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Volterra Integral Equations in a Banach Space

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Abstract

This paper studies a general class of Volterra integro-differential equations of convolution type in a Banach space X. It is shown that formally the solution of the integral equation for a class of initial conditions is equivalent to an abstract differential equation on a functional space over X. This fact is exploited directly or by adapting the differential equation analysis to obtain information about the structure of solutions of the integral equation. This information plus existence-uniqueness theorems for the case of "smooth data" can be used to prove existence of generalized solutions for all data and continuity of solutions with respect to the data.

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1. Introduction

This paper studies an abstract Volterra integro-differential equation of the form

$$\text{(VE)} \qquad x'(t)=\gamma Ax(t)+\int_0^t B(t-s)x(s)ds+f(t), \quad x(0)=x_0$$

in a Banach space X. Here $t\geq 0$ and $'=d/dt$. The operator $A: D(A)\rightarrow X$ is linear with domain $D(A)$ dense in X while γ is a nonnegative real number. Any function of the form $B(t)=b(t)A$ with scalar function $b\in L^1(0,\infty)$ will serve as an example of the class of operators $B(t)$ considered here although a much more general class of operators will be studied.

Present existence, uniqueness theorems for (VE) typically have the following type of assumptions : A is the infinitesimal generator of a C_0-semigroup, x_0 is "smooth" in some sense such as $x_0\in D(A)$, and both $B(t)$ and $f(t)$ satisfy some smoothness properties such as having continuous strong derivatives. In [1] Friedman and Shimbrot used assumptions of this type together with contraction mapping argument in order to obtain existence and uniqueness theorems. It can easily be seen, by reading the proofs, that this type of analysis cannot be used to prove continuity of solutions with respect to the coefficient pair (x_0, f). The purpose here is to prove some general structure theorems concerning (VE). Our results together with known existence-uniqueness theorems can be used to

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prove continuity of solutions w. r. t the pair (x_0, f). They can also be used to obtain existence, uniqueness and continuity results for a more general class of equations.

We showed in [2] that on a finite dimensional space X, solutions of (VE) can be used in order to construct a C_0-semigroup on an appropriate function space. If C is the infinitesimal generator of this semigroup, then all solutions of (VE) are generalized solutions of the abstract differential equation

(DE) $$y' = Cy.$$

The generalized solutions of this abstract differential equation will generate the same C_0-semigroup. The construction of (DE) from (VE) can also formally be carried through in case X is an infinite dimensional space. Here we shall reverse the point of view of [2] in the sense that we shall use (DE) in order to study (VE). Under very mild assumptions on A, B, and f we show that solutions of (VE) are unique whenever they exist. Moreover solutions will exist on a dense set of "sufficiently smooth" pairs (x_0, f). Stronger assumptions of the type used in [1] will imply existence, uniqueness and continuity of solutions whenever x_0 is in $D(A)$ and f is smooth, as well as existence, uniqueness and continuity of generalized solutions for all other values of $x_0 \in X$ and $f \in C([0, \infty); X)$.

Throughout this paper R^+ will denote the interval $[0, \infty)$, R^1 the real line $(-\infty, \infty)$ and X a given real or complex Banach space with norm $\|\ \|$. For any x in $D(A)$, we shall always denote $\|x\|_A = \|x\| + \|Ax\|$. As usual $L^p(I)$ denotes the space of all measurable functions f such that

$$\int_I |f(t)|^p dt < \infty.$$

$C(I; G)$ will denote the set of all continuous functions f defined on I with values in G, and $BC(R^+) = \{f \in C(R^+; X) : \|f(t)\| \text{ is bounded on } R^+\}$. $B^1(I, X)$ will denote the set of all $f: I \to X$ such that f is Bochner integrable on I. The book of Hille and Phillips [3] contains all of the background material on Bochner integrals which will be needed here. This same book plus the book of Krein [4, especially Chapter I] contains all of the background material on semigroups and abstract differential equations which will be necessary to follow our exposition. The text [5] also contains a readable account of part of the necessary theory. For other recent results on various problems related to Volterra integral equations on a Banach space see Dafermos [6], Hannsgen [7, 8, 9] and MacCamy and Wong [10].

2. A Cauchy Problem

It will be assumed throughout this paper that $A : D(A) \to X$ is linear with domain $D(A)$ dense in X while for each $t\geq 0$, $B(t)$ is defined on the domain of A (or on a larger set) with range in X. For each x in $D(A)$, $B(t)x$ is assumed to be strongly measurable on R^+. Moreover $B(t)$ is subordinate to A in the sense that there exists a real valued function $\beta(t)\in L^1(R^+)$ such that for all t in R^+ and all x in $D(A)$

$$\|B(t)x\|\leq\beta(t)\{\|x\|+\|Ax\|\}=\beta(t)\|x\|_A. \tag{2.1}$$

These background assumptions will not be explicitly mentioned in the sequel. From these assumptions it follows that $B(t)x\in B^1(R^+, X)$ for any x in $D(A)$ (see [3, p. 80, Theorem 3.7.4]) and that the Laplace tranform

$$B^*(\lambda)x=\int_0^\infty \exp(-\lambda t)B(t)x\,dt$$

is defined and continuous on the half plane $\mathrm{Re}\,\lambda\geq 0$.

Since the constant γ is nonnegative, the only interesting cases are $\gamma=0$ and $\gamma=1$. When $\gamma=0$, the operator A is used only in order to make precise the domain of definition of $B(t)$ and the subordinate assumption (2.1). We shall assume that $\gamma=1$. If $\gamma=0$, then the results in sections 2 through 6 are easily seen to remain true. However later results will require $\gamma>0$.

Definition. A solution of (VE) on an interval $I=[0, T]$ is a function $x: I\rightarrow D(A)$ such that $x(t)$ is strongly differentiable, $x(t), x'(t)$ and $Ax(t)$ are continuous on I, $x(0)=x_0$ and equation (VE) is satisfied at all points t in I.

Obvious modifications of this definition can be used when the interval I is of the form $[0, T)$ with $0<T\leq+\infty$.

The first order of business is to show that (VE) makes sense for a class of functions which include solutions.

Lemma 2.1. *If $x:[0, T]\rightarrow D(A)$ is any function such that $x(t)$ and $Ax(t)$ are continuous on $[0, T]$, then as a function of s, $B(t-s)x(s)\in B^1([0, t], X)$ for all t in $(0, T]$.*

Proof. Let $I(s, E)$ denote the characteristic function of a set E. Given an integer $M\geq 1$ pick an integer $N>1$ such that if

$$u(s)=\sum_{j=1}^N x(jt/N)I(s, E_j)$$

and $E_j=[(j-1)t/N, jt/N]$, then $\|x(s)-u(s)\|_A\leq 1/M$ on $0\leq s\leq t$. From Corollary 1 in [3, p. 73] it follows that there is a sequence of countable valued functions $\{b_n(s, j, N)\}$ such that

$$\|b_n(s, j, N)-B(t-s)x(jt/N)I(s, E_j)\|<1/n$$

on $[0, t]$ except on a null set $F_n(j, N)$. Then except on the null set $F(n, N)=\bigcup_{j=1}^N F_n(j, N)$ one has

$$\|B(t-s)x(s)-\sum_{j=1}^N b_n(s, j, N)\|$$

$$\le \|B(t-s)\{x(s)-u(s)\}\| + \sum_{j=1}^{N}\|B(t-s)x(jt/N)-b_n(s,j,N)\|$$
$$\le \beta(t-s)/M + N/n.$$

By choosing n so large that $N/n<1/M$ and then letting $M\to\infty$, it can be seen that $B(t-s)x(s)$ is almost everywhere the pointwise limit of a sequence of countable valued functions. This implies strong measurability. This and the estimate

$$\|B(t-s)x(s)\| \le \beta(t-s)\|x(s)\|_A \le \beta(t-s)K$$

with $K=\max\{\|x(s)\|_A : 0\le s\le T\}$ insure that $B(t-s)x(s)\in B^1([0,t],X)$, see [3, p. 79, Def. 3.7.3]. Q. E. D.

The next order of business is to find a suitable class of forcing functions f to be used in (VE). To motivate the choice of the minimum possible class of functions consider the initial value problem

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)\,ds \qquad (t\ge\tau) \tag{2.2}$$

with $x(t)=g(t)$ given on $0\le t\le\tau$. If both $g(t)$ and $Ag(t)$ are defined and continuous on $[0,\tau]$ and if $x(t)$ solves (2.2) for $t\ge\tau$, then the function $z(t)=x(t+\tau)$ will satisfy the relation

$$\begin{aligned} z'(t) &= Ax(t+\tau) + \int_0^t B(t-s)x(s+\tau)\,ds + \int_0^\tau B(t+\tau-s)\,g(s)\,ds \\ &= Az(t) + \int_0^t B(t-s)z(s)\,ds + \int_{-\tau}^0 B(t-s)\,g(s+\tau)\,ds \end{aligned} \tag{2.3}$$

for $t\ge 0$. Thus $z(t)$ solves (VE) for $t\ge 0$ with initial condition $x_0=g(\tau)$ and forcing function

$$f(t) = \int_{-\tau}^0 B(t-s)\,g(s+\tau)\,ds.$$

(In the special case $\tau=0$, $g(t)$ reduces to a constant $g(t)\equiv x_0$ and $f(t)\equiv 0$.)

Problem (2.2) has the obvious advantage that whenever a solution is known on an interval $\tau\le t\le\tau_1$, then $g(t)$ and the solution $x(t)$ can be pieced together to form a new initial condition of the same type on the larger interval $[0,\tau_1]$:

$$g_1(s) = \begin{cases} g(s) & \text{on } 0\le s\le\tau \\ x(s) & \text{on } \tau\le s\le\tau_1 \end{cases}$$

The new initial condition $x(t)=g_1(t)$ on $0\le t\le\tau_1$ is the right initial condition to use if it is desired to continue the solution of (2.2) past τ_1. These considerations motivate the following construction.

Define $C_0(-\infty,0]=\{f:(-\infty,0]\to D(A) : f$ has compact support and both f and Af are continuous$\}$ with norm $\|f\|=\sup\{\|f(t)\|_A : -\infty<t\le 0\}$. Given f in $C_0(-\infty,0]$ let $\varphi(f)$ be the function on R^+ to X defined by

$$\varphi(f)(t) = \int_{-\infty}^0 B(t-s)f(s)\,ds.$$

Define $Y_0=\{(x, F) : F=\varphi(f)$ and $x=f(0)$ for some function f in $C_0(-\infty, 0]\}$.

Lemma 2.2. *Given f in $C_0(-\infty, 0]$, $\varphi(f)$ is well defined, bounded and uniformly continuous on R^+.*

Proof. Let $[-T, 0]$ contain the support of f. An easy modification of the proof of Lemma 2.1 can be used to see that $B(t-s)f(s)$ is in $B^1([-T, 0], X)$ for each $t\geq 0$. The estimate

$$\|\varphi(f)(t)\|=\left\|\int_{-T}^{0} B(t-s)f(s)\,ds\right\|\leq\int_{-T}^{0}\beta(t-s)\|f(s)\|_A ds$$
$$\leq\|f\|\int_{-T}^{0}\beta(t-s)\,ds\leq\|f\|\int_{t}^{\infty}\beta(s)\,ds$$

shows that $\varphi(f)$ is uniformly bounded on R^+ and tends to zero as $t\to\infty$. Put $w(h)=\sup\{\|f(t+h)-f(t)\|_A : 0\geq t+h, t\geq -T\}$ and estimate as follows with $h>0$:

$$\|\varphi(f)(t+h)-\varphi(f)(t)\|=\left\|\int_{-T}^{0} B(t+h-s)f(s)\,ds-\int_{-T}^{0} B(t-s)f(s)\,ds\right\|$$
$$\leq\left\|\int_{-T}^{-h} B(t-s)\{f(s+h)-f(s)\}\,ds\right\|$$
$$+\left\|\int_{-T-h}^{-T} B(t-s)f(s+h)\,ds\right\|$$
$$+\left\|\int_{-h}^{0} B(t-s)f(s)\,ds\right\|$$
$$\leq\int_{-T}^{0}\beta(t-s)\,w(h)\,ds+\int_{-T-h}^{-T}\beta(t-s)\|f\|_A ds$$
$$+\int_{-h}^{0}\beta(t-s)\|f\|_A ds$$
$$\leq w(h)\int_{0}^{\infty}\beta(s)\,ds+2\|f\|_A \sup\left\{\int_{s-|h|}^{s+|h|}\beta(u)\,du : s\geq 0\right\}.$$

The same estimate is true when $h<0$. Thus

$\|\varphi(f)(t+h)-\varphi(f)(t)\|\to 0$ as $|h|\to 0$ uniformly for $t\geq 0$. Q. E. D.

