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Fourier Integral Operators of Multi-Phase and the Fundamental Solution for a Hyperbolic System

By

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Introduction. Let $\mathscr{P}^{0}(\tau)$ $(0\leqq\tau<1)$ denote the set of phase functions $\phi(x,\xi)$ such that $\phi(x,\xi)$ are of class C^{2} in $R^{2n}=R_{x}^{n}\times R_{\xi}^{n}$ and $J(x,\xi)\equiv\phi(x,\xi)-x\cdot\xi$ satisfy

$$\|J\|_{0}\equiv\sum_{|\alpha+\beta|\leqq 2}\sup_{x,\xi}\{|J_{(\beta)}^{(\alpha)}(x,\xi)|/\langle\xi\rangle^{1-|\alpha|}\}\leqq\tau.$$

For $\phi_{j}(x,\xi)\in\mathscr{P}^{0}(\tau_{j}), j=1,2,\cdots,\nu+1,\cdots$, such that $\bar{\tau}_{\infty}\equiv\sum_{j=1}^{\infty}\tau_{j}\leqq\tau_{0}$ $(\leqq 1/8)$, we define $\{X_{\nu}^{j},\Xi_{\nu}^{j}\}_{j=1}^{\nu}(x^{0},\xi^{\nu+1})$ for any ν as the solution of the equation

$$\begin{cases}x^{j}=\nabla_{\xi}\phi_{j}(x^{j-1},\xi^{j}),\\ \xi^{j}=\nabla_{x}\phi_{j+1}(x^{j},\xi^{j+1}), \qquad j=1,2,\cdots,\nu.\end{cases}$$

Then, Theorem 1.4 of the present paper is the fundamental theorem concerning the property of the family of solutions $\{X_{\nu}^{j},\Xi_{\nu}^{j}\}_{j=1}^{\nu}$, $\nu=1,2,\cdots$, whose proof is given in [10]. The multi-product $\Phi_{\nu+1}(x,\xi)=\phi_{1}\#\phi_{2}\#\cdots\#\phi_{\nu+1}(x,\xi)$ of phase functions $\phi_{1},\phi_{2},\cdots,\phi_{\nu+1}$ is defined by

$$\Phi_{\nu+1}(x^{0},\xi^{\nu+1})$$
$$=\sum_{j=1}^{\nu}(\phi_{j}(X_{\nu}^{j-1},\Xi_{\nu}^{j})-X_{\nu}^{j}\cdot\Xi_{\nu}^{j})+\phi_{\nu+1}(X_{\nu}^{\nu},\Xi_{\nu}^{\nu+1}) \qquad (X_{\nu}^{0}=x^{0}).$$

As the subset of $\mathscr{P}^{0}(\tau)$ we define the class $\mathscr{P}_{\rho}(\tau)$ $(1/2<\rho\leqq 1)$ by the class of phase functions $\phi(x,\xi)$ $(\in\mathscr{P}^{0}(\tau))$ such that $J_{(\beta)}^{(\alpha)}(x,\xi)\in S_{\rho}^{1-|\alpha|}$ for $|\alpha+\beta|=2$, and often set $\mathscr{P}(\tau)=\mathscr{P}_{1}(\tau)$, where $S_{\rho}^{m}(-\infty<m<\infty)$ denotes the usual class $S_{\rho,1-\rho}^{m}$ of symbols of pseudo-differential operators $p(X,D_{x})$.

The purpose of the present paper is to represent the product $P_{1,\phi_{1}}P_{2,\phi_{2}}\cdots P_{\nu+1,\phi_{\nu+1}}$ of Fourier integral operators $P_{j,\phi_{j}}$ with phase functions $\phi_{j}\in\mathscr{P}_{\rho}(\tau_{j})$ and symbols $p_{j}(x,\xi)\in S_{\rho}^{m_{j}}$ by a Fourier integral operator $Q_{\nu+1,\Phi_{\nu+1}}$ with phase function $\Phi_{\nu+1}=\phi_{1}\#\phi_{2}\#\cdots\#\phi_{\nu+1}$ and symbol $q_{\nu+1}(x,\xi)$ of class $S_{\rho}^{\bar{m}_{\nu+1}}$ $(\bar{m}_{\nu+1}=m_{1}+m_{2}+\cdots+m_{\nu+1})$. As an application we represent the fundamental solution $E(t,s)$ for a hyperbolic system

$$L=D_t+\mathscr{D}(t)+B(t) \qquad \text{on } [0, T]$$

with characteristics of variable multiplicity by Fourier integral operators of multi-phase, where

$$\mathscr{D}(t)=\begin{bmatrix} \lambda_1(t, X, D_x) & & 0 \\ & \ddots & \\ 0 & & \lambda_l(t, X, D_x) \end{bmatrix}$$
$$(\lambda_j(t, x, \xi) \in \mathscr{B}^\infty([0, T]; S^1), j=1, \cdots, l)$$

and

$$B(t)=(b_{jk}(t, X, D_x))_{k\to 1,\cdots,l}^{j\downarrow 1,\cdots,l}$$
$$(b_{jk}(t, x, \xi) \in \mathscr{B}^\infty([0, T]; S^0), j, k=1, \cdots, l).$$

Here, $S^m=S_1^m$ and $p(t, x, \xi) \in \mathscr{B}^k([0, T]; S_\rho^m)$ means that $p(t, x, \xi)$ belongs to S_ρ^m for any fixed $t \in [0, T]$ and is k-times (S^m-valued) continuously differentiable with respect to t on $[0, T]$.

Using this fundamental solution $\boldsymbol{E}(t, s)$ we can get a generalization of the representation theorems obtained by Ludwig-Granoff [11] and Hata [4] for the solution U of the Cauchy problem

$$\begin{cases} LU=0 & \text{on } [0, T], \\ U|_{t=0}=U_0, \end{cases}$$

and get a theorem concerning the propagation of singularities of the solution U.

The fundamental solution $\boldsymbol{E}(t, s)$ is obtained by the Levi method, and in the series of the successive approximation for $\boldsymbol{E}(t, s)$ each term is represented by Fourier integral operators of multi-phase. We should note that in this process we only solve eiconal equations, and make use of the calculus of Fourier integral operators of multi-phase instead of solving transport equations.

§ 1. Main results on calculus of phase functions

In this section we review the main results obtained in [10] concerning multi-products of phase functions.

Definition 1.1. We say that a C^∞-function $p(x, \xi)$ in $R^{2n}=R_x^n \times R_\xi^n$ belongs to the class S_ρ^m $(=S_{\rho,1-\rho}^m)$ for $1/2<\rho\leqq 1$ and $-\infty<m<\infty$ (c.f., [5] or [7]), when we have for any multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ and $\beta=(\beta_1, \cdots, \beta_n)$

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|+(1-\rho)|\alpha+\beta|}.$$

Here

$$p_{(\beta)}^{(\alpha)}=D_x^\beta\partial_\xi^\alpha p,\quad D_{x_j}=-i\frac{\partial}{\partial x_j},\quad \partial_{\xi_j}=\frac{\partial}{\partial\xi_j}.$$

The class S_ρ^m makes a Fréchet space with semi-norms

$$|p|_l^{(m)}=\max_{|\alpha+\beta|\leqq l}\sup_{x,\xi}\{|p_{(\beta)}^{(\alpha)}(x,\xi)|/\langle\xi\rangle^{m-|\alpha|+(1-\rho)|\alpha+\beta|}\}\qquad (l=0,1,\cdots).$$

We set $S^{-\infty}=\bigcap_{-\infty<m<\infty}S_1^m$ and $S^m=S_1^m$.

Definition 1.2. i) We say that a real-valued C^2-function $\phi(x,\xi)$ in R^{2n} belongs to the class $\mathscr{P}^0(\tau)$ $(0\leqq\tau<1)$ of phase functions, if we have for $J(x,\xi)\equiv\phi(x,\xi)-x\cdot\xi$

$$\|J\|_0\equiv\sum_{|\alpha+\beta|\leqq 2}\sup_{x,\xi}\{|J_{(\beta)}^{(\alpha)}(x,\xi)|/\langle\xi\rangle^{1-|\alpha|}\}\leqq\tau, \tag{1.1}$$

where $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$.

ii) We say that a phase function $\phi(x,\xi)$ of class $\mathscr{P}^0(\tau)$ belongs to the class $\mathscr{P}_\rho(\tau)$ $(1/2<\rho\leqq 1)$, if $J(x,\xi)$ belongs to $S_\rho^1((2))$ in the sense:

$$J_{(\beta)}^{(\alpha)}(x,\xi)\in S_\rho^{1-|\alpha|}\qquad\text{for } |\alpha+\beta|=2 \tag{1.2}$$

(c.f., [8] and compare with [6]).

For $J(x,\xi)\in S_\rho^1((2))$ we introduce semi-norms $\|J\|_l$, $l=1,2,\cdots$, by

$$\|J\|_l=\|J\|_0+\sum_{3\leqq|\alpha+\beta|\leqq 2+l}\sup_{x,\xi}\{|J_{(\beta)}^{(\alpha)}(x,\xi)|/\langle\xi\rangle^{1-|\alpha|+(1-\rho)(|\alpha+\beta|-2)}\}. \tag{1.2$'$}$$

Then, $S_\rho^1((2))$ makes a Fréchet space with these semi-norms.

Remark. In [10] we denoted $\mathscr{P}^0(\tau)$ by $\mathscr{P}(\tau)$. In the present paper we often write $\mathscr{P}_1(\tau)=\mathscr{P}(\tau)$.

