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On the Absolute Continuity of Multi-Functions and Orientor Fields

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0. Introduction.

Let X and Y be given sets and $F(x)$ be a multi-function from X to the collection of non-empty subsets of Y. The so-called selection problem is a problem to determine whether there exists a suitably smooth function $f(x)$, called a selector of $F(x)$, such that $f(x)\in F(x)$ for each $x\in X$. The selection problem has been treated in several papers. In [1], [3], [4] and [5] it has been shown that a continuous selector can be selected from suitably smooth multi-functions. In the papers mentioned above it has been assumed that the multi-function is a convex set valued function. In this note we shall introduce the notion of the absolute-continuity of a multi-function and show that we can select a continuous selector from compact (not necessarily convex) set (in R^m) valued functions, absolutely continuous on an interval in R^1.

A relation of the form

$$\dot{x}(t)\in F(t, x(t))$$

is called a contingent equation, where $F(t, x)$ is a multi-function, called an orientor field. T. Ważewski [7] has combined the control theory and the theory of contingent equations which was originated by M. Hukuhara and developed by Marchaud-Zaremba. The theory of Marchaud-Zaremba is concerned with the convex orientor fields. T. Ważewski [6] investigated the problem without assuming the convexity of the orientor fields. By introducing a new notion of generalized solutions, he has succeeded to extend the results of Marchaud-Zaremba. On the other hand, A. F. Filippov has shown the existence of solutions in case when orientor fields satisfy a Lipschitz condition. The second purpose of this note is to show the existence of solutions for non-convex absolutely continuous orientor fields.

1. Notations and definitions.

We introduce notations and recall some definitions. We denote by

$$\mathrm{Comp}\,(R^m) \text{ (resp. } \mathrm{Conv}\,(R^m))$$

the collection of all non-empty compact (resp. compact and convex) subsets of

m-dimensional Euclidean space R^m. By

$$\text{dist}(x, y) \text{ and } \text{dist}(x, A) = \text{dist}(A, x) = \inf\{\text{dist}(x, y) ; y \in A\}$$

we denote the Euclidean distance of a point x from a point $y \in R^m$ and a set $A \subset R^m$, respectively. For $A, B \in \text{Comp}(R^m)$ we put

$$\text{Dist}(A, B) = \inf\{s > 0 ; A \subset V(B, s), B \subset V(A, s)\},$$

where $V(A, s)$ denotes the closed neighborhood of a set A of radius s, i.e.,

$$V(A, s) = \{x \in R^m ; \text{dist}(x, A) \leqq s\}.$$

This Hausdorff distance can be shown to satisfy the following relation

$$\text{Dist}(A, B) = \sup\{\text{dist}(a, B), \ \text{dist}(b, A) ; a \in A, b \in B\}.$$

∂A denotes the boundary of A. $|A|$ denotes $\text{Dist}(A, O)$, where O is the origin in R^m.

Let $K(x) \in \text{Comp}(R^m)$ be a function defined on a closed set $E \subset R^n$. $K(x)$ is said to be absolutely continuous on E iff for every positive ε there corresponds a positive δ such that

$$\sum_i \text{Dist}(K(x_i), K(x_i')) < \varepsilon$$

holds for all choices of finite points $\{x_i\}, \{x_i'\} \subset E$ with

$$\sum_i \text{dist}(x_i, x_i') < \delta.$$

2. Selection theorems.

In this chapter we treat selection problems. Let $F(t)$ be a multi-function defined on an interval I in R^1.