Given $y=(x, F)$ in Y_0 put $\|y\|=\|x\|+\|F\|$ where $\|F\|=\sup\{\|F(t)\| : t\geq 0\}$. Lemma 2.2 and simple standard arguments will show that Y_0 is a normed linear space. Let Y be the completion of Y_0.

Lemma 2.3. *If (x, F) is in Y, then F is bounded and uniformly continuous on R^+. Y contains all pairs of the form $(x, \varphi(f))$ where x is any point in X and f is any function on $(-\infty, 0]$ to X such that f and Af are uniformly bounded on $(-\infty, 0]$ and piecewise continuous on each compact interval of the form $[-T, 0]$ for $T>0$.*

Proof. The first statement follows easily from Lemma 2.2 and uniform convergence. If x is any point in X, then pick $x_n\in D(A)$ with $x_n\to x$ and put $f_n(t)=(nt+1)x_n$ on $-1/n\leq t\leq 0$ and $f_n(t)=0$ if $t\leq -1/n$. Then $f_n(0)=x_n$ for all n, f_n is in $C_0(-\infty, 0]$ and as $n\to\infty$ one has

$$\left\|\int_{-\infty}^{0} B(t-s) f_n(s)\, ds\right\| \leq \int_{-\frac{1}{n}}^{0} \beta(t-s) \|x\|_A ds \to 0.$$

Thus $(f_n(0), \varphi(f_n)) \to (x, 0)$ and it follows that $(x, 0) \in Y$. In the general case fix (x, F) with $F = \varphi(f)$. Let f have jump discontinuities at points $\{\tau_j\}_{j=1}^{k}$ in $[-N, 0]$. Define $f_n(t) = 0$ if $t \leq -N - 1/n$, $f_n(0) = x$, $f_n(t) = f(t)$ if $-N \leqq t \leq -1/n$ and $|t - \tau_j| \geq 1/n$ for all j. Define $f_n(t)$ linearly in the remainder of the intervals. Then $f_n \in C_0(-\infty, 0]$ for n sufficiently large, $f_n(0) = x$ for all n and $(f_n(0), \varphi(f_n)) \to (x, \varphi(f))$ in Y as $n \to \infty$ and then $N \to \infty$. Q. E. D.

3. The Associated Differential Equations

Given any function g defined on an interval I let g_t denote the translated function $g_t(s) = g(t+s)$ for all appropriate values of s. Given a function x : $[0, s] \to D(A)$ which is continuous in the norm $\| \|_A$ define

$$\left(\int_0^s B_u x(s-u)\, du\right)(t) = \int_0^s B(u+t) x(s-u)\, du$$
$$= \int_0^s B(t+s-u) x(u)\, du$$

for all $t \geq 0$. By Lemma 2.3 the function

$$\int_0^s B_u x(s-u)\, du = \int_0^s B_{s-u} x(u)\, du$$

is bounded and uniformly continuous on R^+. If $x(t)$ is a solution of (VE) on R^+ and if $s \geq 0$, then

$$x_s'(t) = Ax_s(t) + \int_0^t B(t-u) x_s(u)\, du + \int_0^s B(t+s-u) x(u)\, du + f_s(t)$$

or

$$x_s'(t) = Ax_s(t) + \int_0^t B(t-u) x_s(u)\, du + \left(\int_0^s B_u x(s-u)\, du + f_s\right)(t) \tag{3.1}$$

for all $t \geq 0$. (This is essentially the same computation as (2.3) above.) When such a solution exists, define a one-parameter map $U(t)$ by

$$U(t)(x_0, f) = \left(x(t),\, f_t + \int_0^t B_u x(t-u)\, du\right). \tag{3.2}$$

Lemma 3.1. *Suppose solutions of* (VE) *with* (x_0, f) *in* Y, *are unique whenever they exist. If the solution* $x(t)$ *exists for a given pair* (x_0, f) *in* Y, *then* (3.2) *defines a map* $y(t) = U(t)(x_0, f)$ *on* R^+ *to* Y *such that*

$$U(t+s)(x_0, f) = U(t) U(s)(x_0, f) \tag{3.3}$$

for all $t, s \geq 0$.

Proof. Let $\{f_n\}$ be a sequence in $C_0(-\infty, 0]$ such that $(f_n(0), \varphi(f_n)) \to (x_0, f)$ in Y. Then in particular, as $n \to \infty$

$$\int_{-\infty}^{-t} B(s-u) f_n(u+t)\, du = (\varphi(f_n)_t)(s) \to f_t(s)$$

uniformly in $s \geq 0$. If we put $g_n(u) = f_n(u+t)$ on $-\infty < u < -t$ and $g_n(u) = x(t+u)$ on $-t \leq u \leq 0$, then Lemma 2.3 can be applied to see that $(x(t), \varphi(g_n))$ is in Y. Furthermore the last equation above and the identity

$$\left(\int_0^t B_u x(t-u)\, du\right)(s) = \int_{-t}^0 B(s-u)\, x(t+u)\, du \tag{3.4}$$

show that $(x(t), \varphi(g_n)) \to U(t)(x_0, f)$ as $n \to \infty$. Thus $U(t)(x_0, f)$ is in Y for each $t \geq 0$.

To see that $U(t)(x_0, f)$ is continuous in t, note first that the coordinate $x(t)$ is even strongly differentiable, hence continuous. Since f is uniformly continuous on R^+, it is clear that $f_t(s)$ is continuous in t uniformly for $s \geq 0$. The proof of Lemma 2.2 is easily modified to prove that the function (3.4) is continuous. Since (3.3) is essentially just (3.1) rewritten, the proof is complete. Q. E. D.

Formally the function $y(t) = U(t)(x_0, f) = (x(t), F(t, 0))$ with

$$F(t, 0) = f_t + \int_0^t B_u x(t-u)\, du,$$

is a solution of the differential equation

$$y' = Cy \tag{DE}$$

where C is a linear map defined in Y with domain $D(C) = \{(x_0, f) \in Y : x_0 \in D(A), f$ loc. abs. cont. on R^+ and the pair $(Ax_0 + f(0), f' + Bx_0) \in Y\}$ and with

$$C(x_0, f) = (Ax_0 + f(0), f' + Bx_0). \tag{3.5}$$

We will not pursue the question of when this formal calculation can be made precise. Rather we shall reverse the situation to ask when information about (DE) can be used to infer properties of solutions of (VE).

Definition 3.2. By a solution $y(t)$ of (DE) satisfying an initial condition $y(0) = y_0$ we mean a function $y : R^+ \to D(C)$ such that y, y' and Cy are all continuous on R^+, $y(0) = y_0$ and (DE) is satisfied for all t in R^+. Equation (DE) is called well posed if the following two statements are true :

(i) for each $y_0 \in D(C)$, there exists a unique solution $y(t, y_0)$ satisfying the initial condition $y(0, y_0) = y_0$.

(ii) if $\{y_n\}$ is a sequence of points in $D(C)$ with $y_n \to 0$, then for each $t \geq 0$ the corresponding solutions satisfy $y(t, y_n) \to 0$ as $n \to \infty$.

If the continuity condition (ii) is satisfied with limit $y(t, y_n) = 0$ $(n \to \infty)$ existing uniformly in t on compact subsets of R^+, then (DE) is called *uniformaly well posed.*

It is important to determine conditions which guarantee that $D(C)$ is dense as a subset of Y.

Lemma 3.3. *$D(C)$ is dense in Y if A is closed.*

Proof. Let $Y_1 = \{(f(0), \varphi(f)) : f$ and f' are in $C_0(-\infty, 0]\}$. If $(x, F) =$

$(f(0), \varphi(f)) \in Y_1$, then $(x, F) \in D(C)$ and $C(x, F) = (Ax + F(0), \varphi(f'))$. Indeed since

$$\varphi(f)(t) = \int_{-\infty}^{0} B(t-s) f(s) ds = \int_{t}^{\infty} B(u) f(t-u) du,$$

then $\varphi(f)'(t) = -B(t) f(0) + \varphi(f')(t)$. By Lemma 2.3, $(Ax + f(0), \varphi(f)' + Bx) = (Ax + f(0), \varphi(f')) \in Y$.

It will be shown that Y_1 is dense in Y. Let $M(t)$ be a nonnegative, scalar, C^∞ function with support in $[-1, 1]$ such that

$$\int_{-1}^{1} M(t) dt = 1.$$

Given $f \in C_0(-\infty, 0]$ let

$$f_\varepsilon(t) = \varepsilon^{-1} \int_{-\infty}^{\infty} M((t-s)/\varepsilon) f(s) ds.$$

Since A is closed, then

$$A f_\varepsilon(t) = \varepsilon^{-1} \int_{-\infty}^{\infty} M((t-s)/\varepsilon) A f(s) ds.$$

Thus f_ε and Af_ε are in C^1. Since $\|f_\varepsilon - f\|_A \to 0$ as $\varepsilon \to 0$ uniformly for $t \leq 0$, it follows that $(f_\varepsilon(0), \varphi(f_\varepsilon)) \to (f(0), \varphi(f))$ in Y. Thus Y_1 is dense in Y_0. Since $Y_1 \subset D(C)$ and Y_0 is dense in Y (by construction) the conclusion follows.

Q. E. D.

It would be very interesting to obtain other conditions which are sufficient to force the density of $D(C)$. The hypothesis of Lemma 3.3 is very likely too strong, but is also sufficient in order that C is a closed operator.

Lemma 3.4. *The operator C is closed if A is closed.*

Proof. Assume that A is closed. Let $\{(x_n, f_n)\}$ be a sequence in $D(C)$ such that $(x_n, f_n) \to (x, f)$, $Ax_n + f_n(0) \to y$ and $f_n' + Bx_n \to g$ for some y in X and some function g. Since A is closed and $Ax_n \to y - f(0)$, it follows that $x \in D(A)$ and $Ax = y - f(0)$. From the estimates $\|B(t)x_n - B(t)x\| \leq \beta(t) \|x_n - x\|_A$ and $\|x_n - x\|_A \to 0$, it follows that $f_n'(t)$ has limit $g(t) - B(t)x$. Furthermore

$$\int_0^t f_n'(s) ds = f_n(t) - f_n(0) \to f(t) - f(0),$$

so that

$$f(t) = f(0) + \int_0^t \{g(s) - B(s)x\} ds.$$

It is now easy to see that $(x, f) \in D(C)$ and that $C(x, f) = (y, g)$. Q. E. D.

The utility of (DE) is immediately apparent from consideration of the following result.