Definition 1.3. Let ϕ_j belong to $\mathscr{P}^0(\tau_j)$, $j=1,2,\cdots,\nu+1,\cdots$, with $\bar{\tau}_{\nu+1}\equiv\sum_{j=1}^{\nu+1}\tau_j\leqq\bar{\tau}_\infty\equiv\sum_{j=1}^{\infty}\tau_j\leqq\tau_0$ $(\leqq 1/8)$. We define the multi-product $\Phi_{\nu+1}(x,\xi)=\phi_1\#\phi_2\#\cdots\#\phi_{\nu+1}(x,\xi)$ of phase functions $\phi_1,\cdots,\phi_{\nu+1}$ by

$$\Phi_{\nu+1}(x^0,\xi^{\nu+1})=\sum_{j=1}^{\nu}(\phi_j(X_\nu^{j-1},\Xi_\nu^j)-X^j\cdot\Xi_\nu^j)+\phi_{\nu+1}(X_\nu^\nu,\xi^{\nu+1})\qquad (X_\nu^0=x^0), \tag{1.3}$$

where $\{X_\nu^j,\Xi_\nu^j\}_{j=1}^\nu(x^0,\xi^{\nu+1})$ is defined as the solution of the equation

$$\begin{cases}x^j=\nabla_\xi\phi_j(x^{j-1},\xi^j),\\ \xi^j=\nabla_x\phi_{j+1}(x^j,\xi^{j+1}),\qquad j=1,\cdots,\nu\end{cases} \tag{1.4}$$

$$(\nabla_\xi\phi={}^t(\partial_{\xi_1}\phi,\cdots,\partial_{\xi_n}\phi),\ \nabla_x\phi={}^t(\partial_{x_1}\phi,\cdots,\partial_{x_n}\phi)).$$

This definition is justified by the following

Theorem 1.4. *Let ϕ_j belong to $\mathscr{P}^0(\tau_j)$, $j=1, \cdots, \nu+1, \cdots$, with $\bar{\tau}_\infty \leqq \tau_0 \leqq 1/8$. Then, the equation* (1.4) *has the unique C^1-solution $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^\nu (x^0, \xi^{\nu+1})$ in R^{2n}, which satisfies*

$$
(1.5)\quad \begin{cases} \text{i)} \ \sum_{j=1}^{\nu} \sum_{|\alpha+\beta|\leqq 1} \{\langle \xi^{\nu+1}\rangle^{|\alpha|} |\partial_{\xi^{\nu+1}}^\alpha \partial_{x^0}^\beta (X_\nu^j - X_\nu^{j-1})| \\ \qquad + \langle \xi^{\nu+1}\rangle^{-1+|\alpha|} |\partial_{\xi^{\nu+1}}^\alpha \partial_{x^0} (\Xi_\nu^j - \Xi_\nu^{j+1})|\} \\ \qquad \leqq (48n+3)\bar{\tau}_{\nu+1} \qquad (X_\nu^0 = x^0, \Xi_\nu^{\nu+1} = \xi^{\nu+1}), \\ \text{ii)} \ \frac{1}{2}\langle \xi^{\nu+1}\rangle \leqq \langle \Xi_\nu^j \rangle \leqq 2\langle \xi^{\nu+1}\rangle \qquad (j=1, \cdots, \nu), \\ \text{iii)} \ |\nabla_{x^0} X_\nu^j| + \langle \xi^{\nu+1}\rangle^{-1} |\nabla_{x^0} \Xi_\nu^j| + \langle \xi^{\nu+1}\rangle |\nabla_{\xi^{\nu+1}} X_\nu^j| + |\nabla_{\xi^{\nu+1}} \Xi_\nu^j| \leqq 8\sqrt{n} \\ \qquad\qquad (j=1, \cdots, \nu), \end{cases}
$$

where $\nabla_x f = (\partial_{x_j} f_k \,{}^{k\downarrow 1, \cdots, n}_{j\to 1, \cdots, n})$, $\nabla_\xi f = (\partial_{\xi_j} f_k \,{}^{k\downarrow 1, \cdots, n}_{j\to 1, \cdots, n})$ for a function $f = {}^t(f_1, \cdots, f_n)$ (see Theorem 1.7 *of* [10]).

Theorem 1.4′. *In Theorem* 1.4, *furthermore, we assume that $\{J_j/\tau_j\}_{j=1}^\infty$ is bounded in $S_\rho^1((2))$. Then, for any α, β there exist constants $C_{\alpha,\beta}$ and $C'_{\alpha,\beta}$ independent of ν such that*

$$
(1.6)\quad \begin{cases} \text{i)} \ \sum_{j=1}^{\nu} \{\langle \xi^{\nu+1}\rangle^{|\alpha|-(1-\rho)(|\alpha+\beta|-1)} |\partial_{\xi^{\nu+1}}^\alpha \partial_{x^0}^\beta (X_\nu^j - X_\nu^{j-1})| \\ \qquad + \langle \xi^{\nu+1}\rangle^{-1+|\alpha|-(1-\rho)(|\alpha+\beta|-1)} |\partial_{\xi^{\nu+1}}^\alpha \partial_{x^0}^\beta (\Xi_\nu^j - \Xi_\nu^{j+1})| \\ \qquad \leqq C_{\alpha,\beta} \bar{\tau}_{\nu+1} \qquad (|\alpha+\beta| \geqq 2), \\ \text{ii)} \ \langle \xi^{\nu+1}\rangle^{|\alpha|-(1-\rho)(|\alpha+\beta|-1)} |\partial_{\xi^{\nu+1}}^\alpha \partial_{x^0}^\beta X_\nu^j| \\ \qquad + \langle \xi^{\nu+1}\rangle^{-1+|\alpha|-(1-\rho)(|\alpha+\beta|-1)} |\partial_{\xi^{\nu+1}}^\alpha \partial_{x^0}^\beta \Xi_\nu^j| \\ \qquad \leqq C'_{\alpha,\beta} \qquad (|\alpha+\beta| \geqq 2), j=1, \cdots, \nu. \end{cases}
$$

We give only the sketch of the proof of Theorem 1.4 below in four steps. The detailed proof is given in [10].

1) Set

$$
(1.7)\quad \begin{cases} y^j = x^j - x^{j-1}, \quad \eta^j = \xi^j - \xi^{j+1} \quad (j=1, \cdots, \nu), \\ (y, \eta) = (y^1, \cdots, y^\nu, \eta^1, \cdots, \eta^\nu), \\ \bar{y}^j = y^1 + \cdots + y^j, \quad \bar{\bar{\eta}}^j = \eta^j + \cdots + \eta^\nu \quad (j=1, \cdots, \nu). \end{cases}
$$

Then, the equation (1.4) is equivalent to

$$
(1.8)\quad \begin{cases} f_j(y, \eta; x^0, \xi^{\nu+1}) \equiv y^j - \nabla_\xi J_j(x^0 + \bar{y}^{j-1}, \bar{\bar{\eta}}^j + \xi^{\nu+1}) = 0, \\ g_j(y, \eta; x^0, \xi^{\nu+1}) \equiv \eta^j - \nabla_x J_{j+1}(x^0 + \bar{y}^j, \bar{\bar{\eta}}^{j+1} + \xi^{\nu+1}) = 0 \end{cases}
$$

$$
(\bar{y}^0 = 0, \bar{\bar{\eta}}^{\nu+1} = 0; j=1, \cdots, \nu).
$$

Assume that for a fixed $(x^0, \xi^{\nu+1}) \in R^{2n}$ we have a solution (y, η). Then, we have by (1.8),

$$|\eta^j| \leqq \tau_{j+1}\langle \bar{\bar{\eta}}^{j+1} + \xi^{\nu+1}\rangle, \qquad j=1, \cdots, \nu.$$

Hence, using,

$$\langle \bar{\bar{\eta}}^{j+1} + \xi^{\nu+1}\rangle \leqq |\bar{\bar{\eta}}^{j+1}| + \langle \xi^{\nu+1}\rangle \leqq \sum_{k=j+1}^{\nu} |\eta^k| + \langle \xi^{\nu+1}\rangle,$$

we have

$$\begin{aligned}\sum_{j=1}^{\nu} |\eta^j| &\leqq \sum_{j=1}^{\nu} \tau_{j+1}\Big(\sum_{k=j+1}^{\nu} |\eta^k| + \langle \xi^{\nu+1}\rangle\Big) \\ &= \sum_{k=2}^{\nu} \Big(\sum_{j=1}^{k-1} \tau_{j+1}\Big)|\eta^k| + \Big(\sum_{j=1}^{\nu} \tau_{j+1}\Big)\langle \xi^{\nu+1}\rangle \\ &\leqq \bar{\tau}_{\nu+1} \sum_{k=2}^{\nu} |\eta^k| + \bar{\tau}_{\nu+1}\langle \xi^{\nu+1}\rangle.\end{aligned}$$

So we get $\sum_{j=1}^{\nu} |\eta^j| \leqq \frac{1}{2}\langle \xi^{\nu+1}\rangle$. Consequently, if we set for a fixed $(x^0, \xi^{\nu+1}) \in R^{2n}$

$$\Sigma = \Big\{(y, \eta) \in R^{2\nu n};\ \sum_{j=1}^{\nu} |\eta^j| \leqq \tfrac{1}{2}\langle \xi^{\nu+1}\rangle\Big\}, \tag{1.9}$$

we see that the solution (y, η) can be found in Σ if exists.

II) For $(y, \eta) \in \Sigma$ we define the norm $\|(y, \eta)\|$ by

$$\|(y, \eta)\| = \sum_{j=1}^{\nu} \{|y^j| + \langle \xi^{\nu+1}\rangle^{-1} |\eta^j|\}. \tag{1.10}$$

Consider a mapping T: $\Sigma \ni (y, \eta) \to (w, \gamma) = T(y, \eta) \in \Sigma$ defined by

$$\begin{cases} w^j = \nabla_\xi J_j(x^0 + \bar{y}^{j-1}, \bar{\bar{\eta}}^j + \xi^{\nu+1}), \\ \gamma^j = \nabla_x J_{j+1}(x^0 + \bar{y}^j, \bar{\bar{\eta}}^{j+1} + \xi^{\nu+1}), \qquad j=1, \cdots, \nu. \end{cases} \tag{1.11}$$

Then, we see that the mapping T is into and contractive. So the unique solution $(y, \eta) = (y^1, \cdots, y^\nu, \eta^1, \cdots, \eta^\nu)$ of (1.8) is obtained as the fixed point of the mapping T. Then, setting $x^j = x^0 + \bar{y}^j$ and $\xi^j = \bar{\bar{\eta}}^j + \xi^{\nu+1}$ we get the unique solution of (1.4).