Theorem 1. *Let $F(t) \in \text{Conv}(R^m)$ be a function defined on an interval I. Assume that $F(t)$ satisfies a Lipschitz condition, i.e., there exists a positive constant L such that*

$$\text{Dist}(F(t), F(t')) \leqq L|t - t'|, \quad t, t' \in I.$$

Then there can be selected a Lipschitz continuous function $f(t)$ with the Lipschitz constant L such that

$$f(t) \in \partial F(t), \quad t \in I.$$

Proof. Since we can have a global section by joining local sections end to end, it is sufficient to prove this Theorem in case when I is a compact interval $[t_0, T]$. Let D be a subdivision of I as follows,

$$D : t_0 < t_1 < t_2 < \cdots < t_n = T.$$

Let x_0 be any point of $\partial F(t_0)$. Since $F(t_1)$ is a convex set, we can select a point x_1 such that

$$\operatorname{dist}(x_0, x_1) \leqq \operatorname{Dist}(F(t_0), F(t_1)),$$
$$x_1 \in \partial F(t_1).$$

Indeed, in case when x_0 does not belong to the interior of $F(t_1)$, there exists a point x_1 of $\partial F(t_1)$ such that

$$\operatorname{dist}(x_0, x_1) = \operatorname{dist}(x_0, F(t_1)) \leqq \operatorname{Dist}(F(t_0), F(t_1)).$$

On the other hand, let x_0 belong to the interior of $F(t_1)$. Since $x_0 \in \partial F(t_0)$ and $F(t_0)$ is convex, there exists a supporting hyperplane P through x_0. Let H be the open half-space which is divided by P and does not contain any points of $F(t_0)$ and let L be the line through x_0 which is perpendicular to P. Since x_0 is an interior point of $F(t_1)$, the line L through x_0 intersects $\partial F(t_1)$ at two points. A point of those which is contained in H we denote by x_1. By the construction of x_1 the relation

$$\operatorname{dist}(x_0, x_1) = \operatorname{dist}(F(t_0), x_1) \leqq \operatorname{Dist}(F(t_0), F(t_1))$$

holds. We can select $\{x_k\}$ $(k=0, 1, \cdots, n)$ inductively as follows,

$$\operatorname{dist}(x_{k-1}, x_k) \leqq \operatorname{Dist}(F(t_{k-1}), F(t_k)),$$
$$x_k \in \partial F(t_k).$$

We define a polygon $x(t;D)$ corresponding to this subdivision D as follows,

$$x(t;D) = \frac{t_k - t}{t_k - t_{k-1}} x_{k-1} + \frac{t - t_{k-1}}{t_k - t_{k-1}} x_k, \quad \text{if } t \in [t_{k-1}, t_k].$$

Here we note by the construction of $x(t;D)$ that we have the following relation

$$\begin{aligned}\operatorname{dist}(x(t;D), x(t';D)) &= \operatorname{dist}\left(O, \frac{t'-t}{t_k - t_{k-1}} x_{k-1} + \frac{t-t'}{t_k - t_{k-1}} x_k\right) \\ &= \frac{|t-t'|}{t_k - t_{k-1}} \operatorname{dist}(x_k, x_{k-1}) \\ &\leqq \frac{|t-t'|}{t_k - t_{k-1}} L(t_k - t_{k-1}) \\ &= L|t-t'|\end{aligned}$$

and hence

$$\operatorname{dist}(x(t;D),\ x(t';D)) \leqq L|t-t'|, \quad t, t' \in I.$$

Indeed, for $t, t' \in I$ $(t \leqq t')$, there exist positive k, l $(k \leqq l)$ such that $t \in [t_k, t_{k+1}]$, $t' \in [t_l, t_{l+1}]$ and hence from the above relation we have

$$\operatorname{dist}(x(t;D), x(t';D))$$

$$\leqq \text{dist}\,(x(t;D), x_{k+1}) + \text{dist}\,(x_{k+1}, x_{k+2}) + \cdots +$$
$$\text{dist}\,(x_{l-1}, x_l) + \text{dist}\,(x_l, x(t';D))$$
$$\leqq L\{(t_{k+1}-t) + (t_{k+2}-t_{k+1}) + \cdots + (t_l - t_{l-1}) + (t'-t_l)\}$$
$$= L|t'-t|.$$

