$, then since $|\widehat{g}_n(x)| \le \exp[-nd(x,a)]$ for $n \ge 1$, we find that $\lim_{n\to\infty} \widehat{g}_n(x) = 0$. Therefore $\widehat{g}_n \xrightarrow{\pi} 0$ as $n \to \infty$. It follows that $(\widehat{f}_n)$ is a τ-Cauchy sequence.

Now notice that $\lim_{n\to\infty} \widehat{f}_n(x) = I_{\{a\}}(x)$, $x \in E$. Thus $(\widehat{f}_n)$ is not π-convergent to a map in $C_b(E)$. It follows that τ is not sequentially complete. ■

REMARK 2.3. We briefly comment on a different approach that can be used to define the topology τ_0 (see [11]). Consider the Banach space $(\mathcal{M}(E), \|\cdot\|_{\mathcal{M}})$ and its dual space $\mathcal{M}(E)'$. The space $C_b(E)$ can be identified with a closed subspace of $\mathcal{M}(E)'$ by setting $\langle f,\mu\rangle = \int_E f(y)\,\mu(dy)$, $f \in C_b(E)$, $\mu \in \mathcal{M}(E)$. Indeed, by considering Dirac measures, it follows that $\|f\|_0 = \sup_{\|\mu\|_{\mathcal{M}}\le 1} |\langle f,\mu\rangle| = \|f\|_{\mathcal{M}(E)'}$, $f \in C_b(E)$.

The topology τ_0 coincides with the restriction σ_0 of $\sigma(\mathcal{M}(E)', \mathcal{M}(E))$ to $C_b(E)$. Indeed, τ_0 and σ_0 determine the same class of convergent nets. In [11], pp. 495–496, the restriction γ of $\sigma(\mathcal{M}(E)', \mathcal{M}(E))$ to the space of all real Borel bounded functions is considered; it is also proved that γ induces the π-convergence for sequences.

Given a π-semigroup P_t on $C_b(E)$, by the previous theorem it follows that P_t is a semigroup of linear operators which are sequentially continuous on $C_b(E)$ with respect to τ_0. In this paper we will not investigate if the operators P_t, $t \ge 0$, are actually τ_0-continuous on $C_b(E)$ or not. Thus we only consider on $C_b(E)$ the sup norm topology.

REMARK 2.4. The theory of π-semigroups is a development of Cerrai's theory of *weakly continuous semigroups* (see [5] and [6]), where π-convergence for sequences in $C_b(E)$ is replaced by the uniform convergence on each compact set of E (the "$\mathcal{K}$-convergence") and some additional hypotheses are required. Any Markov transition semigroup (see Definition 3.5 for a precise definition) which is weakly continuous in Cerrai's sense is a π-semigroup.

REMARK 2.5. We make some comments on the choice of the space E.

(a) In case E is also a *compact set*, in order that a semigroup P_t of bounded linear operators on $C_b(E)$ is a π-semigroup and also a strongly continuous semigroup, it is enough that P_t satisfies the following two conditions:

(i) there exist $M \ge 1$ and $\omega \ge 0$ such that $\|P_t\|_{\mathcal{L}(C_b(E))} \le Me^{\omega t}$, $t \ge 0$;
(ii′) for any $x \in E$ and $f \in C_b(E)$, $\lim_{t\to 0^+} P_t f(x) = f(x)$.

To see this, first remark that, by the Riesz theorem, $C_b(E)'$ can be isometrically identified with $\mathcal{M}(E)$. Then the assertion follows by combining Theorem 2.2 and [23, §IX.1] ("weak equals strong") (see Remark 2.1.5 of [22] for more details).

(b) Suppose that $E \subset X$, where X is another separable metric space; then it may happen that a semigroup P_t of bounded linear operators on $C_b(E)$ is a π-semigroup on $C_b(E)$ but not on $C_b(\overline{E})$ (see for instance the semigroup in §4.2). Notice that this is impossible for strongly continuous semigroups.

3. The generator of a π-semigroup

DEFINITION 3.1. Let P_t be a π-semigroup on $C_b(E)$. We set $\Delta_h = h^{-1}(P_h - I)$, $h > 0$, and define its *infinitesimal generator* $\mathcal{A} : D(\mathcal{A}) \subset C_b(E) \to C_b(E)$ as follows:

$$\begin{cases} D(\mathcal{A}) = \{f \in C_b(E) : \exists g \in C_b(E),\ \Delta_h f \xrightarrow{\pi} g \text{ as } h \to 0^+\}, \\ \mathcal{A}f(x) := \lim_{h\to 0^+} \Delta_h f(x), \quad f \in D(\mathcal{A}),\ x \in E. \end{cases} \tag{3.1}$$

Let now $\mathcal{L} : D(\mathcal{L}) \subset C_b(E) \to C_b(E)$ be a linear operator. We say that $\mathcal{L}$ is *π-closed* if, for any $(f_n) \subset D(\mathcal{L})$, the following condition holds:

$$f_n \xrightarrow{\pi} f \text{ and } \mathcal{L}f_n \xrightarrow{\pi} g \text{ as } n \to \infty \Rightarrow f \in D(\mathcal{L}) \text{ and } \mathcal{L}f = g. \tag{3.2}$$

A subset $C \subset C_b(E)$ is said to be *π-dense* in $C_b(E)$ if, for any $f \in C_b(E)$, there exists a sequence $(f_n) \subset C$ such that $f_n \xrightarrow{\pi} f$ as $n \to \infty$.

Some basic properties of generators of π-semigroups are stated below.

PROPOSITION 3.2. *Let $\mathcal{A}$ be the generator of a π-semigroup P_t of type ω on $C_b(E)$. Then, for any $f \in D(\mathcal{A})$, we have*

(i) *$P_t f \in D(\mathcal{A})$ and $\mathcal{A}P_t f = P_t\mathcal{A}f$, $t \ge 0$;*
(ii) *for any $x \in E$, the map $[0,\infty[\, \to \mathbb{R}$, $t \mapsto P_t f(x)$, is continuously differentiable and $(d/dt)P_t f(x) = P_t\mathcal{A}f(x)$, $t \ge 0$.*

Proof. (i) Fix $f \in D(\mathcal{A})$ and $t > 0$. There exist $K \ge 0$ and $\delta > 0$ such that $\|\Delta_h f\|_0 \le K$ for any $h \in\,]0,\delta]$. Then $\|P_t\Delta_h f\|_0 \le MKe^{\omega t}$, $h \in\,]0,\delta]$, and applying Definition 2.1(iii), we find

$$\lim_{h\to 0^+} \Delta_h P_t f(x) = \lim_{h\to 0^+} P_t\Delta_h f(x) = P_t\mathcal{A}f(x), \quad x \in E.$$

Thus $P_t f \in D(\mathcal{A})$ and $\mathcal{A}P_t f = P_t\mathcal{A}f$.

(ii) Fix $f \in D(\mathcal{A})$, $x \in E$ and consider the map $t \mapsto P_t f(x)$. The right derivative $(d^+/dt)P_t f(x) = P_t\mathcal{A}f(x)$ exists at any $t \ge 0$. Moreover the map $t \mapsto P_t\mathcal{A}f(x)$ is continuous and so, by a well known lemma of Real Analysis, we obtain the assertion. ■

To proceed with the study of generators of π-semigroups, we need a preliminary lemma. It is basic for the treatment of π-semigroups on $C_b(E)$.