III) We set for any point $(z^0, \{Z_\nu^j, \Psi_\nu^j\}_{j=1}^{\nu}, \psi^{\nu+1})$ with the solution $\{Z_\nu^j, \Psi_\nu^j\}_{j=1}^{\nu}$ for $(z^0, \psi^{\nu+1})$

$$\begin{cases} a_j = \nabla_x \nabla_\xi J_j(Z_\nu^{j-1}, \Psi_\nu^j), \qquad b_j = \nabla_\xi \nabla_\xi J_j(Z_\nu^{j-1}, \Psi_\nu^j), \\ c_{j+1} = \nabla_x \nabla_x J_{j+1}(Z_\nu^j, \Psi_\nu^{j+1}), \qquad d_{j+1} = \nabla_\xi \nabla_x J_{j+1}(Z_\nu^j, \Psi_\nu^{j+1}) \end{cases} \tag{1.12}$$

$$(Z_\nu^0 = z^0,\ \Psi_\nu^{\nu+1} = \psi^{\nu+1}),$$

and set

$$(1.13)\qquad H=\frac{\partial(f_1,\cdots,f_\nu,g_1,\cdots,g_\nu)}{\partial(y^1,\cdots,y^\nu,\eta^1,\cdots,\eta^\nu)}\qquad \text{at } (z^0,\{Z_\nu^j-Z_\nu^{j-1},\Psi_\nu^j-\Psi_\nu^{j+1}\}_{j=1}^\nu,\psi^{\nu+1}).$$

Then, we have by easy calculation

$$H=\left[\begin{array}{ccccc|ccccc}
1 & 0 & \cdots\cdots & & 0 & -b_1 & -b_1 & \cdots\cdots & & -b_1\\
-a_2 & 1 & & & 0 & 0 & -b_2 & & & -b_2\\
\vdots & & \ddots & & 0 & \vdots & & \ddots & & \\
-a_\nu & -a_\nu & \cdots & -a_\nu & 1 & 0 & \cdots\cdots & & 0 & -b_\nu\\
\hline
-c_2 & 0 & \cdots\cdots & & 0 & 1 & -d_2 & \cdots\cdots & & -d_2\\
-c_3 & -c_3 & 0 & \cdots\cdots & 0 & 0 & 1 & -d_3 & \cdots\cdots & -d_3\\
\vdots & & \ddots & & 0 & \vdots & & \ddots & & \\
-c_{\nu+1} & -c_{\nu+1} & \cdots\cdots & & -c_{\nu+1} & 0 & \cdots\cdots & & 0 & 1
\end{array}\right].$$

Hence, we have $\det(H)\neq 0$, since $0<(1-\tau_0)^{2\nu n}\leqq \det(H)\leqq(1+\tau_0)^{2\nu n}$ (see Proposition 1.5 of [10]). So by the implicit function theorem we can prove that the function $\{X_\nu^j-X_\nu^{j-1},\Xi_\nu^j-\Xi_\nu^{j+1}\}_{j=1}^\nu$ and consequently $\{X_\nu^j,\Xi_\nu^j\}_{j=1}^\nu(x^0,\xi^{\nu+1})$ are of class C^1 in a neighborhood of $(z^0,\psi^{\nu+1})$.

IV) Finally we prove (1.5) for the solution of (1.4). Since we have

$$(1.14)\qquad \begin{cases} X_\nu^j-X_\nu^{j-1}=\nabla_\xi J_j(X_\nu^{j-1},\Xi_\nu^j),\\ \Xi_\nu^j-\Xi_\nu^{j+1}=\nabla_x J_{j+1}(X_\nu^j,\Xi_\nu^{j+1}), \qquad j=1,\cdots,\nu,\end{cases}$$

we get

$$(1.15)\qquad \begin{cases} |X_\nu^j-X_\nu^{j-1}|\leqq\tau_j, \qquad |\Xi_\nu^j-\Xi_\nu^{j+1}|\leqq 2\tau_{j+1}\langle\xi^{\nu+1}\rangle,\\ |\Xi_\nu^j-\Xi_\nu^{\nu+1}|\leqq\frac{1}{2}\langle\xi^{\nu+1}\rangle, \qquad j=1,\cdots,\nu\end{cases}$$

and get (1.5)-ii). Applying $\nabla_{x^0,\xi^{\nu+1}}=(\nabla_{x^0},\nabla_{\xi^{\nu+1}})$ to the both sides of (1.14), we have

$$(1.16)\qquad \begin{cases} \nabla_{x^0,\xi^{\nu+1}}(X_\nu^j-X_\nu^{j-1})=\nabla_x\nabla_\xi J_j(X_\nu^{j-1},\Xi_\nu^j)\nabla_{x^0,\xi^{\nu+1}}X_\nu^{j-1}\\ \qquad\qquad +\nabla_\xi\nabla_\xi J_j(X_\nu^{j-1},\Xi_\nu^j)\nabla_{x^0,\xi^{\nu+1}}\Xi_\nu^j,\\ \nabla_{x^0,\xi^{\nu+1}}(\Xi_\nu^j-\Xi_\nu^{j+1})=\nabla_x\nabla_x J_{j+1}(X_\nu^j,\Xi_\nu^{j+1})\nabla_{x^0,\xi^{\nu+1}}X_\nu^j\\ \qquad\qquad +\nabla_\xi\nabla_x J_{j+1}(X_\nu^j,\Xi_\nu^{j+1})\nabla_{x^0,\xi^{\nu+1}}\Xi_\nu^{j+1}.\end{cases}$$

Then, we have (1.5)-i) and iii) by (1.15), (1.16) and (1.5)-ii). Q.E.D.

Proof of Theorem 1.4′. Operating $\partial_{\xi^{\nu+1}}^\alpha D_{x^0}^\beta$ to the both sides of (1.16), we have (1.6) by induction on $|\alpha+\beta|$. Q.E.D.

Theorem 1.5. *Let* $\phi_j\in\mathscr{P}^0(\tau_j)$, $j=1,\cdots,\nu+1,\cdots$, *with* $\bar{\tau}_\infty\leqq\tau_0$ $(\leqq 1/8)$. *Then,* $\Phi_{\nu+1}=\phi_1\#\phi_2\#\cdots\#\phi_{\nu+1}$ *is well defined by* (1.3), *belongs to* $\mathscr{P}^0(c_0\bar{\tau}_{\nu+1})$ *with a constant* c_0

($\geqq 1$) *independent of* ν *and* $\bar{\tau}_{\nu+1}$, *and satisfies*

$$(1.17)\qquad \begin{cases} \nabla_{x^0}\Phi_{\nu+1}(x^0, \xi^{\nu+1}) = (\nabla_x\phi_1)(x^0, \Xi_\nu^1(x^0, \xi^{\nu+1})), \\ \nabla_{\xi^{\nu+1}}\Phi_{\nu+1}(x^0, \xi^{\nu+1}) = (\nabla_\xi\phi_{\nu+1})(X_\nu^\nu(x^0, \xi^{\nu+1}), \xi^{\nu+1}). \end{cases}$$

From this theorem and Theorem 1.4′ we get immediately

Theorem 1.5′. *In Theorem* 1.5 *we assume that* $\{J_j/\tau_j\}_{j=1}^\infty$ *is bounded in* $S_\rho^1((2))$. *Then,* $J_{\nu+1} = \Phi_{\nu+1} - x\cdot\xi$ *is bounded in* $S_\rho^1((2))$ *with respect to* ν.

Proof of Theorem 1.5. Operating ∇_{x^0} and $\nabla_{\xi^{\nu+1}}$ to the both sides of (1.3), we have (1.17) by using the fact that $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^\nu$ is the solution of (1.4). Then, together with Theorem 1.4, we can prove the theorem. Q.E.D.

Theorem 1.6. *Let* $\phi_j \in \mathscr{P}(\tau_j)$, $j=1, \cdots, \nu+1, \cdots$, *such that* $\bar{\tau}_\infty \equiv \sum_{j=1}^\infty \tau_j \leqq c_0^{-1}\tau_0$, *and let* $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^\nu(x^0, \xi^{\nu+1})$ *be the solution of the equation* (1.4). *Setting* $\Phi_{\nu+1} = \phi_1 \# \cdots \# \phi_{\nu+1}$, *define* $\{\tilde{X}_{\nu+1}^{\nu+1}, \tilde{\Xi}_{\nu+1}^{\nu+1}\}(x^0, \xi^{\nu+2})$ *as the solution of the equation*

$$(1.18)\qquad \begin{cases} x^{\nu+1} = \nabla_{\xi^{\nu+1}}\Phi_{\nu+1}(x^0, \xi^{\nu+1}), \\ \xi^{\nu+1} = \nabla_x\phi_{\nu+2}(x^{\nu+1}, \xi^{\nu+2}), \end{cases}$$

and set

$$(1.19)\qquad \{\tilde{X}_{\nu+1}^j, \tilde{\Xi}_{\nu+1}^j\}_{j=1}^\nu(x^0, \xi^{\nu+2}) = \{X_\nu^j, \Xi_\nu^j\}(x^0, \tilde{\Xi}_{\nu+1}^{\nu+1}(x^0, \xi^{\nu+2})).$$

Then, we have

$$(1.20)\qquad \{X_{\nu+1}^j, \Xi_{\nu+1}^j\}_{j=1}^{\nu+1}(x^0, \xi^{\nu+2}) = \{\tilde{X}_{\nu+1}^j, \tilde{\Xi}_{\nu+1}^j\}_{j=1}^{\nu+1}(x^0, \xi^{\nu+2}).$$

Furthermore, we have

$$(1.21)\qquad \begin{cases} \text{i)} & \Phi_{\nu+1}\#\phi_{\nu+2} = \Phi_{\nu+2}, \\ \text{ii)} & (\phi_1\#\phi_2)\#\phi_3 = \phi_1\#(\phi_2\#\phi_3) = \phi_1\#\phi_2\#\phi_3. \end{cases}$$

Proof. Noting (1.17), we see that $\{\tilde{X}_{\nu+1}^j, \tilde{\Xi}_{\nu+1}^j\}_{j=1}^{\nu+1}$ is the solution of (1.4) for ν replaced by $\nu+1$, and we get (1.20). Then, we have (1.21)-i) by (1.3) and (1.20). Similarly we can prove (1.21)-ii). Q.E.D.