Let $\{D_n\}$ be a sequence of subdivisions of I

$$D_n : t_0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n}^{(n)} = T$$

such that

$$\delta(D_n) = \max\{t_{i+1}^{(n)} - t_i^{(n)};\ 0 \leqq i \leqq k_n - 1\}$$

tends to zero as $n \to \infty$. We will denote by $x_n(t)$ the function $x(t;D_n)$ which is constructed as above corresponding to D_n. This $\{x_n(t)\}$ is a family of functions which have the same Lipschitz constant L. Indeed, we have

$$\text{dist}\,(x_n(t), x_n(t')) \leqq L|t-t'|, \quad t, t' \in I.$$

Hence $\{x_n(t)\}$ is an equi-continuous family. By the construction of $x_n(t)$, $x_n(t_0) = x_0$ holds for each n and hence the equi-continuous family $\{x_n(t)\}$ is uniformly bounded on I. Therefore, we can assume without loss of generality that $\{x_n(t)\}$ converges uniformly to a function $x(t)$ continuous on I. This $x(t)$ is a Lipschitz continuous function which has the Lipschitz constant L.

We shall prove that

$$x(t) \in F(t), \quad t \in I.$$

Let t be any point of I. We can select a sequence $\{t_{k_n}^{(n)}\}$ of subdivision points of $\{D_n\}$ such that

$$\lim_{n\to\infty} t_{k_n}^{(n)} = t.$$

Since the relations

$$x_n(t_{k_n}^{(n)}) \in F(t_{k_n}^{(n)}),$$
$$\lim_{n\to\infty} x_n(t_{k_n}^{(n)}) = x(t)$$

hold, we have by the continuity of $F(t)$ that

$$x(t) \in F(t).$$

We shall next prove that

$$x(t) \in \partial F(t)$$

for each $t \in I$. Suppose that for some $\bar{t} \in I$

$$x(\bar{t}) \notin \partial F(\bar{t}).$$

Hence $x(\bar{t})$ is an interior point of $F(\bar{t})$. Therefore, there exists a positive ε_0 such that

$$F(\bar{t}) \supset V(x(\bar{t}), 2\varepsilon_0).$$

Since $F(t)$ is continuous and $\{x_n(t)\}$ is equi-continuous on I, we can find a positive δ such that

$$\text{dist}\,(F(t), F(\bar{t})) < \varepsilon_0, \quad \text{dist}\,(x_n(t), x_n(\bar{t})) < \varepsilon_0$$

for each n and each $t \in I$ satisfying

$$|t - \bar{t}| < \delta.$$

Since an equi-continuous family $\{x_n(t)\}$ converges uniformly to $x(t)$, there exists a sufficiently large positive integer n_0 and a subdivision point $t_{kn_0}^{(n_0)}$ of D_{n_0} such that

$$\text{dist}\,(x_{n_0}(t_{kn_0}^{(n_0)}), x(\bar{t})) < \varepsilon_0, \quad |t_{kn_0}^{(n_0)} - \bar{t}| < \delta.$$

Since $x_{n_0}(t_{kn_0}^{(n_0)}) \in \partial F(t_{kn_0}^{(n_0)})$ and $F(t_{kn_0}^{(n_0)})$ is convex, there exists a supporting hyperplane P through $x_{n_0}(t_{kn_0}^{(n_0)})$. Let H be the open half-space which is divided by P and does not contain any points of $F(t_{kn_0}^{(n_0)})$ and let L be the line through $x_{n_0}(t_{kn_0}^{(n_0)})$ which is perpendicular to P. Since $x_{n_0}(t_{kn_0}^{(n_0)})$ is an interior point of $F(\bar{t})$ because of

$$\text{dist}\,(x_{n_0}(t_{kn_0}^{(n_0)}), x(\bar{t})) < \varepsilon_0$$

and by the choice of $V(x(\bar{t}), 2\varepsilon_0)$, there is a point $x^* \in \partial F(\bar{t})$ which is contained in both L and H. Hence we have