LEMMA 3.3. *Let (Y,μ) be a measurable space (μ is a finite, positive and complete measure). Consider a function $F : Y \times E \to \mathbb{R}$ that satisfies:*

(i) *$F(\cdot,x)$ is measurable for any $x \in E$;*
(ii) *$F(y,\cdot)$ is uniformly continuous, for μ-a.e. $y \in Y$;*
(iii) *there exists $g : Y \to \mathbb{R}$, μ-integrable such that $|F(y,x)| \le g(y)$ for all $x \in E$ and μ-a.e. $y \in Y$.*

Then the map $h : E \to \mathbb{R}$,

$$h(x) = \int_Y F(y,x)\,\mu(dy), \qquad x \in E,$$

is uniformly continuous and bounded.

Proof. The boundedness of h is clear, as is its continuity by the Dominated Convergence Theorem. Let us prove the uniform continuity of h. For any $n \ge 1$, we consider the set $A_n = \{(x,x') \in E \times E : d(x,x') \le 1/n\}$.

To verify the assertion, we prove that

$$\lim_{n\to\infty} \sup_{(x,x')\in A_n} |h(x) - h(x')| = 0. \tag{3.3}$$

For any $n \ge 1$, choose a countable dense set D_n in A_n (thanks to the separability of $E \times E$). Then for μ-a.e. $y \in Y$ we have

$$\sup_{(x,x')\in A_n} |F(y,x) - F(y,x')| = \sup_{(x,x')\in D_n} |F(y,x) - F(y,x')|, \qquad n \ge 1,$$

since for μ-a.e. $y \in Y$, $|F(y,x) - F(y,x')|$ is uniformly continuous on $E \times E$.

Now we remark that, for any $n \ge 1$, since D_n is countable the map

$$Y \to \mathbb{R}, \qquad y \mapsto \sup_{(x,x')\in D_n} |F(y,x) - F(y,x')|,$$

is measurable. Moreover, $\sup_{(x,x')\in D_n} |F(y,x) - F(y,x')| \le 2g(y)$ for all $n \ge 1$ and μ-a.e. $y \in Y$. Thus we get, for any $n \ge 1$,

$$\begin{aligned} \sup_{(x,x')\in A_n} |h(x) - h(x')| &\le \sup_{(x,x')\in A_n} \int_Y |F(y,x) - F(y,x')|\,\mu(dy) \\ &\le \int_Y \sup_{(x,x')\in D_n} |F(y,x) - F(y,x')|\,\mu(dy). \end{aligned} \tag{3.4}$$

Now letting $n \to \infty$ in the last term, by the Dominated Convergence Theorem, we find (3.3). ■

PROPOSITION 3.4. *Let $\mathcal{A}$ be the generator of a π-semigroup P_t of type ω on $C_b(E)$. Then:*

(i) *$D(\mathcal{A})$ is π-dense in $C_b(E)$;*
(ii) *$\mathcal{A}$ is a π-closed operator on $C_b(E)$.*

Proof. Fix any $f \in C_b(E)$ and consider for any $t > 0$ the maps

$$E \to \mathbb{R}, \qquad x \mapsto \int_0^t P_s f(x)\,ds.$$

By Lemma 3.3, they belong to $C_b(E)$. Let us prove that they belong to $D(\mathcal{A})$ for any $t > 0$. First note that

$$P_h\Big(\int_0^t P_s f(\cdot)\,ds\Big)(x) = \int_0^t P_{h+s} f(x)\,ds, \qquad x \in E,\ t \ge 0,\ h \ge 0, \tag{3.5}$$

since $\int_0^t P_s f(\cdot)\,ds$ is the π-limit of a sequence of Riemann sums in $C_b(E)$. Then by standard computations (see for instance [20]), we obtain the assertions. ■

Let us review the important class of Markov transition π-semigroups. We also consider Dynkin's weak generator which is used in the treatment of Markov transition functions and which is similar to our generator of π-semigroups (we refer to [10, Chapter II, §2] for more details).

DEFINITION 3.5. A semigroup T_t of bounded linear operators on $\mathcal{B}_b(E)$, the space of all bounded, real and Borel functions on E, is a *Markov transition semigroup* if it can be represented as follows:

$$T_t f(x) = \int_E f(y)\,p(t,x,dy), \qquad f \in \mathcal{B}_b(E),\ x \in E,\ t \ge 0,$$

where $p(t,x,B)$, for $t > 0$, B a Borel subset of E and $x \in E$, denotes a *Markov transition function* on E. If the transition semigroup T_t on $\mathcal{B}_b(E)$ satisfies the additional conditions:

(v) for any $x \in E$ and $f \in C_b(E)$, the map $t \mapsto T_t f(x)$ is continuous,
(vi) $T_t(C_b(E)) \subset C_b(E)$, $t \ge 0$,

then the restriction of T_t to $C_b(E)$ is a π-semigroup. We call it a *transition π-semigroup* on $C_b(E)$.

Given a Markov transition semigroup T_t on $\mathcal{B}_b(E)$, Dynkin introduces the space $\mathcal{B}_b^0(E) = \{f \in \mathcal{B}_b(E) : \lim_{t\to 0^+} T_t f(x) = f(x) \text{ for } x \in E\}$.

Moreover he defines the *weak generator* $\widetilde{\mathcal{A}}$ of T_t by setting $D(\widetilde{\mathcal{A}}) = \{f \in \mathcal{B}_b(E)$: there exists $g \in \mathcal{B}_b^0(E)$ such that $\lim_{t\to 0^+} t^{-1}[T_t f(x) - f(x)] = g(x)$ for $x \in E$, and there exists $\delta > 0$ such that $\|t^{-1}[T_t f - f]\|_0 \le M$ for any $t \in]0,\delta]\}$. For any $f \in D(\widetilde{\mathcal{A}})$,

$$\widetilde{\mathcal{A}} f(x) := \lim_{t\to 0^+} t^{-1}[T_t f(x) - f(x)], \qquad x \in E.$$

Let P_t be a π-semigroup on $C_b(E)$ such that $\|P_t\|_{\mathcal{L}(C_b(E))} \le Me^{\alpha t}$ for $t \ge 0$ with $\alpha \in \mathbb{R}$ and $M \ge 1$. Consider the operators $(F_\lambda)_{\lambda>\alpha}$ defined as follows:

$$F_\lambda f(x) = \int_0^\infty e^{-\lambda u} P_u f(x)\, du, \qquad f \in \mathcal{C}_b(E),\ x \in E,\ \lambda > \alpha. \tag{3.6}$$

By Lemma 3.3, we deduce that (F_λ) is a family of bounded linear operators on $\mathcal{C}_b(E)$. Moreover $\|F_\lambda f\|_0 \le M(\lambda-\alpha)^{-1}\|f\|_0$ for $f \in \mathcal{C}_b(E)$ and $\lambda > \alpha$.

Let $\mathcal{A}$ be the generator of P_t. By Proposition 3.4, we know in particular that $\mathcal{A}$ is a closed operator. Next we characterize the resolvent operator $R(\lambda, \mathcal{A})$ of $\mathcal{A}$.