Theorem 3.5. *If $y(t) = (x(t), F(t, 0))$ is a solution of* (DE), *then $x(t)$ solves* (VE) *with $x_0 = x(0)$ and $f(t) = F(0, t)$ for all $t \geq 0$.*

Proof. From $x'(t) = Ax(t) + F(t, 0)$ and the continuity of $x'(t)$ and $F(t, s)$, it follows that $Ax(t)$ is continuous on R^{+}. The second coordinate of (DE) can be written as

$$\frac{\partial F}{\partial t}(t, s)=\frac{\partial F}{\partial s}(t, s)+B(s)x(t) \qquad (t, s\geq 0).$$

Since $\partial F/\partial t$ is continuous in (t, s) and since $||B(s)x(t)||\leq\beta(s)||x(t)||_{A}$ with $||x(t)||_{A}$ continuous in t, then it follows from Lemma 2.1 that $g(s)=F(s, t-s)$ is an absolutely continuous function of $s\in[0, t]$ and

$$g'(s)=\frac{\partial F}{\partial t}(s, t-s)-\frac{\partial F}{\partial s}(s, t-s)=B(t-s)x(s).$$

One integration yields

$$F(t, 0)=F(0, t)+\int_{0}^{t}B(t-s)x(s)ds.$$

This and the equation $x'(t)=Ax(t)+F(t, 0)$ give the required conclusion.

Q. E. D.

4. Theorems on Existence, Uniqueness and Continuity

Equation (DE) and Theorem 3.5 will now be used to establish some basic properties of (VE). Given an operator D and a complex number $\lambda, R(\lambda, D)=(D-\lambda I)^{-1}$ will denote the resolvent of D at λ whenever this resolvent exists. Recall that the Laplace transform $B^{*}(\lambda)x$ of $B(t)x$ exists for all x in $D(A)$ and all λ in the half plane Re $\lambda\geq 0$. Define $\rho(\lambda)=(A+B^{*}(\lambda)-\lambda I)^{-1}$ at all points λ such that Re $\lambda\geq 0$ and such that the inverse exists as a bounded linear map on X to X.

Theorem 4.1. *For any λ with* Re $\lambda>0$, *$\rho(\lambda)$ exists if and only if the resolvent $R(\lambda, C)$ exists. When $\rho(\lambda)$ exists with $||\rho(\lambda)||\leq M(\lambda)$, then $||R(\lambda, C)||\leq K(\lambda)$ where the solution of $R(\lambda, C)(z, g)=(x, f)$* is

$$x=\rho(\lambda)\{z+g^{*}(\lambda)\}, \quad f=\int_{0}^{\infty}(B_{u}x-g_{u})\exp(-\lambda u)du \tag{4.1}$$

*and if $\lambda_{1}=$*Re *λ is sufficiently large,*

$$K(\lambda)\leq(1+2\beta^{*}(0))\{\lambda_{1}^{-1}+\beta^{*}(0)(1+(1+|\lambda|)M(\lambda)(1+\lambda_{1})^{-1})\}. \tag{4.2}$$

Proof. If $R(\lambda, C)$ exists and $(C-\lambda I)(x, f)=(z, g)$, then $Ax+f(0)-\lambda x=z$ and $f'+Bx-\lambda f=g$. Thus

$$\begin{aligned}f(t)&=\exp(\lambda t)f(0)+\int_{0}^{t}\exp(\lambda(t-u))\{g(u)-B(u)x\}du\\&=\exp(\lambda t)\{\lambda x-Ax+z\}+\exp(\lambda t)\int_{0}^{\infty}\exp(-\lambda u)\{g(u)-B(u)x\}du\\&\qquad-\int_{t}^{\infty}\exp(\lambda(t-u))\{g(u)-B(u)x\}du,\end{aligned}$$

or

$$f(t)=\exp(\lambda t)\{\lambda x-Ax+z+g^{*}(\lambda)-B^{*}(\lambda)x\} \tag{4.3}$$

$$-\int_0^\infty \exp(-\lambda u)\,\{g_t(u)-B_t(u)x\}\,du.$$

Since f must be bounded on R^+ and since $\mathrm{Re}\,\lambda>0$, the first term on the right in (4.3) must be zero, that is

$$\{\lambda I-A-B^*(\lambda)\}x=-(z+g^*(\lambda)). \tag{4.4}$$

Moreover (4.3) reduces to

$$f=\int_0^\infty \exp(-\lambda u)\,\{B_t x-g_t\}\,du. \tag{4.3$'$}$$

It is assumed that $R(\lambda, C)$ exists and is bounded by some number K. Since $(z, 0)\in Y$, then with $g\equiv 0$ and any z in X, (4.4) has a solution x with $||x||\leq K||z||$. Thus $\rho(\lambda)$ exists and $||\rho(\lambda)||\leq K$.

Conversely assume $\rho(\lambda)$ exists with $||\rho(\lambda)||\leq M$. In this case it is easy to see that (4.1) defines a linear map on Y to Y and that $x\in D(A)$. The computations leading to (4.4) and (4.3$'$) can be reversed to see that $Ax+f(0)-\lambda x=z$ and $f'+Bx-\lambda f=g$. Thus $\rho(\lambda)$ exists if and only if $R(\lambda, C)$ exists.

To estimate the norm of the map defined by (4.1) first note that

$$||x||\leq||\rho(\lambda)||(||z||+||g^*(\lambda)||)\leq M(||z||+||g||/\mathrm{Re}\,\lambda)\leq M(1+\lambda_1^{-1})(||z||+||g||).$$

Since $Ax=z+\lambda x-f(0)$, and $\lambda=\lambda_1+i\lambda_2$, then

$$\left\|\int_0^\infty B(t+u)x\exp(-\lambda u)\,du\right\|\leq\int_0^\infty \exp(-\lambda_1 u)\beta(t+u)\,du\,||x||_A$$
$$=(\beta_t)^*(\lambda_1)(||x||+||Ax||)\leq(\beta_t)^*(\lambda_1)(||x||+||z+\lambda x-f(0)||)$$
$$\leq(\beta_t)^*(\lambda_1)(||x||(1+|\lambda|)+||z||+||f(0)||)$$

for all $t\geq 0$. Therefore

$$||f(t)||=\left\|\int_0^\infty\{B(t+u)x-g(t+u)\}\exp(-\lambda u)\,du\right\|$$
$$\leq||g||\lambda_1^{-1}+(\beta_t)^*(\lambda_1)\{(1+|\lambda|)||x||+||z||+||f(0)||\}.$$

For $\lambda_1=\mathrm{Re}\,\lambda$ sufficiently large, $\beta^*(\lambda_1)\leq 1/2$ so that

$$||f(0)||\leq 2||g||/\lambda_1+2\beta^*(0)\{(1+|\lambda|)||x||+||z||\},$$

and

$$||f||\leq||g||/\lambda_1+\beta^*(0)\{(1+|\lambda|)||x||+||z||\}+\beta^*(0)\{2||g||/\lambda_1$$
$$+2\beta^*(0)(1+|\lambda|)||x||+||z||\}$$
$$\leq(\beta^*(0)+1/\lambda_1)(1+2\beta^*(0))||(z, g)||+\beta^*(0)(1+2\beta^*(0))(1+|\lambda|)||x||.$$

These estimates can be combined to see that (4.2) is true. Q. E. D.

Theorem 4.2. *If A is closed, $\rho(\lambda)$ exists for* $\mathrm{Re}\,\lambda\geq\lambda_0>0$ *and* $||\rho(\lambda)||=O(1+|\lambda|^k)$ *for some $k\geq 0$ as $|\lambda|\to\infty$ with* $\mathrm{Re}\,\lambda\geq\lambda_0$, *then there exists a dense subset Y' of Y such that for all (x_0, f) in Y',* (VE) *has a solution on R^+.*

Proof. Since A is closed, C is closed and $D(C)$ is dense in Y. Moreover the hypothesis and Theorem 4.1 imply that $R(\lambda, C)$ exists for at least one complex value λ (in fact for λ on a half space). Hence $D(C^n)$ is also dense in Y for all positive integers n. The hypotheses and (4.2) imply that there is a constant $L>0$ such that for $\operatorname{Re}\lambda\geq\lambda_0$, $||R(\lambda, C)||\leq L(1+|\lambda|)^{k+1}$. By Theorem 1.5 of Krein [4, p. 34], (DE) has solutions for all initial values in $D(C^{k+4})$. Apply Theorem 3.5 above to obtain existence of solution of (VE) on the same dense set. Q. E. D.

Theorem 4.3. *Suppose there exists $w>0$ such that for any $t\geq 0$, any $\lambda\geq w$, any $T>0$, and any function $x\in C([0, T], D(A))$ with $x(t)=0$ for $t>T$ we have $B(t)x^*(\lambda)=(B(t)x)^*(\lambda)$, that is*

$$B(t)\int_0^T \exp(-\lambda s)x(s)\,ds=\int_0^T \exp(-\lambda s)B(t)x(s)\,ds.$$

If A is closed, $\rho(\lambda)$ exists when λ is real and $\lambda\geq w$ and if

$$\limsup_{\lambda\to\infty}\,(\ln||\rho(\lambda)||/\lambda)=0,$$

then for all (x_0, f) in Y and all $T>0$ the solution of (VE) *is unique on* $[0, T]$ *whenever it exists.*

Proof. The proof is accomplished by modifying a uniqueness proof for differential equations, see Krein [4, pp. 62-63]. Suppose for $x_0=0$ and $f\equiv 0$ that there exists a solution $x(t)$ of (VE) on $[0, T]$. Put $x(t)=0$ for all $t>T$ and integrate by parts in (VE) to obtain

$$\int_0^T \exp(-\lambda u)x(u)\,du=-\lambda^{-1}\exp(-\lambda T)x(T)+\lambda^{-1}\int_0^T \exp(-\lambda u)x'(u)\,du.$$

Rearrange this as follows:

$$\begin{aligned}\lambda\int_0^T \exp(-\lambda u)x(u)\,du+\exp(-\lambda T)x(T)-\int_0^T \exp(-\lambda u)Ax(u)\,du\\ =\int_0^T\int_0^u \exp(-\lambda u)B(u-v)x(v)\,dv\,du\\ =\int_0^T\int_v^T \exp(-\lambda u)B(u-v)x(v)\,du\,dv.\end{aligned}\tag{4.5}$$

Define

$$E(\lambda)=\int_0^\infty\int_T^\infty \exp(-\lambda u)B(u-v)x(v)\,du\,dv,$$

so that the right side of (4.5) can be written as

$$\begin{aligned}&-E(\lambda)+\int_0^\infty\int_v^\infty \exp(-\lambda u)B(u-v)x(v)\,du\,dv\\ &=-E(\lambda)+\int_0^\infty\int_0^\infty \exp(-\lambda u)B(u)\exp(-\lambda v)x(v)\,du\,dv\\ &=-E(\lambda)+\int_0^\infty \exp(-\lambda u)\Big(\int_0^\infty B(u)\exp(-\lambda v)x(v)\,dv\Big)du.\end{aligned}$$

By hypothesis the last expression can be written as

$$-E(\lambda)+\int_0^\infty \exp(-\lambda u)B(u)\Big(\int_0^\infty \exp(-\lambda v)x(v)\,dv\Big)du=-E(\lambda)+B^*(\lambda)x^*(\lambda).$$

Put this expression in the right side of (4.5) and rearrange to obtain

$$\rho(\lambda)\{\exp(-\lambda T)x(T)+E(\lambda)\}=\int_0^T \exp(-\lambda u)x(u)du$$

or

$$\rho(\lambda)\{x(T)+E(\lambda)\exp(\lambda T)\}=\int_0^T \exp(+\lambda u)x(T-u)du$$

This means that

$$\ln\left\|\int_0^T \exp(+\lambda u)x(T-u)du\right\|\Big/\lambda \leq \ln\|\rho(\lambda)\|/\lambda+\ln\|x(T)+\exp(\lambda T)E(\lambda)\|/\lambda.$$

Now

$$\begin{aligned}\exp(\lambda T)\|E(\lambda)\| &\leq \int_0^\infty \int_T^\infty \exp(\lambda(T-u))\beta(u-v)\|x(v)\|_A du dv\\ &\leq \int_T^\infty \exp(\lambda(T-u))\left(\int_0^\infty \beta(u)du\right)(\sup_v \|x(v)\|_A)du\\ &=\lambda^{-1}\beta^*(0)\sup\{\|x(v)\|_A : 0\leq v\leq T\}.\end{aligned}$$

Thus $\limsup_{\lambda\to\infty} \ln\left\|\int_0^T \exp(+\lambda u)x(T-u)du\right\|\Big/\lambda=0$. This implies that $x(T-u)=0$ on $0\leq u\leq T$.