$$\begin{aligned} &\text{Dist}\,(F(\bar{t}), F(t_{kn_0}^{(n_0)})) \geqq \text{dist}\,(x^*, x_{n_0}(t_{kn_0}^{(n_0)})) \\ &\geqq \text{dist}\,(x^*, x(\bar{t})) - \text{dist}\,(x(\bar{t}), x_{n_0}(t_{kn_0}^{(n_0)})) > 2\varepsilon_0 - \varepsilon_0 = \varepsilon_0, \end{aligned}$$

which contradicts

$$\text{Dist}\,(F(\bar{t}), F(t_{kn_0}^{(n_0)})) < \varepsilon_0.$$

Theorem 2. *Let $F(t) \in \text{Comp}\,(R^m)$ be an absolutely continuous function defined on an interval I. Then there can be selected a continuous function $f(t)$ such that*

$$f(t) \in F(t), \quad t \in I$$

Proof. The proof of this theorem follows that of Theorem 1. We use the notations similarly as in the proof of Theorem 1. We construct a sequence of polygons $\{x_n(t)\}$ similarly as in the proof of Theorem 1. However, we can not say that x_k belongs to the boundary of $F(t_k)$, since $F(t_k)$ is not necessarily convex. We shall prove that $\{x_n(t)\}$ is equi-continuous on I. Since $F(t)$ is absolutely continuous, for every positive ε there is a positive δ such that

$$\sum_i \mathrm{Dist}\,(F(t_i),\ F(t_i'))<\varepsilon$$

for all choices of finite points $\{t_i\}$, $\{t_i'\}$ satisfying

$$\sum_i |t_i-t_i'|<\delta.$$

Let t, t' be points of I. For every n there exist positive integers p_n and q_n such that

$$t_{p_n}^{(n)} \leqq t < t_{p_n+1}^{(n)} < \cdots < t_{q_n-1}^{(n)} < t' \leqq t_{q_n}^{(n)}.$$

Since $\delta(D_n)\to 0$ as $n\to\infty$, we can assume without loss of generality that

$$\delta(D_n)<\frac{\delta}{4}, \quad n=1, 2, \cdots.$$

Then, if $|t-t'|<\delta/2$, we have

$$\sum_{i=p_n}^{q_n-1} |t_{i+1}^{(n)}-t_i^{(n)}|<\delta, \quad n=1, 2, \cdots.$$

By the construction of $\{x_n(t)\}$ we have

$$\begin{aligned} \mathrm{dist}\,(x_n(t), x_n(t')) &\leqq \sum_{i=p_n}^{q_n-1} \mathrm{dist}\,(x_n(t_{i+1}^{(n)}), x_n(t_i^{(n)})) \\ &\leqq \sum_{i=p_n}^{q_n-1} \mathrm{Dist}\,(F(t_{i+1}^{(n)}), F(t_i^{(n)})). \end{aligned}$$

Therefore, we have

$$\mathrm{dist}\,(x_n(t), x_n(t'))<\varepsilon, \quad n=1, 2, \cdots$$

for $t, t' \in I$ satisfying

$$|t-t'|<\frac{\delta}{2}.$$

Since an equi-continuous family $\{x_n(t)\}$ is bounded at t_0, we can assume without loss of generality that $\{x_n(t)\}$ converges uniformly to a function $f(t)$ continuous on I. By the absolute continuity of $F(t)$ we conclude that

$$f(t)\in F(t), \quad t\in I.$$

Similarly as in the proof of Theorem 1 and Theorem 2, we can give the proof of the following theorems.