PROPOSITION 3.6. *Let P_t be a π-semigroup with generator $\mathcal{A}$ such that $\|P_t\|_{\mathcal{L}(\mathcal{C}_b(E))} \le Me^{\alpha t}$ for $t \ge 0$ with $\alpha \in \mathbb{R}$ and $M \ge 1$. Consider the operators $(F_\lambda)_{\lambda>\alpha}$ defined in* (3.6). *Then, for any $\lambda > \alpha$, we have*

(i) *$R(\lambda, \mathcal{A}) = F_\lambda$ exists;*

(ii) *$\|R(\lambda, \mathcal{A})^n\|_{\mathcal{L}(\mathcal{C}_b(E))} \le M/(\lambda-\alpha)^n$, $n \ge 1$.*

Proof. (i) First we prove that for $f \in \mathcal{C}_b(E)$ and $\lambda > \alpha$ we have $F_\lambda f \in D(\mathcal{A})$ and

$$(\lambda - \mathcal{A})F_\lambda f = f. \tag{3.7}$$

We fix $f \in \mathcal{C}_b(E)$ and $\lambda > \alpha$ and define the maps $g = F_\lambda f$ and g_T, $g_T(x) = \int_0^T e^{-\lambda u} P_u f(x)\, du$, $x \in E$, $T > 0$. We obtain

$$\lim_{T\to\infty} \|g_T - g\|_0 \le \lim_{T\to\infty} M\|f\|_0 \int_T^\infty e^{(\alpha-\lambda)u}\, du = 0. \tag{3.8}$$

Taking into account that $P_h g_T(x) = \int_0^T e^{-\lambda u} P_{u+h} f(x)\, du$ for $x \in E$, $T > 0$ and $h \ge 0$ (since g_T is a π-limit of Riemann sums in $\mathcal{C}_b(E)$) we have

$$\begin{aligned} \Delta_h g(x) &= \left(\frac{P_h - I}{h}\right) g(x) = \frac{1}{h}\int_0^\infty e^{-\lambda u}[P_{u+h}f(x) - P_u f(x)]\, du \\ &= \Gamma_1 f(x,h) - \Gamma_2 f(x,h) \end{aligned} \tag{3.9}$$

where

$$\Gamma_1 f(x,h) = \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda u} P_u f(x)\, du,$$

$$\Gamma_2 f(x,h) = \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda u} P_u f(x)\, du, \qquad x \in E,\ h > 0.$$

It follows that

$$\|\Delta_h g\|_0 \le M\|f\|_0 e^\lambda \left[\frac{\lambda}{\lambda-\alpha} + e^\alpha\right], \qquad h \in \left]0,1\right].$$

Further

$$\lim_{h\to 0^+} \sup_{x\in E} |\Gamma_1 f(h,x) - \lambda g(x)| = 0. \tag{3.10}$$

Concerning the second term $\Gamma_2 f(h,x)$ we find, for any $x \in E$,

$$\begin{aligned} |\Gamma_2 f(h,x) - f(x)| &\le \left|\Gamma_2 f(h,x) - \frac{1}{h}\int_0^h f(x)\, du\right| \\ &\le \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda u}|P_u f(x) - f(x)|\, du \\ &\quad + \|f\|_0 \frac{e^{\lambda h}}{h}\int_0^h [e^{-\lambda u} - 1]\, du, \end{aligned} \tag{3.11}$$

which tends to 0 as $h \to 0^+$, since $\lim_{t\to 0^+} P_t f(x) = f(x)$, $x \in E$.

Thus we have verified that $\Delta_h g \xrightarrow{\pi} \lambda g - f$ as $h \to 0^+$ and consequently $g \in D(\mathcal{A})$ and $\mathcal{A}g = \lambda g - f$. It follows that $(\lambda - \mathcal{A})F_\lambda f = f$.

Now assume that $l \in D(\mathcal{A})$. We claim that $F_\lambda \mathcal{A} l = \mathcal{A} F_\lambda l$ for $\lambda > \alpha$. This fact and (3.7) will imply that $F_\lambda(\lambda - \mathcal{A})l = l$. We have

$$\begin{aligned} F_\lambda \mathcal{A} l(x) &= \int_0^\infty e^{-\lambda u} P_u \mathcal{A} l(x)\, du = \int_0^\infty e^{-\lambda u} \mathcal{A} P_u l(x)\, du \\ &= \mathcal{A}\int_0^\infty e^{-\lambda u} P_u l(x)\, du = \mathcal{A} F_\lambda l(x), \qquad x \in E. \end{aligned}$$

We have used formula (3.8) and the fact that $\mathcal{A}$ is a π-closed operator on $\mathcal{C}_b(E)$. Thus we have proved that $R(\lambda, \mathcal{A})$ exists for $\lambda > \alpha$ and

$$R(\lambda, \mathcal{A}) f(x) = \int_0^\infty e^{-\lambda u} P_u f(x)\, du, \qquad f \in \mathcal{C}_b(E),\ x \in E. \tag{3.12}$$

(ii) From (3.12) in a standard way (differentiating with respect to λ and using induction) we can obtain, for any $f \in \mathcal{C}_b(E)$, $n \ge 1$, and $\lambda > \alpha$,

$$R(\lambda, \mathcal{A})^n f(x) = \frac{1}{(n-1)!}\int_0^\infty u^{n-1} e^{-\lambda u} P_u f(x)\, du, \qquad x \in E, \tag{3.13}$$

and now (ii) easily follows. ∎

Let now $\mathcal{S}$ be a nontrivial covering of E (see Definition 2.1) and P_t be a π-semigroup. We consider another linear operator $\mathcal{A}_\mathcal{S} : D(\mathcal{A}_\mathcal{S}) \subset \mathcal{C}_b(E) \to \mathcal{C}_b(E)$, defined as follows:

$$\begin{cases} D(\mathcal{A}_\mathcal{S}) = \{f \in D(\mathcal{A}) : \text{for any } S \in \mathcal{S}, \\ \qquad\qquad \lim_{h\to 0^+} \sup_{x\in S} |\Delta_h f(x) - \mathcal{A} f(x)| = 0\}, \\ \mathcal{A}_\mathcal{S} f(x) := \mathcal{A} f(x), \quad f \in D(\mathcal{A}_\mathcal{S}),\ x \in E. \end{cases} \tag{3.14}$$

The proof of Proposition 3.6 can be suitably adapted in order to prove the main result of this section.

THEOREM 3.7. *Let P_t be a π-semigroup in $C_b(E)$ of type ω (denote by $\mathcal{A}$ its generator) and let $\mathcal{S}$ be a nontrivial covering of E. Suppose that formula* (2.3) *is satisfied by $\mathcal{S}$ and P_t. Then $\mathcal{A}_{\mathcal{S}} = \mathcal{A}$.*

Proof. Since $\mathcal{A}$ is an extension of $\mathcal{A}_{\mathcal{S}}$, we only have to prove that $D(\mathcal{A}) \subset D(\mathcal{A}_{\mathcal{S}})$.

To this end, fix $g \in D(\mathcal{A})$ and $\lambda > \omega$. Define $f = (\lambda - \mathcal{A})g$ so that

$$g(x) = \int_0^\infty e^{-\lambda u} P_u f(x)\, du, \quad x \in E.$$

We prove that $g \in D(\mathcal{A}_{\mathcal{S}})$ and $\mathcal{A}_{\mathcal{S}} g = \lambda g - f$.