The condition $B(t)X^*(\lambda)=(B(t)X)^*(\lambda)$ is satisfied when $B(t)=b(t)A$ with b scalar valued since A is assumed to be closed. More generally if $B(t)=\sum_{j=1}^N b_j(t) F_j A$ with b_j scalar and L^1 and with the F_j bounded linear maps, then the assumption will be satisfied when A is closed.

Theorem 4.4. *Suppose A is closed, $\rho(\lambda)$ exists for some λ with* Re $\lambda>0$ *and for each pair (x_0, f) in $D(C)$,* (VE) *has a unique solution $x(t, x_0, f)$ on R^+. Suppose for each x in $D(A)$, $B(t)x$ is strongly differentiable with $B'(t)x$ strongly measurable and $\|B'(t)x\|\leq\gamma(t)\|x\|_A$ for some locally integrable scalar function γ. If*

$$\begin{aligned}\frac{\partial}{\partial t}&\left\{\int_0^t B(t+s-u)x(u, x_0, f)du+f(t+s)\right\}\\ &=f'(t+s)+B(s)x(t, x_0, f)+\int_0^t B'(t+s-u)x(u, x_0, f)du\end{aligned}\tag{4.6}$$

exists uniformly in s whenever $(x_0, f)\in D(C)$, then (DE) *is uniformly well posed.*

Proof. First we show that for each pair (x_0, f) in $D(C)$ equation (DE) has a unique solution on R^+ which satisfies the given initial condition at $t=0$. Once (x_0, f) is fixed put $x(t)=x(t, x_0, f)$ for short. It is clear that the appropriate solution of (DE) should be $(x(t), F(t, 0))$ where

$$F(t, s)=\int_0^t B(t+s-u)x(u)du+f(t+s)=\int_0^t B(s+u)x(t-u)du+f(t+s).\tag{4.7}$$

The only problem is to see that $\left(x'(t), \frac{\partial F}{\partial t}(t, 0)\right)$ lies in Y for each $t\geq 0$. Define

$g(t)=f'(t)+B(t)x_0$. Since (x_0, f) is in $D(C)$, it follows from the definition of $D(C)$ that f' exists *a.e.* and that $g \in BC(R^+)$. From (4.7) it is clear that

$$\begin{aligned}\{F(t+h,s)-F(t,s)\}/h&=h^{-1}\int_t^{t+h} g_s(u)\,du\\ &+h^{-1}\int_0^t \{B(t+h+s-u)-B(t+s-u)\}x(u)\,du\\ &+h^{-1}\int_0^h B(h+s-u)x(t+u)\,du-h^{-1}\int_0^h B(s+t+u)x_0\,du.\end{aligned}$$

From this and assumption (4.6) it follows that

$$\begin{aligned}\frac{\partial F}{\partial t}(t,s)&=g(t+s)+\int_0^t B'(t+s-u)x(u)\,du+B(s)x(t)-B(t+s)x_0\\ &=g_t(s)+g_1(t,s)\end{aligned}$$

where g_1 is defined in the obvious way.

If F is any function such that F' and AF' are continuous on $[0,t]$ and are zero on $(-\infty,0)$, then $(F'(t), \varphi(F'_t)) \in Y$ while

$$\begin{aligned}\varphi(F'_t)(s)&=\int_{-t}^0 B(s-u)F'(t+u)\,du=\int_0^t B(t+s-u)F'(u)\,du\\ &=\int_0^t B'(t+s-u)F(u)\,du+B(s)F(t)-B(t+s)F(0).\end{aligned}$$

Since F can be picked to make $\sup\{\|x(s)-F(s)\|_A : 0\leq s\leq t\}$ arbitrarily small, it follows that $(Ax_0+f(0), g_1(t,0))$ is in Y. From the definitions of the function g and of $D(C)$ we know that $C(x_0,f)=(Ax_0+f(0), g) \in Y$. Since Y is translation invariant, then $(Ax_0+f(0), g_t) \in Y$. Moreover by Lemma 2.3 any point $(x,0) \in Y$. In particular $(x'(t),0)$ and $(Ax_0+f(0),0)$ are in Y. It follows that

$$\begin{aligned}\left(x'(t), \frac{\partial F}{\partial t}(t,0)\right)&=(x'(t), g_t+g_1(t,0))\\ &=(x'(t),0)+(Ax_0+f(0),g_t)+(Ax_0+f(0),g_1(t,0))\\ &\quad-2(Ax_0+f(0),0)\end{aligned}$$

is in Y.

Since A is closed, then C is closed. The existence of $\rho(\lambda)$ implies the existence of the resolvent $R(\lambda, C)$ at the same point λ. The argument above shows that for each initial condition (x_0, f) in $D(C)$ there is a corresponding solution of (DE) satisfying this initial condition. The uniqueness of the solutions of these initial value problems follows from Theorem 3.5 and uniqueness for (VE). All of the hypotheses of Theorem 2.11 of Krein [4, p. 54] are satisfied. The conclusion follows from that theorem. Q. E. D.

The existence of the uniform limit (4.6) can be obtained in several ways. For example if the function $\gamma(t)$ in the hypothesis of Theorem 4.4 is of class $L^1(0,\infty)$, then the expression on the right in (4.6) is bounded and uniformly continuous on any set of the form $\{(t,s) : 0\leq t\leq T, 0\leq s<\infty\}$. In this case

$$(f')_t+Bx(t)+\int_0^t B'_{t+s-u}x(u)\,du$$

defines a continuous map on $[0, T]$ to $BC(R^+, x)$. The integral w.r.t. t of this map will be differentiable in t uniformly for $s\in R^+$. This integral is $F(t,s)-F(0,s)$. Therefore (4.6) will be true.

If (DE) is uniformly well posed, then for any $T>0$ there exists a constant $K>0$ such that

$$\|x(t, x_0, f)\|\leq K\|(x_0, f)\| \tag{4.8}$$

uniformly for $0\leq t\leq T$ and for $(x_0, f)\in D(C)$. A similar inequality is true for the second component of each solution. The hypotheses of Theorem 4.4 are sufficient to guarantee the uniform well posedness of (DE). The continuity (4.8) for the first component only can be proved under conditions weaker than the hypotheses of Theorem 4.4. Some preliminaries will be needed for the proof. Define

$$W=\{(x, f)\in Y : x\in D(A)\}$$

with norm

$$\|(x, f)\|=\|x\|+\|Ax+f(0)\|+\sup\{\|f(t)\| : t\geq 0\}.$$

Clearly W is a normed linear space. If A is closed, it is also a Banach space. Define

$$C(W)=\{(y(t), F(t, 0)) : [0, T]\to W : \text{continuous}\}.$$

If A is closed, this is also a Banach space with the uniform norm.

Theorem 4.5. *Suppose A is closed, $\rho(\lambda)$ exists for some λ with* Re $\lambda>0$ *and for each pair (x_0, f) in $D(C)$,* (VE) *has a unique solution $x(t, x_0, f)$ on R^+. Then the continuity condition* (4.8) *is true.*

Proof. Define $U_1(t)(x_0, f)=x(t, x_0, f)$,

$$U_2(t)(x_0, f)=\int_0^t B_u x(t-u, x_0, f)\,du+f_t$$

and $U(t)(x_0, f)=(U_1(t)(x_0, f), U_2(t)(x_0, f))$ for all $t\geq 0$ and all (x_0, f) in $D(C)$. Then for any fixed $T>0$, $U(0) : D(C)\to C(W)$. Give $D(C)$ the norm $\|(x_0, f)\|_C=\|(x_0, f)\|_Y+\|C(x_0, f)\|_Y$ so that $D(C)$ is a Banach space (see Lemma 3.4).

We claim that U is a closed map on $D(C)$ to $C(W)$. Indeed if $(x_n, f_n)\to(x_0, f)$ in $D(C)$ and $U(t)(x_n, f_n)\to(y(t), G(t, 0))$ in $C(W)$, then put $(x_n(t), F_n(t, 0))=U(t)(x_n, f_n)$. The convergence assumptions imply that $x_n\to x_0$ in X, $x_n(t)\to y(t)$, $Ax_n(t)\to Ay(t)$ and $F_n(t, 0)\to G(t, 0)$ uniformly on $0\leq t\leq T$. Take limits in

$$x_n(t)=x_n+\int_0^t \{Ax_n(s)+F_n(s, 0)\}\,ds$$

to obtain

$$y(t)=x_0+\int_0^t \{Ay(s)+G(s, 0)\}\,ds. \tag{4.9}$$

Note that for any pair (t, s),

$$\left\|F_n(t,s)-\int_0^t B(t+s-u)y(u)du+f_0(t+s)\right\|$$
$$=\left\|f_n(t+s)-f_0(t+s)+\int_0^t B(t+s-u)(x_n(u)-y(u))du\right\|$$
$$\leq \|f_n(t+s)-f_0(t+s)\|+\int_0^t \beta(t+s-u)\|x_n(u)-y(u)\|_A du \to 0.$$

It follows that

$$G(t,s)=\int_0^t B(t+s-u)y(u)du+f_0(t+s).$$

This and (4.9) show that $y(t)$ is the solution of (VE) for the pair (x_0, f_0). This proves that U is closed.

The hypotheses insure that the resolvent $R(\lambda, C)$ exists for the given λ. Given (x_0, f) in $D(C)$, define $(y_0, g)=R(\lambda, C)(x_0, f)\in D(C^2)$. For any $t\geq 0$ compute

$$\begin{aligned}\|U_1(t)(x_0,f)\|&=\|U_1(t)(C-\lambda I)(y_0,g)\|\\ &\leq \|U_1(t)C(y_0,g)\|+|\lambda|\,\|U_1(t)(y_0,g)\|\end{aligned}$$

Let π_1 denote projection onto the first coordinate in W. Since $U(t)$ satisfies (3.3) and $U_1(t)(y_0, g)$ is differentiable, one can compute

$$\begin{aligned}\pi_1 CU(t)(y_0,g)&=U_1'(t)(y_0,g)\\ &=\lim_{h\to 0}(U_1(t+h)-U_1(t))(y_0,g)/h\\ &=\lim_{h\to 0}U_1(t)(U(h)-I)(y_0,g)\\ &=U_1(t)C(y_0,g).\end{aligned}$$

Therefore there exists a constant M_1 such that

$$\|U_1(t)(x_0,f)\|\leq\|\pi_1 CU(t)(y_0,g)\|+|\lambda|\,\|U_1(t)(y_0,g)\|\leq M_1\|U(t)(y_0,g)\|_W$$

uniformly on $0\leq t\leq T$. But $U(\cdot): D(C)\to C(W)$ is closed so there exists $M_j>0$ such that

$$\begin{aligned}\|U_1(t)(x_0,f)\|&\leq M_1M_2\|(y_0,g)\|_C\\ &=M_1M_2(\|y_0,g)\|+\|C(y_0,g)\|)\\ &=M_1M_2(\|(y_0,g)\|+\|(x_0,f)+\lambda(y_0,g)\|)\\ &\leq M_3(\|(x_0,f)\|+\|R(\lambda,C)(x_0,f)\|)\\ &\leq M_4\|(x_0,f)\|.\end{aligned}$$

with M_4 independent of t on $[0, T]$. Q. E. D.