Now consider a hyperbolic operator of the form

$$(1.22)\qquad L_0 = D_t + \lambda(t, x, D_x) \qquad \text{on } [0, T],$$

where $\lambda(t, x, \xi) \in \mathscr{B}^\infty([0, T]; S^1)$ is a real valued function on $[0, T]\times R^{2n}$. For L_0 we consider the eiconal equation

$$(1.23)\qquad \begin{cases} \partial_t\phi + \lambda(t, x, \nabla_x\phi) = 0 \qquad \text{on } [0, T], \\ \phi|_{t=s} = x\cdot\xi. \end{cases}$$

Then, we have

Proposition 1.7. *Let* $\phi=\phi(t, s)=\phi(t, s; x, \xi)$ $(0\leqq s\leqq t\leqq T)$ *be the solution of* (1.23). *Then, there exists a constant* $c>0$ *such that*

$$\text{(1.24)} \quad \begin{cases} \text{i)} & \phi(t, s)\in\mathscr{P}(c(t-s)), \\ \text{ii)} & \{J(t, s)/(t-s)\} \text{ is bounded in } S^1, \end{cases}$$

where $J(t, s)=\phi(t, s; x, \xi)-x\cdot\xi$.

Proof is easily done for a small $T>0$, if we follow the similar procedure to that in Section 3 of [8]. We fix such a T in what follows.

Proposition 1.8. *For the solution* $\phi(t, s)$ *of* (1.23) *we have*

$$\text{(1.25)} \quad \partial_s\phi(t, s; x, \xi)=\lambda(s, \nabla_\xi\phi(t, s; x, \xi), \xi).$$

Proof. Let $(y(\sigma), \eta(\sigma))$ be the bicharacteristic curve of $\lambda(t, X, D_x)$ which passes through the point (x_0, ξ) when $\sigma=s$, where $x_0=\nabla_\xi\phi(t, s; x, \xi)$. Then, $(\partial_s\phi)(\sigma, s; y(\sigma), \xi)$ is independent of σ. So we have

$$\begin{aligned}(\partial_s\phi)(t, s; x, \xi)&=(\partial_s\phi)(t, s; x_0, \xi)|_{t=s} \\ &=\partial_s\phi(s, s; x_0, \xi)-\partial_t\phi(t, s; x_0, \xi)|_{t=s}=-\partial_t\phi(t, s; x_0, \xi)|_{t=s} \\ &=\lambda(s, x_0, \nabla_x\phi(s, s; x_0, \xi))=\lambda(s, \nabla_\xi\phi(t, s; x, \xi), \xi).\end{aligned}$$

Hence we have (1.25). Q.E.D.

Now, take λ_j $(j=1, 2, \cdots, \nu+1, \cdots)$ as λ of (1.22) and let $\phi_j(t, s)$ be the solution of (1.23) corresponding to λ_j. We define $\Phi=\Phi_{1,2,\dots,\nu+1}(t_0, t_1, \cdots, t_{\nu+1})$ by

$$\Phi(t_0, t_1, \cdots, t_{\nu+1})=\phi_1(t_0, t_1)\#\phi_2(t_1, t_2)\#\cdots\#\phi_{\nu+1}(t_\nu, t_{\nu+1}),$$

and define $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^\nu$ $(t_0, t_1, \cdots, t_{\nu+1}; x^0, \xi^{\nu+1})$ as the solution of the equation

$$\text{(1.26)} \quad \begin{cases} x^j=\nabla_\xi\phi_j(t_{j-1}, t_j; x^{j-1}, \xi^j), \\ \xi^j=\nabla_x\phi_{j+1}(t_j, t_{j+1}; x^j, \xi^{j+1}), \qquad j=1, \cdots, \nu. \end{cases}$$

Then, we obtain the following

Theorem 1.9. $\Phi=\Phi(t_0, t_1, \cdots, t_{\nu+1})$ *satisfies*

1.°

$$\text{(1.27)} \quad \begin{cases} \text{i)} & \partial_{t_0}\Phi=-\lambda_1(t_0, x^0, \nabla_{x^0}\Phi), \\ \text{ii)} & \partial_{t_j}\Phi=\lambda_j(t_j, X_\nu^j, \Xi_\nu^j)-\lambda_{j+1}(t_j, X_\nu^j, \Xi_\nu^j) \qquad (j=1, \cdots, \nu), \\ \text{iii)} & \partial_{t_{\nu+1}}\Phi=\lambda_{\nu+1}(t_{\nu+1}, \nabla_{\xi^{\nu+1}}\Phi, \xi^{\nu+1}). \end{cases}$$

2.° *If $t_j = t_{j+1}$ for some j, we have*

$$(1.28)\qquad \begin{aligned}&\Phi_{1,2,\dots,\nu+1}(t_0, \cdots, t_j, t_{j+1}, t_{j+2}, \cdots, t_{\nu+1}; x^0, \xi^{\nu+1})\\ &\quad = \Phi_{1,2,\dots,j,j+2,\dots,\nu+1}(t_0, \cdots, t_j, t_{j+2}, \cdots, t_{\nu+1}; x^0, \xi^{\nu+1})\end{aligned}$$

(*the index* $j+1$ *disappears*).

3.° *If $\lambda_j(t, x, \xi) = \lambda_{j+1}(t, x, \xi)$ (therefore $\phi_j = \phi_{j+1}$) for some j, we have*

$$(1.29)\qquad \begin{aligned}&\Phi_{1,2,\dots,\nu+1}(t_0, \cdots, t_{\nu+1}; x^0, \xi^{\nu+1})\\ &\quad = \Phi_{1,2,\dots,j-1,j+1,\dots,\nu+1}(t_0, \cdots, t_{j-1}, t_{j+1}, \cdots, t_{\nu+1}; x^0, \xi^{\nu+1})\end{aligned}$$

(*the index* j *disappears*).

Proof. 1.° From the definition of Φ and (1.26) we have

$$\begin{aligned}\partial_{t_k}\Phi = \sum_{j=1}^{\nu} \{&\partial_{t_k}\phi_j(t_{j-1}, t_j; X_\nu^{j-1}, \Xi_\nu^j) + \nabla_x\phi_j(t_{j-1}, t_j; X_\nu^{j-1}, \Xi_\nu^j)\cdot\partial_{t_k}X_\nu^{j-1}\\ &+ \nabla_\xi\phi_j(t_{j-1}, t_j; X_\nu^{j-1}, \Xi_\nu^j)\cdot\partial_{t_k}\Xi_\nu^j - \partial_{t_k}X_\nu^j\cdot\Xi_\nu^j - X_\nu^j\cdot\partial_{t_k}\Xi_\nu^j\}\\ &+ \partial_{t_k}\phi_{\nu+1}(t_\nu, t_{\nu+1}; X_\nu^\nu, \xi^{\nu+1}) + \nabla_x\phi_{\nu+1}(t_\nu, t_{\nu+1}; X_\nu^\nu, \xi^{\nu+1})\cdot\partial_{t_k}X_\nu^\nu\\ =\ &\partial_{t_k}\phi_k(t_{k-1}, t_k; X_\nu^{k-1}, \Xi_\nu^k) + \partial_{t_k}\phi_{k+1}(t_k, t_{k+1}; X_\nu^k, \Xi_\nu^{k+1})\end{aligned}$$

$$(X_\nu^0 = x^0,\ \Xi_\nu^{\nu+1} = \xi^{\nu+1}).$$

Then, we have 1° by (1.17), (1.23), (1.25) and (1.26).

2.° If $t_j = t_{j+1}$, we have $\phi_{j+1}(t_j, t_{j+1}: X_\nu^j, \Xi_\nu^{j+1}) = X_\nu^j\cdot\Xi_\nu^{j+1}$, $X_\nu^j = X_\nu^{j+1}$ and $\Xi_\nu^j = \Xi_\nu^{j+1}$. Therefore, we get (1.28) from the definition of Φ.

3.° As $\partial_{t_j}\Phi = 0$ by ii) of 1° if $\lambda_j = \lambda_{j+1}$, we have

$$\begin{aligned}&\Phi_{1,2,\dots,\nu+1}(t_0, \cdots, t_j, t_{j+1}, \cdots, t_{\nu+1}; x^0, \xi^{\nu+1})\\ &\quad = \Phi_{1,2,\dots,\nu+1}(t_0, \cdots, t_{j-1}, t_{j-1}, t_{j+1}, \cdots, t_{\nu+1}; x^0, \xi^{\nu+1}).\end{aligned}$$

Therefore, we get (1.29) by 2.° Q.E.D.