Fix $S \in \mathcal{S}$ and take into account the proof of Proposition 3.6(i). With the same notations we have $\Delta_h g(x) = \Gamma_1 f(h,x) - \Gamma_2 f(h,x)$, $x \in E$, $h > 0$. It follows that

$$(3.15)\qquad \sup_{x\in S} |\Delta_h g(x) - \lambda g(x) + f(x)| \le \sup_{x\in S} |\Gamma_1 f(x,h) - \lambda g(x)| + \sup_{x\in S} |\Gamma_2 f(x,h) - f(x)|.$$

Now by (3.10) we know that $\lim_{h\to 0^+} \sup_{x\in S} |\Gamma_1 f(x,h) - \lambda g(x)| = 0$. Consider the second term of (3.15):

$$(3.16)\qquad \sup_{x\in S} |\Gamma_2 f(x,h) - f(x)| \le \frac{e^{\lambda h}}{h} \sup_{x\in S} \int_0^h e^{-\lambda u} |P_u f(x) - f(x)|\, du + \|f\|_0 \frac{e^{\lambda h}}{h} \int_0^h [e^{-\lambda u} - 1]\, du,$$

which tends to 0 as $h \to 0^+$, since $\lim_{h\to 0^+} \sup_{x\in S} |P_h f(x) - f(x)| = 0$ by our hypothesis. ∎

The next result provides a useful characterization for the domain of a Markov transition C_0-semigroup on $C_b(E)$.

COROLLARY 3.8. *Let P_t be a π-semigroup on $C_b(E)$ with generator $\mathcal{A}$. Suppose in addition that it is a strongly continuous semigroup on $C_b(E)$ and denote by $\mathcal{A}_E$ its generator. Then $\mathcal{A} = \mathcal{A}_E$.*

Proof. We can apply the previous theorem with $\mathcal{S} = \{E\}$. ∎

We conclude the section by making a comparison with other classes of semigroups.

REMARK 3.9. We consider integrated semigroups (see for instance [1] and [15]). By Proposition 3.6, invoking [1, Theorem 4.1], we can state the following result:

Let $\mathcal{A}$ be the generator of a π-semigroup P_t on $C_b(E)$. Then $\mathcal{A}$ also generates a once integrated semigroup S_t on $C_b(E)$. Moreover, for any $f \in \overline{D(\mathcal{A})}$, we have $S_t f = \int_0^t P_r f\, dr$ (the integral is considered in the strong sense).

REMARK 3.10. Here we show that any π-semigroup P_t on $C_b(E)$ is a weakly integrable semigroup on $C_b(E)$ in the Jefferies sense (see [16] and [17]). To this end we verify the Jefferies initial assumptions (S1) and (S2).

Denote by $\langle\cdot,\cdot\rangle$ the duality between $C_b(E)$ and $C_b(E)'$ (the topological dual of $C_b(E)$) and consider the space

$$\Lambda(E) = \{\xi \in C_b(E)' : \text{for any } (u_n) \subset C_b(E),\ u_n \xrightarrow{\pi} u \text{ as } n \to \infty \text{ implies that } \lim_{n\to\infty} \langle \xi, u_n\rangle = \langle \xi, u\rangle\}.$$

It is possible to verify that $\Lambda(E) \ne C_b(E)'$ even if $E = \mathbb{R}$. $\Lambda(E)$ separates the points of $C_b(E)$ and, by Theorem 2.2, $\mathcal{M}(E) \subset \Lambda(E)$. Moreover one verifies that $\Lambda(E)$ is an invariant subspace with respect to the dual semigroup P_t': $C_b(E)' \to C_b(E)'$, $t \ge 0$, and so hypothesis (S1) is satisfied.

Consider hypothesis (S2). For any $\xi \in \Lambda(E)$ and $f \in C_b(E)$, one can check that the map $\mathbb{R}_+ \to \mathbb{R}$, $t \mapsto \langle P_t f, \xi\rangle$, is continuous. Then for $f \in C_b(E)$ and $\lambda > \omega$, we set $g = R(\lambda, \mathcal{A}) f$ (where $\mathcal{A}$ is the generator of P_t and ω its type). We have

$$(3.17)\qquad \langle g, \xi\rangle = \int_0^\infty e^{-\lambda u} \langle P_u f, \xi\rangle\, du, \quad \xi \in \Lambda(E).$$

Indeed, consider, for any $T > 0$, the map g_T defined by

$$g_T(x) = \int_0^T e^{-\lambda u} P_u f(x)\, du, \quad x \in E.$$

Then $g_T \in C_b(E)$ and g_T is a π-limit of Riemann sums in $C_b(E)$. Hence

$$\int_0^T e^{-\lambda u} \langle P_u f, \xi\rangle\, du = \Big\langle \int_0^T e^{-\lambda u} P_u f\, du, \xi \Big\rangle = \langle g_T, \xi\rangle, \quad T > 0.$$

Now letting $T \to \infty$, we get (3.17), since $g_T \to g$ in $C_b(E)$ as $T \to \infty$. Thus formula (3.17) holds and (S2) is verified. Hence we can say, using the Jefferies terminology, that P_t is a $\Lambda(E)$-*semigroup* on $C_b(E)$.

4. Examples of π-semigroups. This section is devoted to describing some basic transition π-semigroups (see Definition 3.5) connected with PDE's with infinitely many variables. Previous results will be applied to give a detailed characterization of their generators.

H stands for a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm denoted by $|\cdot|$. Q will be a positive (i.e. nonnegative and one-to-one) self-adjoint trace class (or nuclear) operator on H ($\mathrm{Tr}(Q)$ will denote the trace of Q).

Fix once and for all an orthonormal basis $\{e_k\}_{k\ge1}$ of H that diagonalizes Q. For any $x\in H$, $Qx=\sum_{k=1}^{\infty}\lambda_k x_k e_k$ with $x_k=\langle x,e_k\rangle$, $\lambda_k>0$, $k\ge1$.

We also consider the Gaussian measure $\mathcal{N}(x,tQ)$ on H, with mean $x\in H$ and covariance operator tQ, $t>0$ (we refer to [8] for more details).

4.1. *The heat semigroup.* We define the heat semigroup O_t on $\mathcal{C}_{\mathrm b}(H)$, associated with the operator Q, by

$$O_tf(x)=\int_H f(x+y)\mathcal{N}(0,tQ)\,dy,\qquad f\in\mathcal{C}_{\mathrm b}(H),\ x\in H,\ t>0. \tag{4.1}$$

It is well known that it is a strongly continuous semigroup on $\mathcal{C}_{\mathrm b}(H)$. O_t is clearly also a transition π-semigroup. In order to characterize its generator, let us first review some function spaces related to O_t.

$\mathcal{C}^1_Q(H)$ is the set of all $f\in\mathcal{C}_{\mathrm b}(H)$ such that:

(i) for any $v\in H$ and $x\in H$, the directional derivative of f at x in direction $Q^{1/2}v$ exists; we denote it by $D_{Q^{1/2}v}f(x)$;

(ii) for any $x\in H$ there exists $D_Qf(x)\in H$ such that

$$D_{Q^{1/2}v}f(x)=\langle D_Qf(x),v\rangle,\qquad v\in H;$$

(iii) the mapping $H\to H$, $x\mapsto D_Qf(x)$, belongs to $\mathcal{C}_{\mathrm b}(H,H)$ (3).

It is easy to prove that, defining the *partial derivatives* $D_kf=D_{e_k}f$, $k\ge1$, for any $f\in\mathcal{C}^1_Q(H)$ we have

$$D_Qf(x)=\sum_{k=1}^{\infty}\sqrt{\lambda_k}\,D_kf(x)e_k,\qquad x\in H.$$

$\mathcal{C}^2_Q(H)$ is the set of all functions in $\mathcal{C}^1_Q(H)$ such that:

(i) for any $v\in H$ and $x\in H$, the directional derivative

$$D_{Q^{1/2}v}(D_Qf)(x)=\lim_{s\to0^+}\frac{D_Qf(x+sQ^{1/2}v)-D_Qf(x)}{s}\quad\text{in } H$$

exists;

(ii) for any $x\in H$, there exists $D^2_Qf(x)\in\mathcal{L}(H)$ such that

$$D_{Q^{1/2}v}(D_Qf)(x)=D^2_Qf(x)(v),\qquad v\in H;$$

(iii) the map $H\to\mathcal{L}(H)$, $x\mapsto D^2_Qf(x)$, belongs to $\mathcal{C}_{\mathrm b}(H,\mathcal{L}(H))$.