It would be interesting to obtain other conditions on the coefficients of (VE) which will insure that (DE) is uniformly well posed. For example if A is closed and there exist constants M and K such that $\rho(\lambda)$ exists and satisfies $\|\rho(\lambda)^n\|\leq M(\lambda-K)^{-n}$ for $n=1, 2, 3, \cdots$ and $\lambda>K$, is (DE) well posed?

5. Well Posed Problems

Defintion 5.1. We say that (VE) is well posed if for each pair (x_0, f) in

$D(C)$, there exists a unique solution $x(t, x_0, f)$ on R^+ and if for any $t \geq 0$, $x(t, x_0, f) \to 0$ as $||(x_0, f)||_Y \to 0$. Equation (VE) will be called uniformly well posed if in addition the convergence of $x(t, x_0, f)$ to zero is uniform in t on compact subsets of R^+.

If follows immediately from Theorem 3.5 that (VE) is well posed (or uniformly well posed) whenever (DE) is. Thus the conclusion of Theorem 4.4 implies that (VE) is uniformly well posed. Theorem 4.5 deals directly with the question of whether (VE) is uniformly well posed without first considering (DE). When (DE) is uniformly well posed, various standard facts concerning solutions of (DE) can be interpreted as results about solutions of (VE). These facts are summarized in the next theorem.

Theorem 5.2. *If* (DE) *is uniformly well posed and if* $x(t) = x(t, x_0, f)$ *is a solution of* (VE) *with* (x_0, f) *in* $D(C)$, *then there exists* $\lambda_0 \geq 0$ *such that the following statements are true*:

(a) *There exists* $M > 0$ *such that* $||x(t)|| \leq \exp(\lambda_0 t) M ||(x_0, f)||$ *for all* $t > 0$.

(b) $\rho(\lambda) = R(\lambda, A + B^*(\lambda))$ *exists whenever* Re $\lambda \geq \lambda_0$ *and both*

$$\rho(\lambda)(x_0 + f^*(\lambda)) = \int_0^\infty \exp(-\lambda t) x(t) dt \tag{5.1}$$

and

$$x(t) = -(2\pi i)^{-1} \int_{\lambda_0 - i\infty}^{\lambda_0 + i\infty} \exp(\lambda t) \rho(\lambda)(x_0 + f^*(\lambda)) d\lambda \tag{5.2}$$

are true.

Proof. The conclusions follow from Theorem 3.5 above and standard results for abstract differential equations, see for example Krein [4, pp. 29–45].

Q. E. D.

Whenever (VE) is well posed, continuity considerations show that for each $t \geq 0$ the solutions $x(t, x_0, f)$ can be extended from $D(C)$ to all (x_0, f) in Y. This extension defines a bounded linear map on Y to Y. Whenever (x_0, f) is in Y but not in $D(C)$, then $x(t, x_0, f)$ is a *generalized solution* of (VE) in the sense that it is the limit of a sequence of actual solutions.

Define $r(t)x_0 = x(t, x_0, 0)$ for all x_0 in $D(A)$ where $x(t, x_0, 0)$ is either a solution or generalized solution of (VE). From (5.1) and (5.2) it can be seen that formally $\rho(\lambda)$ is the Laplace transform of $r(t)$ and that

$$x(t, x_0, f) = r(t) x_0 + \int_0^t r(t-s) f(s) ds. \tag{5.3}$$

We shall determine some conditions which insure that (5.3) is actually correct.

Lemma 5.3. *Suppose* $R(\lambda, A)$ *exists for some* λ *with* Re $\lambda > 0$, *suppose* (VE) *is uniformly well posed, suppose that for all* x_0 *in* $D(A)$, $x(t, x_0, 0)$ *is an actual* (*not a generalized*) *solution of* (VE) *and finally suppose that for* *each* $t\geq 0$, $B(t)$ *maps* $D(A^2)$ *into* $D(A)$. *Then for each* x_0 *in* $D(A)$ *and eaah* $t\geq 0$,

(r) $$r'(t)x_0=Ar(t)x_0+\int_0^t B(t-u)r(u)x_0du$$

and

(r′) $$r'(t)x_0=r(t)Ax_0+\int_0^t r(t-u)B(u)x_0du.$$

Proof. First note that (r) is automatically true from the definition of $r(t)$. Given x_0 in $D(A)$ define

$$a(t)x_0=Ax_0+\int_0^t B(u)x_0du.$$

For all x_0 in $D(A^2)$ define

(5.4) $$T(t)x_0=x_0+\int_0^t r(t-s)a(s)x_0ds.$$

Since A is closed and $x_0\in D(A^2)$, $a(s)x_0\in D(A)$ and is continuous in s. Thus $T'(t)x_0$ exists. From (r) one can compute

$$\begin{aligned}T'(t)x_0&=a(t)x_0+\int_0^t\left\{Ar(t-s)a(s)x_0+\int_0^{t-s}B(t-s-u)r(u)a(s)x_0du\right\}ds\\&=a(t)x_0+A\int_0^t r(t-s)a(s)x_0ds+\int_0^t B(t-s)\int_0^s r(s-u)a(u)x_0duds\\&=a(t)x_0+A\{T(t)x_0-x_0\}+\int_0^t B(t-s)\{T(s)x_0-x_0\}ds\\&=AT(t)x_0+\int_0^t B(t-s)T(s)x_0ds.\end{aligned}$$

By uniqueness of solutions of (VE), $T(t)x_0=r(t)x_0$. This and (5.4) imply that

$$r(t)x_0=x_0+\int_0^t r(s)\left\{Ax_0+\int_0^{t-s}B(u)x_0du\right\}ds,$$

for all x_0 in $D(A^2)$. Since $R(\lambda, A)$ exists and since $D(A)$ is dense in X, it is clear that $D(A^2)$ is $\|\ \|_A-$ dense in $D(A)$. By continuity we see that the same integral equation is true for all x_0 in $D(A)$. By differentiating this integral equation we obtain

$$r'(t)x_0=r(t)Ax_0+\int_0^t r(s)B(t-s)x_0ds$$

for all x_0 in $D(A)$. This is (r′). Q. E. D.

Theorem 5.4. *Assume that* (VE) *is uniformly well posed. If the hypotheses of the last lemma are true, if* $f: R^+\to X$ *is a continuous function and if* $h(t)$ *is a solution of* (VE) *for this* f *and some* x_0 *in* $D(A)$, *then*

(5.5) $$h(t)=r(t)x_0+\int_0^t r(t-s)f(s)ds.$$

Proof. Apply to both sides of

$$h'(s)=Ah(s)+\int_0^s B(s-u)h(u)du+f(s),$$

the operator $r(t-s)$ to obtain

$$r(t-s)h'(s)=r(t-s)Ah(s)+r(t-s)\int_0^s B(s-u)h(u)du+r(t-s)f(s).$$

Since $(d/ds)(r(t-s)h(s))=r(t-s)h'(s)-r'(t-s)h(s)$, then this last equation goes into

$$\frac{d}{ds}\{r(t-s)h(s)\}=r(t-s)f(s)-\int_0^{t-s} r(t-s-u)B(u)h(s)du$$
$$+r(t-s)\int_0^s B(s-u)h(u)du.$$

Integrating w.r.t. s from 0 to t we obtain

$$h(t)-r(t)x_0-\int_0^t r(t-s)f(s)ds$$
$$=\int_0^t r(t-s)\int_0^s B(s-u)h(u)du\,ds-\int_0^t\int_0^{t-s} r(t-s-u)B(u)h(s)du\,ds$$
$$=\int_0^t\int_0^{t-u} r(t-s-u)B(s)h(u)ds\,du-\int_0^t\int_0^{t-s} r(t-u-s)B(u)h(s)du\,ds$$

The last term is zero. Q. E. D.

Corollary 5.5. *If the hypotheses of Lemma* 5.3 *are true, and if* (VE) *is uniformly well posed, then for any* (x_0, f) *in* Y *the generalized solution of* (VE) *can be represented in the form* (5.3).

Proof. Since (5.3) is true for actual solutions, it is true on the dense set of all (x_0, f) in $D(C)$. The conclusion follows by continuity. Q. E. D.

Remark: $B(t)=b(t)A$ with $b\in L^1(R^+)$ a given scalar function is an example of a function $B(t)$ which maps $D(A^2)$ into $D(A)$. This hypothesis seems restrictive. It would be convenient if (5.3) could be proved without it.

6. More General Inhomogeneous Equation

Consider a Cauchy problem of the form

$$x'(t)=Ax(t)+\int_0^t B(t-s)x(s)ds+F(t) \tag{6.1}$$

for $t\geq\tau$ and $x(t)=f(t)$ on $0\leq t\leq\tau$, where $F: R^+\to X$ is continuous and $f: [0,\tau]\to D(A)$ is $\|\ \|_A$–continuous. The space Y constructed in section 2 above was designed to facilitate the study of problem (6.1) when $F\equiv 0$. We shall now enlarge the space Y so that a larger class of functions F can be studied.

Let $BU=\{F: R^+\to X,\ F$ is bounded and uniformly continuous on $R^+\}$ with sup norm $\|F\|=\sup\{\|F(t)\|: 0\leq t<\infty\}$. Define $Y(U)=\{(x,f): x\in X$ and $f\in BU\}$ with norm $\|(x,f)\|=\|x\|+\|f\|$. Again put

$$C(x,f)=(Ax+f(0), f'+Bx)$$

but with $D(C)=\{(x,f): x\in D(A), f'$ exists *a.e.* and $f'+Bx\in BU\}$. Lemma 3.1 is still true with the same proof. The proof of Lemma 3.3 needs some modification.

Lemma 6.1. *$D(C)$ is dense in $Y(U)$ if A is closed.*

Proof. The set Y_1 constructed in the proof of Lemma 3.3 was dense in Y. A simple mollifyer argument will show that $BU' = \{f \in BU : f' \in BU\}$ is dense in BU. Thus the set $\{(x, f+F) : (x, f) \in Y_1$ and $F \in BU'\}$ is contained in $D(C)$. Lemma 2.2 can be used to see that this set is dense in $Y(U)$. Q. E. D.

The remaining results and definitions in sections 3, 4 and 5 are the same with only minor modifications of the proofs (such as replacing Y by $Y(U)$). This extension of the theory will allow the study (6.1) for any F in BU. If we wish to study (6.1) on a bounded set $0 \leq t \leq T$, then it is clear that $B(t)$ and $F(t)$ can be defined outside of the interval of interest in such a way that the theory can be applied to the resulting problem.

7. Examples of Well Posed Problems

Consider an initial value probelm of the form

$$x'(t) = Ax(t) + \int_0^t b(t-s)Ax(s)ds + f(t), \quad x(0) = x_0. \tag{7.1}$$

As before assume $A: D(A) \to X$ is linear with dense domain. Assume $\gamma = 1$ (the case $\gamma = 0$ will not work). Assume that $b(t)$ is a scalar function and that the following additional assumptions hold:

(A 1) A is the infinitesimal generator of a C_0-semigroup $V(t)$, that is a semigroup such that $V(t)x_0$ is continuous in x_0 uniformly for t on compact subsets of R^+.

(A 2) $b \in C^1(R^+) \cap L^1(R^+)$.

(A 3) $f(t)$ is strongly continuously differentiable on R^+.