Corollary. *For $\Phi_{1,2} = \phi_1\#\phi_2$ we have*

$$(1.30)\qquad \begin{aligned}\partial_{t_0}\Phi_{1,2} + \partial_{t_1}\Phi_{1,2} = &-\lambda_2(t_0, x^0, \nabla_{x^0}\Phi_{1,2})\\ &+ \{\lambda_2(t_0, x^0, \nabla_{x^0}\Phi_{1,2}) - \lambda_1(t_0, x^0, \nabla_{x^0}\Phi_{1,2})\}\\ &- \{\lambda_2(t_1, X_1^1, \Xi_1^1) - \lambda_1(t_1, X_1^1, \Xi_1^1)\}.\end{aligned}$$

Furthermore, if the Poisson bracket:

$$(1.31)\qquad \{\tau+\lambda_1, \tau+\lambda_2\} \equiv \partial_t\lambda_1 - \partial_t\lambda_2 + \nabla_x\lambda_1\cdot\nabla_\xi\lambda_2 - \nabla_x\lambda_2\cdot\nabla_\xi\lambda_1 = 0,$$

then we have

$$(1.32)\qquad \partial_{t_0}\Phi_{1,2} + \partial_{t_1}\Phi_{1,2} = -\lambda_2(t_0, x^0, \nabla_{x^0}\Phi_{1,2}),$$

where τ is the dual variable of t.

Proof. Let $(y(\sigma), \eta(\sigma))$ be the bicharacteristic curve of $\lambda_1(t, X, D_x)$ which passes through the point $(x^0, \nabla_x\phi_1(t_0, t_1; x^0, \Xi_1^1))$ when $\sigma=t_0$. Then, we have (1.32), since we have $y(t_1)=X_1^1$, $\eta(t_1)=\Xi_1^1$ and $\lambda_2(\sigma, y(\sigma), \eta(\sigma))-\lambda_1(\sigma, y(\sigma), \eta(\sigma))$ is independent of σ. Q.E.D.

Theorem 1.10. *Let $\lambda_j \in \mathscr{B}^\infty([0, T]; S^1)$ $(j=1, \cdots, \nu+1)$ and let $\phi_j(t, s)$ be the solutions of (1.23) corresponding to λ_j. Set*

$$(1.33)\qquad \begin{cases}\Phi_{1,2}(t, \theta, s)=\phi_1(t, \theta)\#\phi_2(\theta, s),\\ \tilde{\Phi}_{1,2}(t, \theta, s)=\phi_2(t, \theta)\#\phi_1(\theta, s),\end{cases}$$

and set for $1\leqq k\leqq\nu$

$$(1.34)\qquad \begin{cases}\Phi_{1,\dots,\nu+1}(t_0, \cdots, t_{\nu+1})\\ \qquad =\phi_1(t_0, t_1)\#\cdots\#\phi_k(t_{k-1}, t_k)\#\phi_{k+1}(t_k, t_{k+1})\#\cdots\#\phi_{\nu+1}(t_\nu, t_{\nu+1}),\\ \tilde{\Phi}_{1,\dots,\nu+1}(t_0, \cdots, t_{\nu+1})\\ \qquad =\phi_1(t_0, t_1)\#\cdots\#\phi_{k+1}(t_{k-1}, t_k)\#\phi_{k+1}(t_k, t_{k+1})\#\cdots\#\phi_{\nu+1}(t_\nu, t_{\nu+1}).\end{cases}$$

Then, we have

$$(1.35)\qquad \tilde{\Phi}_{1,2}(t, \theta, s)=\Phi_{1,2}(t, t-\theta+s, s)$$

if the Poisson bracket $\{\tau+\lambda_1, \tau+\lambda_2\}=0$. Furthermore, we have

$$(1.36)\qquad \begin{aligned}&\tilde{\Phi}_{1,\dots,\nu+1}(t_0, \cdots, t_{k-1}, t_k, t_{k+1}, \cdots, t_{\nu+1})\\ &\quad=\Phi_{1,\dots,\nu+1}(t_0, \cdots, t_{k-1}, t_{k-1}-t_k+t_{k+1}, t_{k+1}, \cdots, t_{\nu+1}).\end{aligned}$$

if $\{\tau+\lambda_k, \tau+\lambda_{k+1}\}=0$.

Proof. By 1° of Theorem 1.9 we have

$$\partial_t\tilde{\Phi}_{1,2}(t, \theta, s)=-\lambda_2(t, x^0, \nabla_x\tilde{\Phi}_{1,2}),$$

and by (1.32) we have

$$\begin{aligned}&\partial_t\Phi_{1,2}(t, t-\theta+s, s)\\ &\quad=(\partial_t\Phi_{1,2})(t, t-\theta+s, s)+(\partial_\theta\Phi_{1,2})(t, t-\theta+s, s)\\ &\quad=-\lambda_2(t, x^0, \nabla_x\Phi_{1,2}(t, t-\theta+s, s)).\end{aligned}$$

Hence, $\tilde{\Phi}_{1,2}(t, \theta, s)$ and $\Phi_{1,2}(t, t-\theta+s, s)$ satisfy the same differential equation. On the other hand

$$\tilde{\Phi}_{1,2}(\theta, \theta, s)=\phi_1(\theta, s)=\Phi_{1,2}(\theta, s, s).$$

So we get (1.35).

Using (1.35) we can write

$$\begin{aligned}
&\tilde{\Phi}_{1,\cdots,\nu+1}(t_0, \cdots, t_{k-1}, t_k, t_{k+1}, \cdots, t_{\nu+1}) \\
&\quad = \phi_1(t_0, t_1)\# \cdots \#(\phi_{k+1}(t_{k-1}, t_k)\#\phi_k(t_k, t_{k+1}))\# \cdots \#\phi_{\nu+1}(t_\nu, t_{\nu+1}) \\
&\quad = \phi_1(t_0, t_1)\# \cdots \#(\phi_k(t_{k-1}, t_{k-1}-t_k+t_{k+1})\#\phi_{k+1}(t_{k-1}-t_k+t_{k+1}, t_{k+1}))\# \\
&\qquad\qquad \cdots \#\phi_{\nu+1}(t_\nu, t_{\nu+1}). \\
&\quad = \Phi_{1,\cdots,\nu+1}(t_0, \cdots, t_{k-1}, t_{k-1}-t_k+t_{k+1}, t_{k+1}, \cdots, t_{\nu+1}).
\end{aligned}$$

Hence, we get (1.36). Q.E.D.

Finally, we note that, for $\phi_j \in \mathscr{P}_1(\tau_j)$ ($\bar{\tau}_\infty \equiv \sum_{j=1}^\infty \tau_j \leqq \tau_0$) such that $\{J_j/\tau_j\}_{j=1}^\infty$ is bounded in S^1, we can find $\tilde{\phi}_j(t, x, \xi) \in \mathscr{P}_1(ct)$ for a constant $c>0$ and $\tilde{\lambda}_j(t, x, \xi) \in \mathscr{B}^1([0, T]; S^1)$, $j=1, 2, \cdots$, such that "$\partial_t\tilde{\phi}_j + \tilde{\lambda}_j(t, x, \nabla_x\tilde{\phi}_j)=0$ on $[0, T]$, $\tilde{\phi}_j|_{t=0}=x\cdot\xi$, $\tilde{\phi}_j(\tau_j, x, \xi)=\phi_j(x, \xi)$" and $\{\tilde{J}_j/\tau_j\}_{j=1}^\infty$, $\{\tilde{\lambda}_j\}_{j=1}^\infty$ are bounded in S^1, where $\tilde{J}_j=\tilde{\phi}_j - x\cdot\xi$. This fact can be shown by setting $\tilde{\phi}_j(t, x, \xi)=x\cdot\xi+a(t/\tau_j)J_j(x, \xi)$ and using the discussion in [1], where $a(t)$ is a C^1-function in R^1 such that $a(t)=0$ $(0\leqq t\leqq 1/2)$, $=1$ $(t\geqq 1)$.

§ 2. Fourier integral operators of multi-phase

Definition 2.1 (c. f., [7]). Let $a(\eta, y)$ be a C^∞-function in $R^{2n}=R^n_\eta \times R^n_y$ satisfying for any multi-index α, β

$$|\partial_\eta^\alpha \partial_y^\beta a(\eta, y)| \leqq C_{\alpha,\beta}\langle\eta\rangle^{m+\delta|\beta|}\langle y\rangle^\tau \qquad (-\infty<m<\infty,\ 0\leqq\tau,\ 0\leqq\delta<1).$$

We define the oscillatory integral for $a(\eta, y)$ by

$$\begin{aligned}
O_s-&\iint e^{-iy\cdot\eta}a(\eta, y)dy\,d\!\!{}^-\!\eta \\
&\equiv \lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi(\varepsilon\eta, \varepsilon y)a(\eta, y)dy\,d\!\!{}^-\!\eta \\
&= \iint e^{-iy\cdot\eta}\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\{\langle\eta\rangle^{2l}\langle D_y\rangle^{2l}a(\eta, y)\}dy\,d\!\!{}^-\!\eta,
\end{aligned}$$

where $d\!\!{}^-\!\eta=(2\pi)^{-n}d\eta$, $\chi(\eta, y) \in \mathscr{S}$ in R^{2n} such that $\chi(0, 0)=1$, and l and l' are integers satisfying $-2l(1-\delta)+m<-n$ and $-2l'+\tau<-n$.