(3) Let $(E,\|\cdot\|_E)$ and $(F,\|\cdot\|_F)$ be two Banach spaces and $S\subset E$. We denote by $\mathcal{C}_{\mathrm b}(S,F)$ the Banach space of all uniformly continuous and bounded functions from S into F, endowed with the usual sup norm $\|f\|_0=\sup_{x\in S}\|f(x)\|_F$.

Setting $D_{e_h}(D_kf)=D_{hk}f$, $h,k\ge1$, we can easily see that

$$\langle D^2_Qf(x)u,v\rangle=\sum_{h,k=1}^{\infty}\sqrt{\lambda_h\lambda_k}D_{hk}f(x)u_kv_h,\qquad x\in H,\ u,v\in H,\ f\in\mathcal{C}^2_Q(H).$$

In a similar way it is possible to define the spaces $\mathcal{C}^n_Q(H)$ and the differential operators D^n_Q for $n\ge3$, and also $\mathcal{C}^\infty_Q(H)=\bigcap_{n\ge1}\mathcal{C}^n_Q(H)$. Every $\mathcal{C}^n_Q(H)$, $n\ge1$, turns out to be a Banach space, endowed with the norm

$$\|f\|_{n,Q}=\|f\|_0+\sum_{j=1}^{n}\|D^j_Qf\|_0,\qquad f\in\mathcal{C}^n_Q(H).$$

Some comments about the above spaces are in order. The space $\mathcal{C}^1_Q(H)$ has recently been introduced by Cannarsa and Da Prato [3]. The spaces $\mathcal{C}^n_Q(H)$, $n\ge2$, are introduced in [21]; they are a slight modification of those considered in [3].

The spaces $\mathcal{C}^n_Q(H)$ are related to the differentiability along the reproducing kernel space of the Gaussian measure $\mathcal{N}(0,Q)$. This type of differentiability was considered by Gross in the more general setting of the *abstract Wiener spaces* (we refer to [13] and [18] for a detailed exposition). $H_0=Q^{1/2}H$ is called the *reproducing kernel of* $\mathcal{N}(0,Q)$; it is a Hilbert space endowed with the inner product $\langle u,v\rangle_{H_0}:=\langle Q^{-1/2}u,Q^{-1/2}v\rangle_H$, $u,v\in H_0$. If $p_t=\mathcal{N}(0,tQ)$, $t>0$, and $i:H_0\to H$ denotes the natural embedding then (H_0,H,i) is an abstract Wiener space with Wiener measure p_t.

Now it is not difficult to verify that $\mathcal{C}^1_Q(H)$ coincides with the space of all functions $g\in\mathcal{C}_{\mathrm b}(H)$ such that g is H_0-differentiable on H in the Gross sense (see [13, §3]), and the H_0-derivative $D_{H_0}g$ is in $\mathcal{C}_{\mathrm b}(H,H_0)$ (we only remark that $D_{H_0}f(x)=Q^{1/2}D_Qf(x)$, $f\in\mathcal{C}^1_Q(H)$, $x\in H$). The same happens for higher order H_0-derivatives (an analysis of these connections is given in [22, Appendix A]).

$\mathcal{L}_1(H)$ stands for the subspace of $\mathcal{L}(H)$ of all trace class operators. It is a Banach space endowed with the norm $\|T\|_1=\mathrm{Tr}(\sqrt{T^*T})$, $T\in\mathcal{L}_1(H)$.

Let us introduce the following linear operator:

$$\begin{cases} D(\mathcal{A}_0)=\{f\in\mathcal{C}^2_Q(H): D^2_Qf(x)\in\mathcal{L}_1(H)\text{ for }x\in H,\text{ and}\\ \qquad\qquad D^2_Qf\in\mathcal{C}_{\mathrm b}(H,\mathcal{L}_1(H))\};\\ \mathcal{A}_0:D(\mathcal{A}_0)\to\mathcal{C}_{\mathrm b}(H),\quad \mathcal{A}_0f(x):=\frac12\mathrm{Tr}[D^2_Qf(x)],\\ \qquad\qquad f\in D(\mathcal{A}_0),\ x\in H.\end{cases} \tag{4.2}$$

In terms of the orthonormal basis $\{e_k\}_{k\ge1}$ we have, for any $f\in D(\mathcal{A}_0)$,

$$\mathcal{A}_0f(x)=\frac12\mathrm{Tr}[D^2_Qf(x)]=\frac12\sum_{k=1}^{\infty}\lambda_kD_{kk}f(x),\qquad x\in H.$$

We denote by $\mathcal{A}_H$ the generator of O_t when considered as a C_0-semigroup, and by $\mathcal{A}$ its generator as a π-semigroup on $C_b(H)$. By Corollary 3.8, $\mathcal{A} = \mathcal{A}_H$. Using this fact we prove the main result of this section. It asserts that $D(\mathcal{A}_0)$ is a core for $\mathcal{A}_H$.

THEOREM 4.1. *Let O_t be the heat semigroup on $C_b(H)$ defined by* (4.1) *with generator $\mathcal{A}$. Then:*

(i) *$\mathcal{A}$ is an extension of $\mathcal{A}_0$;*
(ii) *$D(\mathcal{A}_0)$ is dense in $D(\mathcal{A})$ with respect to the graph norm.*

Proof. Statement (ii) follows from (i) and [13, Theorem 3 and Corollary 3.2]. Let us prove the first assertion.

(i) Fix $\widehat{f} \in D(\mathcal{A}_0)$. We have to show that $\widehat{f} \in D(\mathcal{A})$ and $\mathcal{A}\widehat{f}(x) = \mathcal{A}_0\widehat{f}(x)$, $x \in H$. We split up the proof into several steps.

STEP 1. Denote by $P^n : H \to \mathbb{R}^n$, $n \geq 1$, the finite-dimensional projections $P^n x = \sum_{k=1}^n x_k e_k$, $x \in H$.

Let us introduce, for any $n \geq 1$ and $t > 0$, the approximating operators $O_t^n : C_b(H) \to C_b(H)$, defined as follows:

$$O_t^n f(x) = \int_H f(x + P^n y)\mathcal{N}(0, tQ)\, dy, \quad f \in C_b(H),\ x \in H. \tag{4.3}$$

It is easy to check, using the standard properties of Gaussian measures, that O_t^n is a strongly continuous semigroup of bounded linear operators on $C_b(H)$, for any $n \geq 1$. We now prove that $O_t^n f \to O_t f$ in $C_b(H)$ as $n \to \infty$, uniformly in t on bounded sets of $[0, \infty[$ (a similar statement is proved in [4, Theorem 3.1]).