Let λ_0 be a real number such that $R(\lambda, A) = (A - \lambda I)^{-1}$ exists for all λ in the half plane $\mathrm{Re}\lambda \geq \lambda_0$. If $\lambda_0 > 0$, then put $z(t) = x(t)\exp(-\lambda_0 t)$ so that $z'(t) = -\lambda_0 z(t) + \exp(-\lambda_0 t)x'(t)$ and

$$z'(t) = (A - \lambda_0 I)z(t) + \int_0^t \exp(-\lambda_0(t-s))b(t-s)Az(s)ds + \exp(-\lambda_0 t)f(t) \tag{7.1'}$$

for $t \geq 0$ with $z(0) = x_0$. The coefficients of (7.1′) also satisfy (A1)-(A3). These two problems are equivalent from the point of view of existence, uniqueness and continuity. Thus without loss of generality we shall assume:

(A 4) $R(\lambda, A) = (A - \lambda I)^{-1}$ exists for Re $\lambda \geq 0$.

Lemma 7.1. *If* (A1), (A3) *and* (A4) *are true, then for any $x_0 \in D(A)$,*

$$u'(t) = Au(t) + f(t), \quad u(0) = x_0$$

has a unique, strongly differentiable solution on R^+ with all of $u(t)$, $u'(t)$ and $Au(t)$ continuous. This solution can be represented in either of the forms:

$$u(t) = V(t)x_0 + \int_0^t V(t-s)f(s)ds$$

or

$$u(t)=V(t)(x_0+A^{-1}f(0))-A^{-1}f(t)+\int_0^t V(t-s)A^{-1}f'(s)ds.$$

For a proof see for example Krein [4, p. 135]. The next result is similar to one claimed in [1, p. 152]. The proof of the result uses the same techniques as those used in [1].

Lemma 7.2. *If* (A1)-(A4) *are true and if* x_0 *is a point in* $D(A)$, *then* (7.1) *has a unique solution on* R^+.

Proof. Fix any $T>0$. Define $d(t_0)=\{v: v$ maps $[0, t_0]$ into $D(A)$ and both $v(t)$ and $Av(t)$ are continuous$\}$. Since A is closed, it is easy to see that $d(t_0)$ with norm $|||v|||=\sup\{||v(t)||_A: 0\le t\le t_0\}$ is a Banach space. Given v in $d(t_0)$ define

$$S_0v(t)=\int_0^t b(t-s)Av(s)ds$$

on $0\le t\le t_0$. The s_0v is strongly continuously differentiable with

$$(S_0v)'(t)=b(0)Av(t)+\int_0^t b'(t-s)Av(s)ds.$$

From Lemma 7.1 it follows that for any v in $d(t_0)$,

$$u'(t)=Au(t)+\{S_0v(t)+f(t)\},\quad u(0)=x_0$$

has a unique solution $u=Sv$ which is again in $d(t_0)$.

Since Sv is linear, we can decide whether or not it is a contraction map by computing the norm when $x_0=0$ and $f(t)\equiv 0$. If $|||v|||\le 1$, then

$$||S_0v(t)||\le\int_0^t|b(t-s)|\,||Av(s)||ds\le\int_0^t|b(s)|ds,$$

while $x_0=0$, $f\equiv 0$ and Lemma 7.1 imply that

$$\begin{aligned}||Sv(t)||&=\left\|\int_0^t V(t-s)S_0v(s)ds\right\|\\&\le\int_0^t||V(t-s)||\,||S_0v(s)||ds\\&\le\int_0^t M\int_0^s|b(u)|duds=M\int_0^t u|b(u)|du\end{aligned}$$

where M is a bound for $||V(t)||$ on $[0, T]$ and $[0, t_0]\subset[0, T]$. The second representation in Lemma 7.1 can be used to see that

$$ASv(t)=A\int_0^t V(t-s)A^{-1}(S_0v)'(s)ds+AV(t)A^{-1}(S_0v)(0)-S_0v(t).$$

Since A is closed and A commutes with $V(t)$, then

$$\begin{aligned}||ASv(t)||&=\left\|\int_0^t V(t-s)(S_0v)'(s)ds+S_0v(0)-S_0v(t)\right\|\\&\le\int_0^t|b(s)|ds+\left\|\int_0^t V(t-s)\left\{b(0)Av(s)+\int_0^s b'(s-u)Av(u)du\right\}ds\right\|\end{aligned}$$

$$\leq \int_0^t |b(s)|ds + \int_0^t M\left\{|b(0)| + \int_0^s |b'(s-u)|du\right\}ds$$
$$\leq \int_0^t |b(s)|ds + M|b(0)|t + M\int_0^t u|b'(u)|du.$$

Therefore

$$||S|| \leq \int_0^{t_0} |b(u)|du + M\left\{|b(0)|t_0 + \int_0^{t_0} u(|b(u)| + |b'(u)|)du\right\} < 1$$

for t_0 sufficiently small. The contraction mapping theorem implies the existence and uniqueness of a solution of (7.1) on $[0, t_0]$.

Translate (7.1) by t_0 to see that $y(t) = x(t+t_0)$ must satisfy

$$y(t) = \left\{f(t+t_0) + \int_0^{t_0} b(t+t_0-s)Ax(s)ds\right\}$$
$$+Ay(t) + \int_0^t b(t-s)Ay(s)ds,$$

with $y(0) = x(t_0)$ in $D(A)$. Since the term in brackets is strongly continuously differentiable, the same argument can be repeated to obtain a solution on $[t_0, 2t_0]$, $[2t_0, 3t_0]$, $\cdots$ until $[Nt_0, T]$ where $(N+1)t_0 > T$. Since T is an arbitrary positive number, this proves the existence and uniqueness on R^+. Q. E. D.

Theorem 7.3. *If* (A1)-(A3) *are true, then* (7.1) *is uniformly well posed on either of the spaces* Y *or* $Y(U)$. *If in addition* $b' \in L^1(R^+)$, *then the corresponding differential equation* (DE) *is uniformly well posed.*

Proof. We apply Theorem 4.5. Since $R(\lambda, A)$ exists for Re λ large, then A is closed. Moreover for Re λ sufficiently large

$$A + b^*(\lambda)A - \lambda I = \{A - \lambda I/(1+b^*(\lambda))\}(1+b^*(\lambda))$$

and

$$\rho(\lambda) = (A + b^*(\lambda)A - \lambda I)^{-1} = R(\lambda(1+b^*(\lambda))^{-1}, A)(1+b^*(\lambda))^{-1}$$

exists as a bounded linear map from X to X. Since $b(t) \in C^1(R^+)$, if $(x_0, f) \in D(C)$, then $f'(t) + b(t)Ax$ is continuous and $f'(t)$ must be continuous. Thus the last lemma implies the existence and uniqueness of solution of (7.1) for all pairs (x_0, f) in $D(C)$. All hypotheses of Theorem 4.5 are true, (A4 without loss of generality), so continuity of $x(t, x_0, f)$ follows.

If in addition $b' \in L^1(R^+)$, then $b \in BU$ and

$$F_0(t) = (f')_t + b(\cdot)Ax(t)\int_0^t (b')_u Ax(t-u)du$$

defines a continuous map from R^+ to BU. Since

$$f_t + \int_0^t b_u Ax(t-u)du = f + \int_0^t F_0(v)dv,$$

then (4.6) is trivial to verify. The second conclusion follows from Theorem 4.4. Q. E. D.

Corollary 7.4. *Under the hypotheses of the last theorem, each solution or* *generalized solution of* (7.1) *can be written in the form*

$$x(t)=r(t)x_0+\int_0^t r(t-s)f(s)ds$$

where $r(t)x_0$ *is the solution of* (7.1) *when* $f(t)\equiv 0$.

Proof. Corollary 5.5 applies. Q.E.D.

The first part of Theorem 7.3 can be generalized as follows. Let A and f satisfy (A1) and (A3). Let $b_j(t)\in C^1(R^+)$ with b_j and b'_j in $L^1(R^+)$, let E_j: $X\to X$ be a bounded linear map, and let $A_j:D(A)\to X$ be linear maps with $||A_jx||\le K||x||_A$. Then the equation

$$x'(t)=Ax(t)+E_0x(t)+\int_0^t\left(\sum_{j=1}^{N-1}b_j(t-s)A_j\right)x(s)ds+\int_0^t b_N(t-s)E_Nx(s)ds+f(t) \tag{7.2}$$

has a unique solution on R^+ for each initial condition x_0 in $D(A)$. The proof is essentially the same as the proof of Lemma 7.2. The proof of the first part of Theorem 7.3 generalizes once the following lemma is proved.

Lemma 7.5. *Under the hypotheses above*

$$\rho(\lambda)=\left\{A+\sum_{j=1}^{N-1}b_j{}^*(\lambda)A_j+E_0+b_N{}^*(\lambda)E_N-\lambda I\right\}^{-1}$$

exists for λ *positive and sufficiently large.*

Proof. Given y in X we want to find an x so that

$$\left\{A-\lambda I+E_0+\sum_{j=1}^{N-1}b_j{}^*(\lambda)A_j+b_N{}^*(\lambda)E_N\right\}x=y.$$

For λ real and large, say $\lambda\ge\lambda_0$, $R(\lambda,A)$ exists and satisfies a bound $||R(\lambda,A)||\le M\lambda^{-1}$. Put $z=(A-\lambda I)x$ or $x=R(\lambda,A)z$ so that

$$(A-\lambda I)x=y-\left(E_0+\sum_{j=1}^{N-1}b_j{}^*(\lambda)A_j+b_N{}^*(\lambda)E_N\right)x$$

becomes

$$z=y-(E_0+\sum b_j{}^*(\lambda)A_j+b_N{}^*(\lambda)E_N)R(\lambda,A)z. \tag{7.3}$$

Therefore

$$\begin{aligned}||z||&\le||y||+(||E_0||+|b_N{}^*(\lambda)|\,||E_N||)||R(\lambda,A)||\,||z||+\sum|b_j{}^*(\lambda)|\,||A_jR(\lambda,A)z||\\&\le||y||+(||E_0||+|b_N{}^*(\lambda)|\,||E_N||)(M/\lambda)||z||\\&\quad+\sum|b_j{}^*(\lambda)|K(||AR(\lambda,A)z||+||R(\lambda,A)z||)\end{aligned}$$

Since $||AR(\lambda,A)||=||I+\lambda R(\lambda,A)||\le 1+M$ if $\lambda\ge\lambda_0$, then

$$||z||\le||y||+(M_1/\lambda)||z||+\sum|b_j{}^*(\lambda)|K(1+M+M/\lambda)||z||\le||y||+||z||/2$$

for λ sufficiently large. By the contraction mapping theorem there exists a continuous linear map $F(\lambda)$ such that $z=F(\lambda)y$ is the unique fixed point of (7.3) in X. Since $z=(A-\lambda I)x$, then $x=R(\lambda,A)F(\lambda)y$ solves the original equation and $\rho(\lambda)=R(\lambda,A)F(\lambda)$. Q.E.D.

Corollary 7.6. *Under the hypotheses above,* (7.2) *is uniformly well posed* *on either* Y *or on* $Y(U)$.

Next we consider an example in Hilbert space. Our results are different then but motivated by recent results of Hannsgen [9]. Most of our hypotheses will be stronger than those of Hannsgen but our hypotheses on f will be weaker. The necessary background material about spectral resolution of the identity can be found in [11].

Lemma 7.7. *Let* X *be a Hilbert space with inner product* $(\ ,\ )$ *and let* A *a densely defined, closed, symmetric, linear map with* $(Ax, x) \le \lambda_0(x, x)$ *for some real number* λ. *Let* $\{E_\lambda\}$ *be the spectral resolution of the identity corresponding to* A *and assume that*

$$B(t)x = \int_{-\infty}^{\lambda_0} a(t, \lambda)\, dE_\lambda x \quad (a \text{ scalar valued})$$

with $B(t)x$ *strongly continuously differentiable when* x *is in* $D(A)$. *Finally suppose there exist two scalar functions* β_0 *and* β_1 *in* $L^1(R^+)$ *such that* $|a(t, \lambda)| \le \beta_0(t)|\lambda|$ *and* $|\partial a(t, \lambda)/\partial t| \le \beta_1(t)|\lambda|$ *for all* $t \ge 0$ *and all* $\lambda \le \lambda_0$. *Then for any* x_0 *in* $D(A)$ *and any* f *satsisfying* (A3) *the initial value problem* (VE) (*with* $\gamma = 1$) *has a unique solution on* R^+.