Definition 2.2. For $\phi(x, \xi) \in \mathscr{P}_\rho(\tau)$ and $p(x, \xi) \in S^m_\rho$ we define the Fourier integral operator $P_\phi=p_\phi(X, D_x)$ with symbol $\sigma(P_\phi)=p(x, \xi)$ and phase function $\phi(x, \xi)$ by

$$P_\phi u(x)=O_s-\iint e^{i(\phi(x,\xi)-y\cdot\xi)}p(x, \xi)u(y)dy\,d\!\!{}^-\!\xi \qquad \text{for } u \in \mathscr{S}. \tag{2.1}$$

Theorem 2.3 (c.f., [2], [6]). *Let* $\phi_j \in \mathscr{P}_\rho(\tau_j)$ $(\tau_1+\tau_2 \leqq 1/8)$, $p_j \in S_\rho^{m_j}$ $(j=1, 2)$ *and* $\{\dot{x}^1, \dot{\xi}^1\}(x^0, \xi^2)$ *be the solution of*

$$\text{(2.2)} \qquad \begin{cases} x^1 = \nabla_\xi \phi_1(x^0, \xi^1), \\ \xi^1 = \nabla_x \phi_2(x^1, \xi^2). \end{cases}$$

Define $\phi_{1,2}(x, \xi)$ *by*

$$\text{(2.3)} \qquad \phi_{1,2}(x^0, \xi^2) \equiv \phi_1 \# \phi_2(x^0, \xi^2) = \phi_1(x^0, \dot{\xi}^1) - \dot{x}^1 \cdot \dot{\xi}^1 + \phi_2(\dot{x}^1, \xi^2)$$

and set

$$\text{(2.4)} \qquad q(x^0, \xi^2) = O_s - \iint e^{i\psi} p_1(x^0, \xi^1) p_2(x^1, \xi^2) dx^1 d\xi^1,$$

where

$$\text{(2.5)} \qquad \begin{aligned} \psi &= \psi(x^0, x^1; \xi^1, \xi^2) \\ &= \phi_1(x^0, \xi^1) - x^1 \cdot \xi^1 + \phi_2(x^1, \xi^2) - \phi_{1,2}(x^0, \xi^2). \end{aligned}$$

Then, we have $q \in S_\rho^{m_1+m_2}$ *and* $P_{1,\phi_1} P_{2,\phi_2} = Q_{\phi_{1,2}}$. *Furthermore,* $q(x^0, \xi^2)$ *has an asymptotic expansion* $\sum_{j=0}^\infty q_j(x^0, \xi^2)$ *with the following properties:*

i) $q_j(x^0, \xi^2)$ *has the form*

$$\text{(2.6)} \qquad \begin{aligned} &q_j(x^0, \xi^2) \\ &= \sum_{|\alpha+\beta| \leqq 2j} \gamma_{j,\alpha,\beta}(x^0, \xi^2) p_1^{(\alpha)}(x^0, \dot{\xi}^1) p_{2(\beta)}(\dot{x}^1, \xi^2), \end{aligned}$$

where

$$\text{(2.7)} \qquad \gamma_{j,\alpha,\beta}(x^0, \xi^2) = \gamma_{j,\alpha,\beta}(x^0, \xi^2; \phi_1, \phi_2) \in S_\rho^{-(2\rho-1)j+|\alpha|-(1-\rho)|\alpha+\beta|},$$

and for any integer l *there exists a constant* C *depending on* j, l *and* $|J_1|_{l'}^{(1)}$ *and* $|J_2|_{l'}^{(1)}$ (*for some* l') *such that*

$$\text{(2.8)} \qquad |\gamma_{j,\alpha,\beta}|_l^{(-(2\rho-1)j+|\alpha|-(1-\rho)|\alpha+\beta|)} \leqq C.$$

ii) *For any* N *there exists* $r_N(x, \xi) \in S_\rho^{m_1+m_2-(2\rho-1)N}$ *such that*

$$\text{(2.9)} \qquad P_{1,\phi_1} P_{2,\phi_2} - \sum_{j=0}^{N-1} Q_{j,\phi_{1,2}} = R_{N,\phi_{1,2}},$$

and $r_N(x, \xi)$ *satisfies the following: for any integer* l *there exist an integer* l' *depending on* $|m_1|, |m_2|, N, l$ *and a constant* C *depending on* $|m_1|, |m_2|, N, l$ *and* $|J_1|_{l''}^{(1)}$ *and* $|J|_{l''}^{(1)}$ (*for some* l'') *such that*

$$\text{(2.10)} \qquad |r_N|_l^{(m_1+m_2-(2\rho-1)N)} \leqq C |p_1|_{l'}^{(m_1)} |p_2|_{l'}^{(m_2)}.$$

Proof. I) We write

$$\phi_1(x^0, \xi^1) - x^1 \cdot \xi^1 + \phi_2(x^1, \xi^2) = \phi_{1,2}(x^0, \xi^2) + \psi \tag{2.11}$$

with ψ of (2.5). Then, it is easy to see that $P_{1,\phi_1} P_{2,\phi_2} = Q_{\phi_{1,2}}$ for $q(x, \xi)$ defined by (2.4). Since ϕ_2 belongs to $\mathscr{P}_\rho(\tau_2)$ and $0 \leqq \tau_2 \leqq 1/8$, we have

$$|\nabla_{x^1} \psi| \geqq |\xi^1 - \xi^2| - \tfrac{1}{8}\langle \xi^2 \rangle. \tag{2.12}$$

Let $\chi(\xi)$ be a C^∞-function in R^n such that

$$\chi(\xi) = 1 \quad (|\xi| \leqq \tfrac{1}{4}), \quad = 0 \quad (|\xi| \geqq \tfrac{1}{2}),$$

and set $\chi_\infty = 1 - \chi((\xi^1 - \xi^2)/\langle \xi^2 \rangle)$. Then, by (2.12) we have

$$|\nabla_{x^1} \psi| \geqq \tfrac{1}{2} |\xi^1 - \xi^2| \geqq \tfrac{1}{8} \langle \xi^2 \rangle \qquad \text{on supp } \chi_\infty. \tag{2.13}$$

Setting $T_1 = (1 + |\nabla_{\xi^1} \psi|^2)^{-1} (1 - i \nabla_{\xi^1} \psi \cdot \nabla_{\xi^1})$ and $T_2 = -i |\nabla_{x^1} \psi|^{-2} \cdot \nabla_{x^1} \psi \cdot \nabla_{x^1}$, we write for a fixed $l_0 > n$ and large l

$$\begin{aligned} q_\infty(x^0, \xi^2) &\equiv O_s - \iint e^{i\psi} \chi_\infty p_1(x^0, \xi^1) p_2(x^1, \xi^2) dx^1 d\xi^1 \\ &= \iint e^{i\psi} (T_2^t)^l (T_1^t)^{l_0} \{\chi_\infty p_1(x^0, \xi^1) p_2(x^1, \xi^2)\} dx^1 d\xi^1, \end{aligned}$$

where T_j^t $(j = 1, 2)$ are the transport operators of T_j. Then, we see that

$$q_\infty(x^0, \xi^2) \in S^{-\infty}. \tag{2.14}$$

II) For $\chi_0 = \chi((\xi^1 - \xi^2)/\langle \xi^2 \rangle)$ we consider

$$q_0(x^0, \xi^2) = O_s - \iint e^{i\tilde{\psi}} \chi_0 p_1(x^0, \xi^1) p_2(x^1, \xi^2) dx^1 d\xi^1. \tag{2.15}$$

Then, by the change of variables: $x^1 = \dot{x}^1 + y$, $\xi^1 = \dot{\xi}^1 + \eta$, we can write

$$q_0(x^0, \xi^2) = O_s - \iint e^{-i\psi} \tilde{\chi}_0 p_1(x^0, \dot{\xi}^1 + \eta) p_2(\dot{x}^1 + y, \xi^2) dy d\eta, \tag{2.16}$$

where

$$\tilde{\chi}_0 = \tilde{\chi}_0(\eta; x^0, \xi^2) = \chi((\dot{\xi}^1 + \eta - \xi^2)/\langle \xi^2 \rangle) \tag{2.17}$$

and

$$\begin{aligned} \tilde{\psi} &= \tilde{\psi}(y, \eta; x^0, \xi^2) = -\psi(x^0, \dot{x}^1 + y; \dot{\xi}^1 + \eta, \xi^2) \\ &= y \cdot \eta - (\phi_1(x^0, \dot{\xi}^1 + \eta) - \dot{x}^1 \cdot \eta - \phi_1(x^0, \dot{\xi}^1)) - (\phi_2(\dot{x}^1 + y, \xi^2) \\ &\quad - y \cdot \dot{\xi}^1 - \phi_2(\dot{x}^1, \xi^2)). \end{aligned} \tag{2.18}$$

Since $(\dot{x}^1, \dot{\xi}^1)$ is the solution of (2.2), we have

$$|\dot{x}^1-x^0|\leqq\tau_1\leqq\tfrac{1}{8},\qquad |\dot{\xi}^1-\xi^2|\leqq\tau_2\langle\xi^2\rangle\leqq\tfrac{1}{8}\langle\xi^2\rangle. \tag{2.19}$$

By the definition of $\tilde{\chi}_0$ we have

$$|\dot{\xi}^1-\xi^2+\eta|\leqq\tfrac{1}{2}\langle\xi^2\rangle\qquad \text{on supp } \tilde{\chi}_0. \tag{2.20}$$

Hence, from (2.19) and (2.20) we have

$$|\dot{\xi}^1+\theta\eta-\xi^2|\leqq\theta|\dot{\xi}^1+\eta-\xi^2|+(1-\theta)|\dot{\xi}^1-\xi^2|\leqq\tfrac{1}{2}\langle\xi^2\rangle\qquad \text{on supp } \tilde{\chi}_0. \tag{2.21}$$

Consequently, using $\langle\xi+\xi'\rangle\lessgtr\langle\xi\rangle\pm|\xi'|$ $(\xi, \xi'\in R^n)$, we have

$$\tfrac{1}{2}\langle\xi^2\rangle\leqq\langle\dot{\xi}^1+\theta\eta\rangle\leqq2\langle\xi^2\rangle\quad(0\leqq\theta\leqq1)\quad \text{on supp } \tilde{\chi}_0. \tag{2.22}$$

Now, using (2.2), we write

$$\begin{cases}\nabla_\eta\tilde{\psi}=y-\displaystyle\int_0^1\nabla_\xi\nabla_\xi J_1(x^0, \dot{\xi}^1+\theta\eta)d\theta\cdot\eta,\\[2ex] \nabla_y\tilde{\psi}=\eta-\displaystyle\int_0^1\nabla_x\nabla_x J_2(\dot{x}^1+y, \xi^2)d\theta\cdot y.\end{cases}$$