Let $f \in C_b(H)$, $T > 0$ and denote by ω_f the modulus of continuity of f. We have, for any $n \geq 1$ and $t \in [0, T]$,

$$\begin{aligned} &\sup_{x \in H} |O_t^n f(x) - O_t f(x)| \\ &\leq \sup_{x\in H} \int_H |f(x + \sqrt{t}\, P^n y) - f(x + \sqrt{t}\, y)| \mathcal{N}(0, Q)\, dy \\ &\leq \int_H \omega_f(\sqrt{t}\, |P^n y - y|)\mathcal{N}(0, Q)\, dy \leq \int_H \omega_f(\sqrt{T}\, |P^n y - y|)\mathcal{N}(0, Q)\, dy. \end{aligned} \tag{4.4}$$

Notice that the map $H \to \mathbb{R}$, $y \mapsto \omega_f(|y|)$, is continuous and bounded. Letting $n \to \infty$ in the last term of (4.4), we obtain the assertion by the Dominated Convergence Theorem.

STEP 2. We verify that

$$\frac{d}{dt} O_t^n \widehat{f}(x) = \frac{1}{2} O_t^n \Big(\sum_{k=1}^n \lambda_k D_{kk} \widehat{f} \Big)(x), \quad x \in H,\ t \geq 0.$$

Fix $x \in H$. Differentiating (4.3) with respect to t and using the standard properties of Gaussian measures, we obtain

$$\begin{aligned} \frac{d}{dt} O_t^n \widehat{f}(x) &= \frac{d}{dt} \int_H \widehat{f}(x + \sqrt{t}\, P^n y)\mathcal{N}(0, Q)\, dy \\ &= \frac{1}{2} \sum_{k=1}^n \lambda_k D_k \Big(\int_H D_k \widehat{f}(x + P^n y)\mathcal{N}(0, tQ)\, dy \Big) \\ &= \frac{1}{2} \sum_{k=1}^n \lambda_k \int_H D_{kk} \widehat{f}(x + P^n y) y_k \mathcal{N}(0, tQ)\, dy \\ &= O_t^n \Big(\frac{1}{2} \sum_{k=1}^n \lambda_k D_{kk} \widehat{f} \Big)(x), \quad t \geq 0,\ n \geq 1. \end{aligned} \tag{4.5}$$

STEP 3. We set

$$F_n(x) = \sum_{k=1}^n \lambda_k D_{kk} \widehat{f}(x), \quad n \geq 1,\ x \in H.$$

We prove that (F_n) is a sequence of uniformly bounded and equi-uniformly continuous functions in $C_b(H)$.

We use the fact that, for any $T \in \mathcal{L}_1(H)$ and $A \in \mathcal{L}(H)$, one has $TA \in \mathcal{L}_1(H)$ and $|\mathrm{Tr}(TA)| \leq \|TA\|_1 \leq \|A\|_{\mathcal{L}(H)} \|T\|_1$. Since $D_Q^2 \widehat{f} \in C_b(H, \mathcal{L}_1(H))$, we denote by $\omega_{D_Q^2 \widehat{f}}$ the modulus of continuity of $D_Q^2 \widehat{f}$. Moreover we set $\sup_{x\in H} \|D_Q^2 \widehat{f}(x)\|_1 = \|D_Q^2 \widehat{f}\|_0$. This way we find

$$\begin{aligned} |F_n(x)| &= \Big| \sum_{k=1}^n \langle P^n D_Q^2 \widehat{f}(x) e_k, e_k \rangle \Big| = |\mathrm{Tr}(P^n D_Q^2 \widehat{f}(x))| \\ &\leq \|P^n\|_{\mathcal{L}(H)} \|D_Q^2 \widehat{f}(x)\|_1 \leq \|D_Q^2 \widehat{f}\|_0, \quad x \in H,\ n \geq 1. \end{aligned} \tag{4.6}$$

Therefore (F_n) is uniformly bounded. The equicontinuity follows from the inequality

$$\begin{aligned} |F_n(x) - F_n(z)| &= |\mathrm{Tr}(P^n [D_Q^2 \widehat{f}(x) - D_Q^2 \widehat{f}(z)])| \\ &\leq \omega_{D_Q^2 \widehat{f}}(|x - z|), \quad x, z \in H,\ n \geq 1. \end{aligned}$$

STEP 4. We show that $\widehat{f} \in D(\mathcal{A}) = D(\mathcal{A}_H)$ and $\mathcal{A}\widehat{f} = \mathcal{A}_0 \widehat{f}$.

Fix $x \in H$. For any $n \geq 1$, we consider $O_t^n \widehat{f}(x)$ as a function of t. By the first step we know in particular that for fixed $x \in H$, $\lim_{n\to\infty} O_t^n \widehat{f}(x) = O_t \widehat{f}(x)$, $t \geq 0$.

Moreover, by the second and third steps, we have, for any $n \geq 1$,

$$\Big| \frac{d}{dt} O_t^n \widehat{f}(x) \Big| = \frac{1}{2} |O_t^n F_n(x)| \leq \frac{1}{2} \|D_Q^2 \widehat{f}\|_0, \quad t \geq 0. \tag{4.7}$$

Once we have proved that

$$\lim_{n\to\infty}\frac{d}{dt}O_t^n\widehat{f}(x)=\lim_{n\to\infty}\frac{1}{2}O_t^nF_n(x)=O_t\mathcal{A}_0\widehat{f}(x),\qquad t>0, \tag{4.8}$$

using the fact that $O_{(\cdot)}^n\widehat{f}(x)\in\mathcal{C}^1([0,\infty[)$ for any $n\ge 1$, $O_{(\cdot)}\mathcal{A}_0\widehat{f}(x)\in\mathcal{C}([0,\infty[)$ and appealing to (4.7), we deduce that $O_{(\cdot)}\widehat{f}(x)\in\mathcal{C}^1([0,\infty[)$, by a well known lemma of Real Analysis. Moreover we find

$$\frac{d}{dt}O_t\widehat{f}(x)=O_t\mathcal{A}_0\widehat{f}(x),\qquad t\ge 0, \tag{4.9}$$

and, by the Mean Value Theorem,

$$\sup_{y\in H}\left|\frac{O_t\widehat{f}(y)-\widehat{f}(y)}{t}\right|\le\|\mathcal{A}_0\widehat{f}\|_0,\qquad t>0. \tag{4.10}$$

By (4.9) and (4.10) we conclude that $\widehat{f}\in D(\mathcal{A})$ and further that $\mathcal{A}\widehat{f}=\mathcal{A}_0\widehat{f}$.

Thus it remains to check (4.8). We have

$$\begin{aligned}
&\left|\frac{d}{dt}O_t^n\widehat{f}(x)-O_t\mathcal{A}_0\widehat{f}(x)\right|\\
&\quad=\left|\frac{1}{2}O_t^nF_n(x)-O_t\mathcal{A}_0\widehat{f}(x)\right|\\
&\quad\le\frac{1}{2}\int_H|F_n(x+P^ny)-F_n(x+y)|\mathcal{N}(0,tQ)\,dy\\
&\qquad+\frac{1}{2}\int_H\Big|\sum_{k=n+1}^{\infty}\lambda_kD_{kk}\widehat{f}(x+y)\Big|\mathcal{N}(0,tQ)\,dy\\
&\quad\le\frac{1}{2}\int_H\Big[\omega_{D_Q^2\widehat{f}}(|P^ny-y|)\\
&\qquad+\sum_{k=n+1}^{\infty}\lambda_kD_{kk}\widehat{f}(x+y)\Big]\mathcal{N}(0,tQ)\,dy,\qquad t\ge 0,\ n\ge 1.
\end{aligned}$$

Letting $n\to\infty$ we get $|(d/dt)O_t^n\widehat{f}(x)-O_t\mathcal{A}_0\widehat{f}(x)|\to 0$, by the Dominated Convergence Theorem. The proof of (i) is complete. ∎

4.2. *A Dirichlet problem in a half space of H.* In this subsection the fact that H is finite-dimensional or infinite-dimensional has no relevance for the statements or proofs.