Proof. Recall that for any function g

$$Gx = \int_{-\infty}^{\lambda_0} g(\lambda)\, dE_\lambda x$$

is a linear map defined for those and only those x such that

$$||Gx||^2 = \int_{-\infty}^{\lambda_0} |g(\lambda)|^2 dE_\lambda(x, x) < \infty.$$

Moreover when $g(\lambda) = \lambda$, then $G = A$. Thus if x is in $D(A)$, then

$$||B(t)x||^2 = \left\| \int_{-\infty}^{\lambda_0} a(t, \lambda)\, dE_\lambda x \right\|^2 \le \beta_0(t)^2 \int_{-\infty}^{\lambda_0} |\lambda|^2 dE_\lambda(x, x) = \beta_0(t)^2 ||Ax||^2 < \infty.$$

Similarly $||B'(t)x|| \le \beta_1(t)||Ax||$ when $x \in D(A)$.

The proof of Lemma 7.7 proceeds by the same general outline as the proof of Lemma 7.2. First note that when $\lambda_0 \ge 0$, then (VE) can be replaced by the equivalent problem

$$z'(t) = (A - \lambda_1 I)z(t) + \int_0^t \exp(-\lambda_1(t-s))B(t-s)z(s)\,ds + \exp(-\lambda t)f(t)$$

where $\lambda_1 > \lambda_0$. Define $d(t_0)$, S_0 and S as before. When $|||v||| \le 1$, the estimate on $S_0 v$ is replaced by

$$\begin{aligned} ||S_0 v(t)|| &= \left\| \int_0^t \exp(\lambda_1(s-t)) \int_{-\infty}^{\lambda_0} a(t-s, \lambda)\, dE_\lambda v(s)\,ds \right\| \\ &\le \int_0^t \exp(\lambda_1(s-t))\beta_0(t-s)||Av(s)||ds \\ &\le \int_0^t \exp(\lambda_1(s-t))\beta_0(t-s)\,ds. \end{aligned}$$

The rest of the estimates are modified in a similar way to complete the proof. Q. E. D.

Theorem 7.8. *If the hypotheses of Lemma* 7.7 *are true,* (VE) *is uniformly well posed.*

Proof. In order to apply Theorem 4.5 it remains only to show that $\rho(\lambda)$ exists for at least one λ. It is easy to see that $|(B^*(\lambda)x, x)| \leq \beta_0^*(\lambda)(x, x)$. Thus

$$(\{A+B^*(\lambda)\}x, x) = (Ax, x) + (B^*(\lambda)x, x) \leq (\lambda_0+\beta_0^*(\lambda))(x, x).$$

Since $\beta_0 \in L^1(R^+)$, then $\beta_0^*(\lambda) \to 0$ as $\lambda \to +\infty$. Therefore $\rho(\lambda) = (A+B^*(\lambda) - \lambda I)^{-1}$ exists for λ real and sufficiently large. Q. E. D.

8. When A Generates an Analytic Semigroup

We shall continue the theme of the last section here. Under stronger assumptions on A in (7.1) it will be shown that the smoothness hypotheses on $b(t)$ can be weakened. Assume the following:

(A5) A generates an analytic, uniformly continuous semigroup $V(t)$.

(A6) $b \in L^1(R^+)$ and for any $T>0$, the integral

$$t^{-1}\int_0^t |b(u)|du \in L^1(0, T),$$

the integrals

$$\int_0^t s^{-1}\int_0^s |b(t-u)|duds, \int_0^t \int_0^{t-s} |b(s)-b(s+u)|u^{-1}dsdu$$

exist uniformly on $0 \leq t \leq T$ and the last integral above tends to zero as $t \to 0$.

(A7) (x_0, f) is in $D(C)$.

The following result is known, see for example Krein [4, p. 138].

Lemma 8.1. *If A satisfies* (A5), *x_0 is in $D(A)$ and the integral*

$$\int_0^t ||f(t-u)-f(t)||u^{-1}du \tag{8.1}$$

converges uniformly on $0 \leq t \leq T$, then

$$u'(t) = Au(t) + f(t), \quad u(0) = x_0$$

has a unique solution on $[0, T]$. *This solution can be written in the form*

$$u(t) = V(t)x_0 + \int_0^t V(t-s)f(s)ds.$$

Lemma 8.2. *If* (A5)-(A7) *are true, then* (7.1) *has a unique solution on* R^+.

Proof. Define $d(t_0), S_0v$ and Sv as in the proof of Lemma 7.2. For v in $d(t_0)$ it will be shown that S_0v satisfies condition (8.1) above. The definition of S_0 allows us to compute

$$\int_\varepsilon^t ||S_0v(t-u)-S_0v(t)||u^{-1}du = \int_\varepsilon^t u^{-1}\left|\left|\int_0^{t-u} b(t-u-s)Av(s)ds\right.\right.$$

$$
\begin{aligned}
&\quad -\int_0^t b(t-s)Av(s)ds\Big\|du \\
(8.2)\qquad &\le |||v|||\left\{\int_\varepsilon^t\int_0^{t-u}|b(t-u-s)-b(t-s)|u^{-1}dsdu\right. \\
&\quad \left.+\int_\varepsilon^t u^{-1}\int_0^u |b(u-s)|dsdu\right\} \\
&\le |||v|||\left\{\int_\varepsilon^t\int_0^{t-u}|b(s)-b(s+u)|u^{-1}duds+\int_\varepsilon^t u^{-1}\int_0^u|b(s)|dsdu\right\}.
\end{aligned}
$$

This computation and (A6) prove the assertion.

Next note that f satisfies (8.1). Indeed if $f'+bAx_0=g$, then

$$
\begin{aligned}
\int_0^t ||f(t-s)-f(t)||s^{-1}ds &= \int_0^t\Big\|\int_0^{t-s}(g(u)-b(u)Ax_0)du \\
&\quad -\int_0^t(g(u)-b(u)Ax_0)du\Big\|s^{-1}ds \\
&=\int_0^t\Big\|s^{-1}\int_{t-s}^t(g(u)-b(u)Ax_0)du\Big\|ds \\
&\le\int_0^t||g||ds+\int_0^t s^{-1}\int_{t-s}^t|b(u)|duds||Ax_0|| \\
&\le||g||t+\int_0^t s^{-1}\int_0^s|b(t-u)|duds||Ax_0||.
\end{aligned}
$$

Assumption (A6) implies that the last integral exists uniformly for $t\in[0, T]$. These assertions and Lemma 8.1 imply that $S: d(t_0)\to d(t_0)$.

In order to see that S is contracting it is enough to show that if $x_0=0$ and $f\equiv 0$, then $||S||<1$. In this case, if $|||v|||=1$, we have

$$
\begin{aligned}
||Sv(t)|| &= \Big\|\int_0^t V(t-s)\int_0^s b(s-u)Av(u)duds\Big\| \\
&\le\int_0^t||V(t-s)||\int_0^s|b(s-u)|\,||Av(u)||duds \\
&\le M\int_0^t\int_0^s|b(u)|duds=M\int_0^t u|b(u)|du
\end{aligned}
$$

where M is a bound for $||V(t)||$ on $[0, T]$. Since V is an analytic semigroup, we can assume for the same constant M that $||AV(t)||\le Mt^{-1}$ on $[0, T]$, see Krein [4, p. 75, Theorem 3.9]. Since

$$
\begin{aligned}
ASv(t)=\int_0^t AV(s)S_0v(t-s)ds &= \int_0^t AV(s)\{S_0v(t-s)-S_0v(t)\}ds \\
&\quad+\int_0^t AV(s)S_0v(t)ds,
\end{aligned}
$$

then

$$
||ASv(t)||\le\int_0^t Ms^{-1}||S_0v(t-s)-S_0v(t)||ds+||(V(t)-I)S_0v(t)||.
$$

Let $\varepsilon\to 0^+$ in (8.2) and use the result to see that

$$||ASv(t)|| \leq M\int_0^t\int_0^{t-s}|b(u)-b(u+s)|s^{-1}duds + M\int_0^t s^{-1}\int_0^s |b(u)|duds$$
$$+(1+M)||S_0 v(t)||.$$

Since $|||v|||=1$, then

$$||S_0 v(t)|| = \left\|\int_0^t b(t-s)Av(s)ds\right\| \leq \int_0^t |b(u)|du.$$

From these estimates it follows that

$$||S|| \leq M\int_0^{t_0} u|b(u)|du + (1+M)\int_0^{t_0}|b(u)|du + M\int_0^{t_0}s^{-1}\int_0^s |b(u)|du + \eta(t_0)$$

where

$$\eta(t_0) = \sup\left\{\int_0^t\int_0^{t-s}|b(u)-b(u+s)|s^{-1}duds = 0 \leq t \leq t_0\right\}.$$

Clearly $||S||<1$ if t_0 is small. Thus there is a unique solution on $[0, t_0]$. As in the proof of Lemma 7.2 the solution will be extended across the interval $[0, T]$ by translations. Q. E. D.

Remark: If near $t=0$, $b(t)=t^{-a}$ for some $a\in(0,1)$, the (A6) will be true. First note that

$$s^{-1}\int_0^s s^{-a}ds = s^{-1}s^{1-a}(1-a)^{-1} = (1-a)^{-1}s^{-a}$$

is in $L^1(0, T)$. Since $b(t)=t^{-a}\in L^p(0, T)$ for any $p\in(1, a^{-1})$, then the integral of $b(t)$ is locally Hölder continuous with some exponent r. Therefore when $\varepsilon>0$

$$\int_\varepsilon^t\int_0^s s^{-1}(t-u)^{-a}duds = \int_\varepsilon^t s^{-1}\{(t-s)^{1-a}-t^{1-a}\}(1-a)^{-1}ds$$
$$\leq \int_\varepsilon^t s^{-1}Ks^r ds \leq (K/r)t^r$$

and

$$\int_\varepsilon^t\int_0^{t-u}|b(s)-b(s+u)|u^{-1}dsdu = \int_\varepsilon^t u^{-1}\{(t-u)^{1-a}+u^{1-a}-t^{1-a}\}(1-a)^{-1}du$$
$$\leq \int_\varepsilon^t u^{-1}\{(t-u)^{1-a}-t^{1-a}\}(1-a)^{-1}du + t^{1-a}(1-a)^{-2}$$
$$\leq \int_\varepsilon^t Ku^{r-1}du + t^{1-a}(1-a)^{-2}$$
$$\leq (K/r)t^r + t^{1-a}(1-a)^{-2} \to 0 \quad (t\to 0).$$

These estimates can be modified to include the more general case $b(t)=\sum_{j=1}^n t^{-a_j}g_j(t)$ with $0<a_j<1$ and $g_j(t)$ locally Hölder continuous.

Theorem 8.3. *If* (A5)-(A7) *are true and* $b\in L^1(0,\infty)$, *then* (7.1) *is uniformly well posed on* Y *or on* $Y(U)$.

Proof. Given Lemma 8.2 it remains to show that $\rho(\lambda)$ exists for at least one λ with Re $\lambda>0$. This was already shown in the proof of Theorem 7.3 under weaker assumptions on A. Q. E. D.