Then, by (2.22) we have

$$\begin{cases}|\nabla_\eta\tilde{\psi}|\geqq|y|-4\tau_1|\eta|\langle\xi^2\rangle^{-1}\geqq|y|-\tfrac{1}{2}\langle\xi^2\rangle^{-1}|\eta|,\\ |\nabla_y\tilde{\psi}|\geqq|\eta|-2\tau_2|y|\langle\xi^2\rangle\geqq|\eta|-\tfrac{1}{4}\langle\xi^2\rangle|y|\qquad \text{on supp } \tilde{\chi}_0.\end{cases}$$

Hence, we have

$$\begin{aligned}\langle\xi^2\rangle^2|\nabla_\eta\tilde{\psi}|^2+|\nabla_y\tilde{\psi}|^2&\geqq\tfrac{1}{2}(\langle\xi^2\rangle|\nabla_\eta\tilde{\psi}|+|\nabla_y\tilde{\psi}|)^2\\ &\geqq\tfrac{1}{2}\cdot\tfrac{1}{4}(\langle\xi^2\rangle|y|+|\eta|)^2\geqq\tfrac{1}{8}(\langle\xi^2\rangle^2|y|^2+|\eta|^2)\qquad \text{on supp } \tilde{\chi}_0.\end{aligned} \tag{2.23}$$

Set for any fixed δ $(0<\delta<1)$

$$\chi'_\infty=1-\chi((\langle\xi^2\rangle^2|y|^2+|\eta|^2)/(\delta\langle\xi^2\rangle^\rho)^2).$$

Then, by (2.23) and the definition of χ we have

$$(\langle\xi^2\rangle|\nabla_\eta\tilde{\psi}|)^2+|\nabla_y\tilde{\psi}|^2\geqq\frac{\delta^2}{32}\langle\xi^2\rangle^{2\rho}\qquad \text{on supp } \chi'_\infty\tilde{\chi}_0. \tag{2.24}$$

Hence, setting

$$T_3=i(\langle\xi^2\rangle^2|\nabla_\eta\tilde{\psi}|^2+|\nabla_y\tilde{\psi}|^2)^{-1}(\langle\xi^2\rangle^2\nabla_\eta\tilde{\psi}\cdot\nabla_\eta+\nabla_y\tilde{\psi}\cdot\nabla_y),$$

we write for large l

$$\begin{aligned} &q_{0,\infty}(x^0, \xi^2) \\ &\equiv O_s - \iint e^{-i\tilde{\Psi}} \chi'_\infty \tilde{\chi}_0 p_1(x^0, \dot{\xi}^1+\eta) p_2(\dot{x}^1+y, \xi^2) dy d\eta \\ &= \iint e^{-i\tilde{\Psi}} (T_3^t)^l \{\chi'_\infty \tilde{\chi}_0 p_1(x^0, \dot{\xi}^1+\eta) p_2(\dot{x}^1+y, \xi^2)\} dy d\eta. \end{aligned}$$

Then, using (2.20)–(2.22), (2.24) and $2\rho>1$, we see that

$$q_{0,\infty}(x^0, \xi^2) \in S^{-\infty} \qquad \text{for any fixed } 0<\delta<1. \tag{2.25}$$

III) Setting

$$\tilde{\chi}'_0 = \tilde{\chi}'_0(y, \eta; \xi^2) = \chi((\langle\xi^2\rangle^2 |y|^2 + |\eta|^2)/(\delta\langle\xi^2\rangle^\rho)^2), \tag{2.26}$$

we consider

$$\begin{aligned} &q'_0(x^0, \xi^2) \\ &= \iint e^{-i\tilde{\Psi}} \tilde{\chi}_0 \tilde{\chi}'_0 p_1(x^0, \dot{\xi}^1+\eta) p_2(\dot{x}^1+y, \xi^2) dy d\eta. \end{aligned} \tag{2.27}$$

We write by using (2.2)

$$\tilde{\Psi} = y\cdot\eta - A\eta\cdot\eta - By\cdot y, \tag{2.28}$$

where

$$\begin{cases} A = A(\eta; x^0, \xi^2) = \displaystyle\int_0^1 (1-\theta) \nabla_\xi \nabla_\xi J_1(x^0, \dot{\xi}^1+\theta\eta) d\theta, \\ B = B(y; x^0, \xi^2) = \displaystyle\int_0^1 (1-\theta) \nabla_x \nabla_x J_2(\dot{x}^1+\theta y, \xi^2) d\theta. \end{cases} \tag{2.29}$$

We define symmetric matrices

$$F = F(y, \eta; x^0, \xi^2) \quad \text{and} \quad G = G(y, \eta; x^0, \xi^2)$$

by

$$\begin{cases} F = A + FBF, \\ G = B + GAG, \end{cases} \tag{2.30}$$

and define the change of variables: $(y, \eta) \to (z, \gamma)$ by

$$\begin{cases} y = z + F\gamma, \\ \eta = Gz + \gamma. \end{cases} \tag{2.31}$$

Then, noting that A and B are symmetric, we see that we can write

$$\tilde{\psi}=(I+FG-2AG-2FB)z\cdot\gamma, \tag{2.32}$$

where I is the unit matrix.

Now, we investigate the properties of F and G.

Set for a fixed (x^0, ξ^2)

$$\Sigma=\Big\{p=(p_{jk}(y,\eta;x^0,\xi^2));\ n\times n \text{ matrices of elements } p_{jk} \text{ such that} \\ \|p(y,\eta;x^0,\xi^2)\|\equiv\sum_{j,k=1}^{n}|p_{jk}(y,\eta;x^0,\xi^2)|\leqq\frac{1}{5}\Big\} \tag{2.33}$$

for $(y,\eta)\in\Lambda$ defined by

$$\Lambda=\Lambda(\xi^2,\delta)=\Big\{(y,\eta);\ (\langle\xi^2\rangle^2|y|^2+|\eta|^2)\leqq\frac{\delta^2}{2}\langle\xi^2\rangle^{2\rho}\Big\},$$

and consider the mapping T: $\Sigma\to\Sigma$ defined by

$$Tp=A\langle\xi^2\rangle+pB\langle\xi^2\rangle^{-1}p. \tag{2.34}$$

For small $0<\delta<1$ we have

$$\|A\|\leqq\frac{5}{4}\tau_1\langle\xi^2\rangle^{-1}\leqq\frac{5}{32}\langle\xi^2\rangle^{-1},\qquad \|B\|\leqq 2\tau_2\langle\xi^2\rangle\leqq\frac{1}{4}\langle\xi^2\rangle. \tag{2.35}$$

Hence, we have

$$\|Tp\|\leqq\|A\langle\xi^2\rangle\|+\|p\|\,\|B\langle\xi^2\rangle^{-1}\|\,\|p\|\leqq\tfrac{1}{3}\qquad (p\in\Sigma), \tag{2.36}$$

and

$$\|Tp-Tp'\|\leqq\|(p-p')B\langle\xi^2\rangle^{-1}p\|+\|p'B\langle\xi^2\rangle^{-1}(p-p')\| \\ \leqq\frac{1}{32}\|p-p'\|+\frac{1}{32}\|p-p'\|\leqq\frac{1}{16}\|p-p'\|\qquad (p,p'\in\Sigma). \tag{2.37}$$

This means that the mapping T: $\Sigma\to\Sigma$ is into and contractive, so that we have a unique fixed point $p_0=p_0(y,\eta;x^0,\xi^2)$.

Let D_μ denote one of the differential operators $\partial_y, \partial_\eta, \partial_{x^0}, \partial_{\xi^2}$. Then, we have formally

$$D_\mu^k p_0=D_\mu^k(A\langle\xi^2\rangle)+(D_\mu^k p_0)(B\langle\xi^2\rangle^{-1}p_0)+(p_0B\langle\xi^2\rangle^{-1})D_\mu^k p_0 \\ +\sum_{\substack{k_1+k_2+k_3=k\\ k_1\neq k,\ k_3\neq k}} D_\mu^{k_1}p_0\cdot D_\mu^{k_2}(B\langle\xi^2\rangle^{-1})\cdot D_\mu^{k_3}p_0.$$

Using this we have by induction

$$\|\partial_y^\beta\partial_\eta^\alpha\partial_{x^0}^{\beta'}\partial_{\xi^2}^{\alpha'}p_0\| \\ \leqq C_{\alpha,\beta',\alpha,\beta'}\langle\xi^2\rangle^{-|\alpha+\alpha'|+(1-\rho)|\alpha+\alpha'+\beta+\beta'|}\qquad ((y,\eta)\in\Lambda). \tag{2.38}$$

Then, setting $F=p_0\langle\xi^2\rangle^{-1}$ and $G=p_0\langle\xi^2\rangle$, we see that the equation (2.30) has the unique solution (F, G) such that

$$\|F\|\leqq\frac{1}{5}\langle\xi^2\rangle^{-1},\quad \|G\|\leqq\frac{1}{5}\langle\xi^2\rangle\quad ((y,\eta)\in\Lambda), \tag{2.39}$$

and

$$\begin{cases} \text{i)}\ \ \|\partial_y^{\beta}\partial_\eta^{\alpha}\partial_{x^0}^{\beta'}\partial_{\xi^2}^{\alpha'}F\| \\ \qquad\leqq C'_{\alpha,\alpha',\beta,\beta'}\langle\xi^2\rangle^{-1-|\alpha+\alpha'|+(1-\rho)|\alpha+\alpha'+\beta+\beta'|}, \\ \text{ii)}\ \ \|\partial_y^{\beta}\partial_\eta^{\alpha}\partial_{x^0}^{\beta'}\partial_{\xi^2}^{\alpha'}G\| \\ \qquad\leqq C'_{\alpha,\alpha',\beta,\beta'}\langle\xi^2\rangle^{1-|\alpha+\alpha'|+(1-\rho)|\alpha+\alpha'+\beta+\beta'|}\qquad ((y,\eta)\in\Lambda). \end{cases} \tag{2.40}$$