We define an open half space of H with respect to the orthonormal basis $\{e_k\}_{k\ge1}$, previously fixed. Each element x of H will be identified with its coordinates with respect to that basis. We set

$$H_+:=\{x=(x_1,x')\in H:x_1>0\}.$$

Let H' be the Hilbert subspace generated by $\{e_k\}_{k\ge2}$. We set $Q'x'=\sum_{k=2}^{\infty}\lambda_kx_k'e_k$, $x'=(x_k')\in H'$. Moreover we have $H_+=\mathbb{R}_+\times H'$, where $\mathbb{R}_+=(0,\infty)$.

Now we construct a semigroup P_t, associated with the following infinite-dimensional Dirichlet problem (see [21] for details):

$$\begin{cases}\lambda\psi(x)-\frac{1}{2}\mathrm{Tr}[QD^2\psi(x)]=f(x), & x\in H_+,\ \lambda>0,\\ \psi(z)=0, & z\in\partial H_+,\end{cases} \tag{4.11}$$

where $f\in\mathcal{C}_{\mathrm{b}}(H_+)$ and ∂H_+ denotes the boundary of H_+.

For any $g\in\mathcal{C}_{\mathrm{b}}(H_+)$, we set $Eg(x)=g(x)$ if $x=(x_1,x')$ with $x_1\ge0$, $Eg(x)=-g(-x_1,x')$ if $x=(x_1,x')$ with $x_1<0$. Define

$$\begin{aligned}P_tf(x)&:=\int_H Ef(x+\sqrt{t}\,y)\mathcal{N}(0,Q)\,dy\\
&=\int_{\mathbb{R}_+\times H'}f(y_1,y')D(x_1,t\lambda_1)\otimes\mathcal{N}(x',tQ')(dy_1,dy'),\end{aligned}$$

for $f\in\mathcal{C}_{\mathrm{b}}(H_+)$, $t>0$ and $x\in H_+$, where

$$\begin{aligned}&D(x_1,t\lambda_1)(dy_1)\\
&\quad=\left(\frac{e^{-(x_1-y_1)^2/(2t\lambda_1)}-e^{-(x_1+y_1)^2/(2t\lambda_1)}}{\sqrt{2\pi t\lambda_1}}\right)dy_1,\qquad\lambda_1>0,\ x_1>0,\end{aligned}$$

and $\mathcal{N}(x',tQ')$ is a Gaussian measure on H'. Notice that $D(x_1,t\lambda_1)\otimes\mathcal{N}(x',tQ')(H_+)<1$, $x\in H_+$, $t>0$.

It is possible to verify that P_t is a semigroup of contractions on $\mathcal{C}_{\mathrm{b}}(H_+)$. Clearly P_t is a transition π-semigroup on $\mathcal{C}_{\mathrm{b}}(H_+)$ but it is not a strongly continuous semigroup. Indeed the *maximal subspace* on which P_t is a strongly continuous semigroup is $\mathcal{C}_0(H_+)=\{f\in\mathcal{C}_{\mathrm{b}}(H_+):f(z)=0\text{ for }z\in\partial H_+\}$.

It is possible to prove that $P_tf\in\mathcal{C}_0(H_+)$ for any $f\in\mathcal{C}_{\mathrm{b}}(H_+)$ and $t>0$ (see [21, §3.6]). This implies that P_t is not a π-semigroup on $\mathcal{C}_{\mathrm{b}}(\overline{H}_+)$ (compare with (b) of Remark 2.5).

Let $\mathcal{T}$ be the generator of P_t. Notice that $\mathcal{T}$ coincides (by Proposition 3.6) with the generator introduced in [21] by the pointwise Laplace transform of P_t. Now we consider the following subsets of H_+: $H_+^\eta=\{(x_1,x')\in H_+:x_1\ge\eta\}$, $\eta>0$.

PROPOSITION 4.2. *For any $f\in\mathcal{C}_{\mathrm{b}}(H_+)$ we have:*

(i) $\lim_{s\to0^+}P_sf=f$ *uniformly on each* H_+^η, *for any* $\eta>0$;
(ii) $\lim_{s\to0}P_{s+t}f=P_tf$ *uniformly on* H_+, *for any* $t>0$.

Proof. (i) Let us fix $\eta>0$ and prove that

$$\lim_{s\to0^+}\sup_{x\in H_+^\eta}|P_sf(x)-f(x)|=0. \tag{4.12}$$

Thanks to the separability of H, we can choose a countable dense subset D^η of H_+^η. Since $P_sf-f\in\mathcal{C}_{\mathrm{b}}(H_+)$ for any $s\ge0$, formula (4.12) is equivalent to $\lim_{s\to0^+}\sup_{x\in D^\eta}|P_sf(x)-f(x)|=0$.

We introduce the functions $F_s : H \to \mathbb{R}$, $s \ge 0$, by

$$F_s(y) := \sup_{x\in D^\eta} |Ef(x+\sqrt{s}\,y) - Ef(x)|, \quad s \ge 0,\ y \in H. \tag{4.13}$$

It turns out that $\|F_s\|_0 \le 2\|f\|_0$ and F_s is a Borel function on H for all $s \ge 0$. Furthermore, thanks to the uniform continuity of f, we get

$$\lim_{s\to 0^+} F_s(y) = 0, \quad y \in H. \tag{4.14}$$

Now since

$$\sup_{x\in D^\eta} |P_s f(x) - f(x)| \le \int_H F_s(y)\mathcal{N}(0,Q)\,dy,$$

letting $s \to 0^+$ on the right-hand side, by the Dominated Convergence Theorem, we obtain (4.12). Thus (i) is proved.

(ii) Fix $t > 0$ and $f \in \mathcal{C}_b(H_+)$. Since $P_t f \in \mathcal{C}_0(H_+)$, it follows that $\lim_{r\to 0^+} P_r P_t f = P_t f$ in $\mathcal{C}_b(H_+)$. Hence to verify the assertion it remains to check that

$$\lim_{s\to 0^-} P_s P_t f = P_t f \quad \text{in } \mathcal{C}_b(H_+).$$

To this purpose we have, for any $-t/2 \le s \le 0$,

$$\|P_t f - P_{s+t} f\|_0 = \|P_{s+t/2}(P_{t/2-s} f - P_{t/2} f)\|_0 \le \|P_{t/2-s} f - P_{t/2} f\|_0,$$

so that

$$\lim_{s\to 0^-} \|P_t f - P_{s+t} f\|_0 \le \lim_{h\to 0^+} \|P_{t/2+h} f - P_{t/2} f\|_0 = 0. \ \blacksquare$$

Let us introduce the family $\mathcal{P} = \{H_+^\eta\}_{\eta>0}$. Similarly to (3.14), we can define the linear operator $\mathcal{T}_\mathcal{P} : D(\mathcal{T}_\mathcal{P}) \subset \mathcal{C}_b(H_+) \to \mathcal{C}_b(H_+)$ as follows:

$$\begin{cases} D(\mathcal{T}_\mathcal{P}) = \{f \in D(\mathcal{T}) : \text{ for any } H_+^\eta \in \mathcal{P}, \\ \qquad\qquad \lim\limits_{h\to 0^+} \sup\limits_{x\in H_+^\eta} |h^{-1}(P_h f(x) - f(x)) - \mathcal{T} f(x)| = 0\}, \\ \mathcal{T}_\mathcal{P} f(x) = \mathcal{T} f(x), \quad f \in D(\mathcal{T}_\mathcal{P}),\ x \in H_+. \end{cases} \tag{4.15}$$

By the previous proposition and by Theorem 3.7 we deduce that $\mathcal{T}_\mathcal{P} = \mathcal{T}$.