Theorem 3. *Let $F(t)\in \mathrm{Conv}\,(R^m)$ be an absolutely continuous function defined on an interval I. Then there can be selected a continuous function $f(t)$ such that*

$$f(t) \in \partial F(t), \quad t \in I.$$

Theorem 4. *Let $F(t) \in \mathrm{Comp}\,(R^m)$ be a Lipschitz continuous function with the Lipschitz constant L defined on an interval I. Then there can be selected a Lipschitz continuous function $f(t)$ with the Lipschitz constant L such that*

$$f(t) \in F(t), \quad t \in I.$$

3. Existence theorem of solutions for orientor fields.

In this chapter we give an existence theorem of solutions for non-convex orientor fields. Let I be an interval $[t_0, T]$.

Theorem 5. *Let $F(t, x) \in \mathrm{Comp}\,(R^m)$ be an absolutely continuous function defined on $I \times R^m$ and let there exist a positive number M such that*

$$|F(t, x)| \leqq M \quad \text{on } I \times R^m.$$

Then, for each $x_0 \in R^m$ there exists a continuously differentiable function $x(t)$ such that

$$\frac{dx(t)}{dt} \in F(t, x(t)) \quad \text{on } I,$$

$$x(t_0) = x_0.$$

Proof. We divide I by a subdivision

$$D: t_0 < t_1 < t_2 < \cdots < t_k = T.$$

For any $u_0 \in F(t_0, x_0)$ we put

$$x_1 = x_0 + (t_1 - t_0) u_0.$$

Next, we take out $u_1 \in F(t_1, x_1)$ in such a way that

$$\mathrm{dist}\,(u_0, u_1) = \mathrm{dist}\,(u_0, F(t_1, x_1)),$$

and put

$$x_2 = x_1 + (t_2 - t_1) u_1.$$

By the construction of u_1, the relation

$$\mathrm{dist}\,(u_0, u_1) \leqq \mathrm{Dist}\,(F(t_0, x_0), F(t_1, x_1))$$

holds. Successively, we can construct $\{u_i\}$, $\{x_i\}$, $i = 0, 1, \cdots, k$, in such a way that

$$u_i \in F(t_i, x_i),$$
$$\mathrm{dist}\,(u_{i-1}, u_i) \leqq \mathrm{Dist}\,(F(t_{i-1}, x_{i-1}), F(t_i, x_i)),$$
$$x_i = x_{i-1} + (t_i - t_{i-1}) u_{i-1}$$

and we define in each interval $[t_i, t_{i+1}]$, $i = 0, 1, \cdots, k-1$,

$$x(t; D) = x_i + (t - t_i) u_i,$$
$$u(t; D) = u_i + (t - t_i)(t_{i+1} - t_i)^{-1}(u_{i+1} - u_i).$$

Here we have by the similar computation as in the proof of Theorem 1 that

$$\text{dist}\,(u(t;D), u(t';D)) \leqq \text{dist}\,(u_i, u_{i+1}), \quad t, t' \in [t_i, t_{i+1}],$$
$$\text{dist}\,(x(t;D), x(t';D)) \leqq M|t-t'|, \quad t, t' \in I$$

hold by the construction of $u(t;D)$ and $x(t;D)$. In this way we have constructed continuous functions $x(t;D)$ and $u(t;D)$ on the interval I.

Let $\{D_n\}$ be a sequence of subdivisions of I

$$D_n : t_0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n}^{(n)} = T$$

such that

$$\delta(D_n) = \max\{t_{i+1}^{(n)} - t_i^{(n)} ;\ 0 \leqq i \leqq k_n - 1\}$$

tends to zero as $n \to \infty$. By $x_n(t)$ and $u_n(t)$ we will denote $x(t;D_n)$ and $u(t;D_n)$, respectively.