9. Some Perturbation Theory

Consider (7.1) and a perturbation equation of the form

$$z'(t)=Az(t)+A_1z(t)+\int_0^t \{b(t-s)Az(s)+b_2(t-s)A_2z(s)\}ds+f(t). \tag{9.1}$$

Theorem 9.1. *Suppose A satisfies* (A1), *b and b_2 satisfy* (A2), *A_1 is bounded and linear on X and $A_2:D(A)\to X$ is a linear map such that $||A_2x||\le K_2 ||x||_A$ for all x in $D(A)$. Then* (9.1) *is uniformly well posed on Y or $Y(U)$. Moreover if $r(t)x_0$ is the solution of* (7.1) *with $x_0\in X$ and $f\equiv 0$, then solutions of* (9.1) *satisfy*

$$z(t)=r(t)x_0+\int_0^t r(t-s)\left\{A_1z(s)+\int_0^t b_2(s-u)A_2z(u)du+f(s)\right\}ds. \tag{9.2}$$

Proof. Apply Corollaries 7.6 and 7.4. Q.E.D.

Theorem 9.2. *Suppose A satisfies* (A5), *b and b_2 satisfy* (A6), *$A_2:D(A)\to X$ is a linear map such that $||A_2x||\le K_2||x||_A$ for all x in $D(A)$ and suppose for each $\varepsilon>0$ there exists a continuous φ_ε on X such that*

$$||A_1x||\le\varphi_\varepsilon(x)+\varepsilon||Ax|| \quad (x \text{ in } D(A)).$$

Then (9.1) *is uniformly well posed on Y or $Y(U)$ and* (9.2) *is true.*

Proof. By Theorem 7.2 in [4, p.149] the operator $A_3=A+A_1$ satisfies (A5). If $||x||_3=||x||+||A_3x||$, then by the closed graph theorem there exists a constant $K_3>0$ such that $||x||_A\le K_3||x||_3$ for all x in $D(A)=D(A_3)$. Thus

$$w'(t)=(A+A_1)w(t)+\int_0^t b(t-s)(A+A_1)w(s)ds+f(t) \tag{9.3}$$

is uniformly well posed (by the results in section 8). A simple modification of Lemma 8.2 will prove existence and uniqueness on $D(C)$ when (9.3) is perturbed by terms

$$-\int_0^t b(t-s)A_1w(s)ds+\int_0^t b_2(t-s)A_2w(s)ds.$$

The existence of $\rho(\lambda)$ follows from the proof of Lemma 7.5. Therefore Theorem 4.5 applies. Q.E.D.

Now consider a well posed equation (VE) and a perturbed equation

$$z'(t)=Az(t)+A_1z(t)+\int_0^t \{B(t-s)+b_2(t-s)A_2\}z(s)ds+f(t). \tag{9.4}$$

Theorem 9.3. *Suppose* (VE) *is well posed on $Y(U)$, A is closed, $B(t)x$ is strongly continuously differentiable when x is in $D(A)$ and* (5.3) *is true. Let $\rho(\lambda)$ exist for some value λ_1 with* Re *$\lambda_1>0$, let $A_1:X\to X$ be a bounded linear map and let $A_2:D(A)\to X$ be a linear map such that $||A_2x||\le K_2||x||_A$ on $D(A)$. If b_2 satisfies* (A2), *then* (9.4) *has a unique solution for all initial conditions on $D(C_1)$ where*

$$C_1(x,f)=((A+A_1)x+f(0),f'+(B_u+b_{2u}A_2)x).$$

Proof. The hypotheses imply that for any (x, f) in $D(C_1)$, $f\in C^1(R^+)$.

Let $d(t_0)=\{v:[0,t_0]\to D(A):v, v'$ and Av are continuous$\}$ with norm $|||v|||=\sup\{||v(s)||_A: 0\le s\le t_0\}+\sup\{||v'(s)||: 0\le s\le t_0\}$. Given $v\in d(t_0)$ let Sv denote the solution of

$$z'(t)=Az(t)+\int_0^t B(t-s)z(s)ds+A_1v(t)+\int_0^t b_2(t-s)A_2v(s)ds+f(t)$$

with $z(0)=x_0$ and $(x_0, f)\in D(C_1)$. Since (VE) is uniformly well posed on $Y(U)$, Sv exists and is in $d(t_0)$. Moreover Theorem 5.4 implies that

$$Sv(t)=x(t)+\int_0^t r(t-s)\left\{A_1v(s)+\int_0^s b_2(s-u)A_2v(u)du\right\}ds$$

where

$$x(t)=r(t)x_0+\int_0^t r(t-s)f(s)ds.$$

In order to see that S is contracting assume $x_0=0$, $f(t)\equiv 0$ and $|||v|||=1$. If $K=\sup||r(t)||$ on $[0,t_0]$, then

$$||Sv(t)||\le\int_0^t ||r(t-s)||\left\{||A_1||\,|||v|||+\int_0^s |b_2(s-u)|K_2|||v|||du\right\}ds$$
$$\le K||A_1||t_0+KK_2\int_0^{t_0}|b_2(u)|udu.$$

For t_0 small, the last number of the right is at most 1/4.

Let $F(t)=A_1v(t)+\int_0^t b_2(t-s)A_2v(s)ds$ so that

$$||F(0)||\le||A_1||\,|||v|||+\int_0^t|b_2(t-s)|K_2|||v|||ds\le||A_1||+K_2\int_0^{t_0}|b_2(s)|ds,$$

$$F'(t)=A_1v'(t)+b_2(0)A_2v(t)+\int_0^t b_2'(t-s)A_2v(s)ds,$$

and

$$||F'(t)||\le||A_1||+|b_2(0)|K_2+K_2\int_0^{t_0}|b_2'(s)|ds$$

Note that

$$Sv(t)=\int_0^t r(t-s)\left\{F(0)+\int_0^s F'(u)du\right\}ds$$
$$=\int_0^t r(s)F(0)ds+\int_0^t\int_0^{t-s} r(u)F'(s)duds.$$

Since A is closed, the A applied to the first term on the right equals

$$\int_0^t Ar(s)F(0)ds=\int_0^t\left\{r'(s)F(0)-\int_0^s B(s-u)r(u)F(0)du\right\}ds$$
$$=(r(t)-I)F(0)-\int_0^t\int_0^s B(s-u)r(u)F(0)duds.$$

It follows that

$$\left\|A\int_0^t r(s)F(0)ds\right\|\le\left\{(K+1)+\int_0^t\int_0^s \beta(s-u)Kduds\right\}||F(0)||$$

$$\leq \left\{K+1+K\int_0^{t_0} u\beta(u)\,du\right\} ||F(0)||.$$

If A is applied to the second term, we get

$$\int_0^t \int_0^{t-s} Ar(u)F'(s)\,duds$$

$$=\int_0^t \int_0^{t-s} \left\{r'(u)F'(s)-\int_0^u B(u-\sigma)r(\sigma)F'(s)\,d\sigma\right\} duds$$

$$=\int_0^t (r(t-s)-I)F'(s)\,ds-\int_0^t \int_0^{t-s} \int_0^u B(u-\sigma)r(\sigma)F'(s)\,d\sigma duds,$$

so that

$$\left\|A\int_0^t \int_0^{t-s} r(u)F'(s)\,duds\right\| \leq \int_0^{t_0} (K+1)||F'(s)||ds$$

$$+\int_0^t \int_0^{t-s} \int_0^u \beta(u-\sigma)K||F'(s)||d\sigma duds$$

$$\leq \sup_{0\leq s\leq t_0}||F'(s)|| \left\{t_0(K+1)+K\int_0^{t_0}\int_0^{t_0} u\beta(u)\,duds\right\}.$$

These estimates imply that $||ASv(t)||\leq 1/4$ if t_0 is sufficiently small.

$$(Sv)'(t)=\left\{\int_0^t r(s)F(0)\,ds+\int_0^t \int_0^{t-s} r(u)F'(s)\,duds\right\}'$$

$$=r(t)F(0)+\int_0^t r(t-s)F'(s)\,ds,$$

$$||(Sv)'(t)||\leq K||F(0)||+K\int_0^{t_0}||F'(s)||ds\leq 1/4$$

if t_0 is small and $|||v|||=1$. By combining these estimates it follows that for t_0 small, $x_0=0$, $f(t)\equiv 0$, $|||v|||=1$ we have $|||Sv|||\leq 3/4$. Since S is contracting, there exists a unique solution of (9.4) on $[0, t_0]$. Existence and uniqueness on any finite interval can be proved from this contraction argument and translation. Q. E. D.

Corollary 9.4. *Assume the hypotheses of the last theorem. Suppose there exist* λ_0 *and* M *such that* $\rho(\lambda)=(A+B^*(\lambda)-\lambda I)^{-1}$ *exists and*

(9.5) $$||A\rho(\lambda)||\leq M \text{ and } ||\rho(\lambda)||\leq N(\lambda) \quad (\lambda\geq\lambda_0)$$

where $N(\lambda)\to 0$ *as* $\lambda\to\infty$. *Then* (9.4) *is uniformly well posed on* $Y(U)$.

Proof. In order to apply Theorem 4.5 it remains to show that $\rho_1(\lambda)=(A+A_1+B^*(\lambda)+b_2^*(\lambda)A_2-\lambda I)^{-1}$ exists. Given y we want to solve

$$(A+B^*(\lambda)-\lambda+b_2^*(\lambda)A_2+A_1)x=y$$

or

$$(A+B^*(\lambda)-\lambda I)x=y-(A_1+b_2^*(\lambda)A_2)x.$$

Let $z=(A+B^*(\lambda)-\lambda I)x$ so that

$$z=y-(A_1+b_2^*(\lambda)A_2)\rho(\lambda)z.$$

Now

$$||(A_1+b_2^*(\lambda)A_2)\rho(\lambda)||\leq||A_1||\;||\rho(\lambda)||+|b_2^*(\lambda)|K(||A\rho(\lambda)||+||\rho(\lambda)||)$$

$$\leq ||A_1|| N(\lambda) + |b_2{}^*(\lambda)| K(M+N(\lambda)) \leq 1/2$$

if λ is sufficiently large. Thus $z=G(\lambda)y$ exists and $x=\rho(\lambda)G(\lambda)y$ solves the original problem. In particular $\rho_1(\lambda)=\rho(\lambda)G(\lambda)$ exists for λ real and large.

Q. E. D.

It would be intersting to know whether or not conditions (9. 5) are automatically true when (VE) is uniformly well posed. They are automatically true when the problem (DE) corresponding to (VE) is uniformly well posed.

Theorem 9.5. *Assume the hypotheses of Theorem* 9. 4. *Suppose the problem* (DE) *corresponding to* (VE) *is uniformly well posed on* $Y(U)$. *Then* (9. 4) *is uniformly well posed on* $Y(U)$.

Proof. Recall that $||R(\lambda, C)|| \leq M_0(\lambda-\lambda_0)^{-1}$ for $\lambda > \lambda_0$ where M_0 and λ_0 are fixed real constants. Also recall that $CR(\lambda, C) = I + \lambda R(\lambda, C)$. These remarks and (4. 1) imply that for any z in X,

$$||\rho(\lambda)z|| \leq ||R(\lambda, C)(z, 0)|| \leq M_0(\lambda-\lambda_0)^{-1}||z||,$$

and $A\rho(\lambda)z + B^*(\lambda)\rho(\lambda)z = z + \lambda\rho(\lambda)z$. The triangle inequality implies that

$$||A\rho(\lambda)|| - \beta^*(\lambda)(||A\rho(\lambda)|| + ||\rho(\lambda)||) \leq ||A\rho(\lambda) + B^*(\lambda)\rho(\lambda)||$$
$$= ||I + \lambda\rho(\lambda)|| \leq 1 + \lambda M_0(\lambda-\lambda_0)^{-1}.$$

For λ real and sufficiently large $1-\beta^*(\lambda) < 1$ so that

$$||A\rho(\lambda)|| \leq (1-\beta^*(\lambda))^{-1}(1 + M_0\lambda(\lambda-\lambda_0)^{-1} + \beta^*(\lambda)M_0(\lambda-\lambda_0)^{-1}) \leq M_1.$$

Thus Corollary 9. 4 applies. Q. E. D.

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