4.3. *The Ornstein–Uhlenbeck semigroup.* Let S_t be a strongly continuous semigroup on H, and let M be a self-adjoint and nonnegative bounded operator on H. For all $t \ge 0$ we define the bounded linear operators $M(t)$ by

$$M(t)x = \int_0^t S_u M S_u^* x\,du, \quad x \in H,$$

where S_t^* is the adjoint semigroup of S_t. Suppose that for each $t > 0$, $M(t)$ is a trace class operator. Under this assumption, there exist the Gaussian measures $\mathcal{N}(S_t x, M(t))$, $t > 0$, $x \in H$. The *Ornstein–Uhlenbeck semigroup* on $\mathcal{C}_b(H)$, associated with S_t and M, is defined as follows:

$$U_t f(x) = \int_H f(S_t x + y)\mathcal{N}(0, M(t))\,dy, \quad f \in \mathcal{C}_b(H),\ x \in H,\ t > 0. \tag{4.16}$$

This semigroup has been intensively studied, under various assumptions (see for instance [4]–[8]).

Unless $S_t = I$, for any $t \ge 0$, U_t is not a strongly continuous semigroup on $\mathcal{C}_b(H)$ (see [5, §6]). However U_t turns out to be a transition π-semigroup on $\mathcal{C}_b(H)$. To see this, it is enough to verify that the map $t \mapsto U_t f(x)$ is continuous for any $f \in \mathcal{C}_b(H)$ and $x \in H$. Actually a stronger assertion holds: *for any compact set K in H, and $f \in \mathcal{C}_b(H)$, one has*

$$\lim_{h\to 0} \sup_{x\in K} |U_{t+h} f(x) - U_t f(x)| = 0, \quad t \ge 0. \tag{4.17}$$

This result was proved by Cerrai [5, Proposition 6.2 and Lemma 6.3] in case S_t is a semigroup of negative type (this hypothesis can be removed with few changes in Cerrai's proof). For a different proof see [22, §3.3.1].

Denote by $\mathcal{U}$ the generator of the π-semigroup U_t. By Proposition 3.6, $\mathcal{U}$ coincides with the generator introduced in [5] by the pointwise Laplace transform of U_t.

We can introduce another linear operator $\mathcal{U}_\mathcal{K} : D(\mathcal{U}_\mathcal{K}) \subset \mathcal{C}_b(H) \to \mathcal{C}_b(H)$, defined as follows:

$$\begin{cases} D(\mathcal{U}_\mathcal{K}) = \{f \in D(\mathcal{U}) : \text{for any } K \in \mathcal{K}, \\ \qquad\qquad \lim\limits_{h\to 0^+} \sup\limits_{x\in K} |h^{-1}(U_h f(x) - f(x)) - \mathcal{U} f(x)| = 0\}, \\ \mathcal{U}_\mathcal{K} f(x) = \mathcal{U} f(x), \quad f \in D(\mathcal{U}_\mathcal{K}),\ x \in H. \end{cases} \tag{4.18}$$

By (4.17) and by Theorem 3.7, we deduce that $\mathcal{U}_\mathcal{K} = \mathcal{U}$.

Finally, we mention that there exist Markov transition semigroups on $\mathcal{C}_b(H)$ associated with non-Gaussian transition functions, which satisfy condition (4.17). Among these semigroups there are the *Mehler semigroups*, studied in [12], where also (4.17) is proved. Thus also for the Mehler semigroups, as for the Ornstein–Uhlenbeck semigroups, we can define a generator in three different equivalent ways: by a pointwise Laplace transform (as in [12, §4]), by a pointwise limit of an incremental ratio of the semigroup (as in (3.1)) and also by a uniform limit on compact sets of H of the same incremental ratio (see (4.18)).

5. Possible extensions. We have presented the theory of π-semigroups in the space $\mathcal{C}_b(E)$ for convenience. However, it is possible to extend this theory to more general function spaces. Here we briefly indicate how to proceed.

Let $B(E)$ be the Banach space of all bounded real functions on E, endowed with the sup norm. We consider any linear subspace $\Theta(E)$ of $B(E)$ that has the following two properties.

(i) $\Theta(E)$ is closed in $B(E)$ (with respect to the norm topology).

(ii) For any $T > 0$ and for any map $G : [0,T] \times E \to \mathbb{R}$ satisfying:

(a) $G(\cdot, x)$ is a Borel map on $[0,T]$ for any $x \in E$;

(b) $G(s,\cdot) \in \Theta(E)$ for any $s \in [0,T]$;
(c) $\sup_{s\in[0,T]} \|G(s,\cdot)\|_0 < \infty$,

the map $g : E \to \mathbb{R}$ given by $g(x) = \int_0^T G(s,x)\,ds$, $x \in E$, belongs to $\Theta(E)$.

Conditions (i) and (ii) are similar to those introduced by Dynkin [10, p. 57]. Moreover the space $\mathcal{C}_b(E)$ satisfies these assumptions (see Lemma 3.3).

By (i), $(\Theta(E), \|\cdot\|_0)$ is a Banach space. On $\Theta(E)$ we can define π-convergence for sequences as in $\mathcal{C}_b(E)$ and also π-semigroups of bounded linear operators (through Definition 2.1 with $\mathcal{C}_b(E)$ replaced by $\Theta(E)$). Let P_t be a π-semigroup on $\Theta(E)$ of type ω. The following two basic facts about P_t can be deduced from (i) and (ii).

(1) For any $f \in \Theta(E)$ and $T > 0$, the map $x \mapsto \int_0^T P_t f(x)\,dt$ belongs to $\Theta(E)$.

(2) For any $f \in \Theta(E)$ and $\lambda > \omega$, the map g defined by

$$g(x) = \int_0^\infty e^{-\lambda u} P_u f(x)\,du, \qquad x \in E,$$

belongs to $\Theta(E)$.

Clearly to obtain (1) and (2) it is enough to assume in hypothesis (a) of (ii) that the map $G(\cdot, x)$ is continuous on $[0,T]$ for any $x \in E$. Our generality is motivated by future applications to the Cauchy problem for π-semigroups.

We emphasize that $\Theta(E)$ can also be the space $\mathcal{BC}(E)$ of all continuous, real and bounded functions on E. All results of Sections 2 and 3 can be adapted to the space $\mathcal{BC}(E)$. It is easy to see that the heat semigroup O_t is a π-semigroup on $\mathcal{BC}(H)$. Moreover, by the standard properties of weak convergence of Gaussian measures, U_t is also a π-semigroup on $\mathcal{BC}(H)$. The semigroup P_t, associated with the Dirichlet problem considered in §4.2, is not a π-semigroup on $\mathcal{BC}(\overline{H}_+)$ (the same happens with the space $\mathcal{C}_b(\overline{H}_+)$, compare with (b) of Remark 2.5). P_t is a π-semigroup on the Banach space of all functions in $\mathcal{BC}(H_+)$ which can be extended to maps belonging to $\mathcal{BC}(\overline{H}_+)$, endowed with the sup norm.

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