We shall next prove that $\{u_n(t)\}$ is equi-continuous on I. Since $F(t, x)$ is absolutely continuous, for every positive ε there is a positive δ such that

$$\sum_i \text{Dist}\,(F(t_i, x_i), F(t_i', x_i')) < \varepsilon$$

for all choices of finite points $\{(t_i, x_i)\}$, $\{(t_i', x_i')\}$ satisfying

$$\sum_i |t_i - t_i'| < \delta, \quad \sum_i \text{dist}\,(x_i, x_i') < M\delta.$$

Let t, t' be points of I. For every n there exist positive integers p_n and q_n such that

$$t_{p_n}^{(n)} \leqq t < t_{p_n+1}^{(n)} < \cdots < t_{q_n-1}^{(n)} < t' \leqq t_{q_n}^{(n)}.$$

Since $\delta(D_n) \to 0$ as $n \to \infty$, we can assume without loss of generality that

$$\delta(D_n) < \frac{\delta}{4}, \quad n = 1, 2, \cdots.$$

Then, if $|t-t'| < \delta/2$, then we have

$$\sum_{i=p_n}^{q_n-1} |t_{i+1}^{(n)} - t_i^{(n)}| < \delta, \quad n = 1, 2, \cdots.$$

By the construction of $\{u_n(t)\}$ we have

$$\begin{aligned} \text{dist}\,(u_n(t), u_n(t')) &\leqq \sum_{i=p_n}^{q_n-1} \text{dist}\,(u_n(t_{i+1}^{(n)}), u_n(t_i^{(n)})) \\ &\leqq \sum_{i=p_n}^{q_n-1} \text{Dist}\,(F(t_{i+1}^{(n)}, x_n(t_{i+1}^{(n)})), F(t_i^{(n)}, x_n(t_i^{(n)}))) \end{aligned}$$

and

$$\sum_{i=p_n}^{q_n-1} \text{dist}\,(x_n(t_{i+1}^{(n)}), x_n(t_i^{(n)})) \leqq \sum_{i=p_n}^{q_n-1} M|t_{i+1}^{(n)} - t_i^{(n)}| < M\delta.$$

Therefore, we have

$$\text{dist}\,(u_n(t), u_n(t')) < \varepsilon, \quad n=1, 2, \cdots$$

for $t, t' \in I$ satisfying

$$|t-t'| < \frac{\delta}{2}.$$

Hence we have proved that $\{u_n(t)\}$ is equi-continuous. By the construction of $u_n(t)$, $u_n(t_0)=u_0$ holds for each n and hence the equi-continuous family $\{u_n(t)\}$ is uniformly bounded on I. Therefore, we can assume without loss of generality that $\{u_n(t)\}$ converges uniformly to a function $u(t)$ continuous on I. For each $t \in I$ we can find an interval

$$[t_{k_n}^{(n)}, t_{k_n+1}^{(n)}]$$

such that

$$t \in [t_{k_n}^{(n)}, t_{k_n+1}^{(n)}]$$

and in this interval

$$\frac{dx_n(t)}{dt} = u_n(t_{k_n}^{(n)}),$$

$$\text{dist}\left(\frac{dx_n(t)}{dt},\ u_n(t)\right) \leqq \text{dist}\,(u_n(t_{k_n}^{(n)}), u_n(t_{k_n+1}^{(n)}))$$

hold. Hence we have that

$$\lim_{n\to\infty} \frac{dx_n(t)}{dt} = u(t)$$

uniformly on I and hence $\{x_n(t)\}$ converges to a function $x(t)$ continuous on I with

$$\frac{dx(t)}{dt} = u(t), \ t \in I,$$

$$x(t_0) = x_0.$$

By passing n to infinity in the relation

$$\frac{dx_n(t)}{dt} = u_n(t_{k_n}^{(n)}) \in F(t_{k_n}^{(n)}, x_n(t_{k_n}^{(n)})), \ t \in [t_{k_n}^{(n)}, t_{k_n+1}^{(n)}],$$

we have by the continuity of $F(t, x)$ in (t, x) that

$$\frac{dx(t)}{dt} \in F(t, x(t)), \ t \in I.$$

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