From (2.31) we have

$$y-F\eta=z-FGz,\qquad \eta-Gy=\gamma-GF\gamma.$$

So we have

$$\begin{cases} z=(I-FG)^{-1}(y-F\eta)=(y-F\eta)+\sum_{k=1}^{\infty}(FG)^k(y-F\eta), \\ \gamma=(I-GF)^{-1}(\eta-Gy)=(\eta-Gy)+\sum_{k=1}^{\infty}(GF)^k(\eta-Gy). \end{cases} \tag{2.41}$$

From (2.39)–(2.41) we have

$$\begin{cases} \|\nabla_y z-I\|\leqq\frac{5}{24}+\delta C,\qquad \|\nabla_\eta z\|\leqq\left(\frac{5}{24}+\delta C\right)\langle\xi^2\rangle^{-1}, \\ \|\nabla_y\gamma\|\leqq\left(\frac{5}{24}+\delta C\right)\langle\xi^2\rangle,\qquad \|\nabla_\eta\gamma-I\|\leqq\frac{5}{24}+\delta C \end{cases} \tag{2.42}$$

and

$$\begin{cases} \|\partial_y^{\beta}\partial_\eta^{\alpha}\partial_{x^0}^{\beta'}\partial_{\xi^2}^{\alpha'}z\| \\ \qquad\leqq C''_{\alpha,\alpha',\beta,\beta'}\langle\xi^2\rangle^{-|\alpha+\alpha'|+(1-\rho)(|\alpha+\alpha'+\beta+\beta'|-1)}, \\ \|\partial_y^{\beta}\partial_\eta^{\alpha}\partial_{x^0}^{\beta'}\partial_{\xi^2}^{\alpha'}\gamma\| \\ \qquad\leqq C''_{\alpha,\alpha',\beta,\beta'}\langle\xi^2\rangle^{1-|\alpha+\alpha'|+(1-\rho)(|\alpha+\alpha'+\beta+\beta'|-1)} \\ \qquad\qquad\qquad (|\alpha+\alpha'+\beta+\beta'|\geqq 1,\ (y,\eta)\in\Lambda), \end{cases} \tag{2.43}$$

where constants $C, C''_{\alpha,\alpha',\beta,\beta'}$ are independent of $0<\delta<1$.

Now, set

$$\begin{cases} Z=Z(y,\eta;x^0,\xi^2)=(I+FG-2AG-2FB)z, \\ \Gamma=\Gamma(y,\eta;x^0,\xi^2)=\gamma\qquad ((y,\eta)\in\Lambda). \end{cases} \tag{2.44}$$

Then, we have (2.42) for Z and Γ by replacing 5/24 by 5/12 and C by some constant

C', and have (2.43) for constants $C'''_{\alpha,\alpha',\beta,\beta'}$. Hence, we have for a small fixed $0<\delta<1$

$$\det\left(\frac{\partial(Z,\Gamma)}{(y,\eta)}\right)\geqq\frac{1}{6} \qquad ((y,\eta)\in\Lambda(\xi^2,\delta)), \tag{2.45}$$

and

$$\begin{cases} \|\partial_y^\beta\partial_\eta^\alpha\partial_{x^0}^{\beta'}\partial_{\xi^2}^{\alpha'}Z\| \\ \quad\leqq C''_{\alpha,\alpha',\beta,\beta'}\langle\xi^2\rangle^{-|\alpha+\alpha'|+(1-\rho)(|\alpha+\alpha'+\beta+\beta'|-1)}, \\ \|\partial_y^\beta\partial_\eta^\alpha\partial_{x^0}^{\beta'}\partial_{\xi^2}^{\alpha'}\Gamma\| \\ \quad\leqq C'_{\alpha,\alpha',\beta,\beta'}\langle\xi^2\rangle^{1-|\alpha+\alpha'|+(1-\rho)(|\alpha+\alpha'+\beta+\beta'|-1)} \\ \qquad\qquad (|\alpha+\alpha'+\beta+\beta'|\geqq 1,\ (y,\eta)\in\Lambda). \end{cases} \tag{2.46}$$

Then, noting $(y,\eta)\in\Lambda$ in supp $\tilde{\chi}'_0$, we have by (2.32)

$$q'_0(x^0,\xi^2)=\iint e^{-iZ\cdot\Gamma}p'_0(Z,\Gamma;x^0,\xi^2)dZd\Gamma, \tag{2.47}$$

where

$$\begin{aligned} &p'_0(Z,\Gamma;x^0,\xi^2) \\ &=\left\{\tilde{\chi}_0\tilde{\chi}'_0p_1(x^0,\dot{\xi}^1+\eta)p_2(\dot{x}^1+y,\xi^2)\det\left(\frac{\partial(Z,\Gamma)}{\partial(y,\eta)}\right)^{-1}\right\}_{\substack{y=y(Z,\Gamma)\\ \eta=\eta(Z,\Gamma)}} \end{aligned} \tag{2.48}$$

for the inverse $(y,\eta)=(y,\eta)(Z,\Gamma)$ of $(Z,\Gamma)=(Z,\Gamma)(y,\eta)$.

IV) From the theory of the oscillatory integrals we have $q'_0(x^0,\xi^2)\in S_\rho^{m_1+m_2}$ and have the asymptotic expansion

$$q'_0(x^0,\xi^2)\sim\sum_\alpha\frac{1}{\alpha!}D_Z^\alpha\partial_\Gamma^\alpha p'_0(0,0;x^0,\xi^2). \tag{2.49}$$

Then, noting $(y,\eta)=(0,0)$ for $(Z,\Gamma)=(0,0)$, we have

$$q'_0(x^0,\xi^2)\sim\sum_{j=0}^\infty q_j(x^0,\xi^2)$$

for $q_j(x^0,\xi^2)$ of the form (2.6). Hence, from (2.14) and (2.25) the proof is complete Q.E.D.

Remark. For $\tilde{\psi}=Z\cdot\Gamma$ we have

$$\begin{aligned} \partial_{y_j}\partial_{y_k}\tilde{\psi}&=\partial_{y_j}Z\cdot\partial_{y_k}\Gamma+\partial_{y_k}Z\cdot\partial_{y_j}\Gamma, \\ \partial_{y_j}\partial_{\eta_k}\tilde{\psi}&=\partial_{y_j}Z\cdot\partial_{\eta_k}\Gamma+\partial_{\eta_k}Z\cdot\partial_{y_j}\Gamma, \\ \partial_{\eta_j}\partial_{\eta_k}\tilde{\psi}&=\partial_{\eta_j}Z\cdot\partial_{\eta_k}\Gamma+\partial_{\eta_k}Z\cdot\partial_{\eta_j}\Gamma \end{aligned}$$

at $(y,\eta)=(0,0)$. Hence, we have

$$\det\left(\frac{\partial(Z,\Gamma)}{\partial(y,\eta)}\right)^2=\det\left(\begin{pmatrix}{}^t(\partial_y Z) & {}^t(\partial_y \Gamma)\\ {}^t(\partial_\eta Z) & {}^t(\Gamma_\eta \Gamma)\end{pmatrix}\begin{pmatrix}\partial_y Z & \partial_\eta Z\\ \partial_y \Gamma & \partial_\eta \Gamma\end{pmatrix}\right)$$
$$(2.50)\qquad =\det\begin{pmatrix}\nabla_y\nabla_y\tilde{\psi} & \nabla_y\nabla_\eta\tilde{\psi}\\ \nabla_\eta\nabla_y\tilde{\psi} & \nabla_\eta\nabla_\eta\tilde{\psi}\end{pmatrix}=\det\begin{pmatrix}\nabla_x\nabla_x\phi_2(\dot{x}^1,\xi^2) & I\\ I & \nabla_\xi\nabla_\xi\phi_1(x^0,\dot{\xi}^1)\end{pmatrix}$$

at $(y,\eta)=(0,0)$.

Theorem 2.4. *Let $p_j\in S_\rho^{m_j}$ and $\phi_j\in\mathscr{P}_\rho(\tau_j)$ $(j=1,2,\cdots)$. Assume that $\bar{M}_\infty\equiv\sum_{j=1}^\infty|m_j|<\infty$, $\bar{\tau}_\infty\equiv\sum_{j=1}^\infty\tau_j\leqq\tau_0/(8c_0)$ with a constant τ_0 in Theorem 1.4 and a constant c_0 of Theorem 1.5, and $\{J_j/\tau_j\}_{j=1}^\infty$ is a bounded set in $S_\rho^1((2))$. Set $\Phi_1=\phi_1$ and $\Phi_\nu=\phi_1\#\cdots\#\phi_\nu$ $(\nu\geqq 2)$.*

Define

$$q_{j_1,\dots,j_\nu}(x^0,\xi^{\nu+1})\in S_\rho^{\bar{m}_{\nu+1}-(2\rho-1)\bar{j}_\nu}\qquad(\bar{m}_{\nu+1}=m_1+\cdots+m_{\nu+1},\ \bar{j}_\nu=j_1+\cdots+j_\nu),$$

inductively, by

$$(2.51)\qquad q_{j_1}(x^0,\xi^2)=\sum_{|\alpha^1+\beta^1|\leqq 2j_1}\gamma_{j_1,\alpha^1,\beta^1}(x^0,\xi^2;\Phi_1,\phi_2)p_1^{(\alpha^1)}(x^0,\Xi_1^1)p_{2(\beta^1)}(X_1^1,\xi^2),$$

and

$$(2.52)\qquad \begin{aligned}q_{j_1,\dots,j_\nu}(x^0,\xi^{\nu+1})=&\sum_{|\alpha^\nu+\beta^\nu|\leqq 2j_\nu}\gamma_{j_\nu,\alpha^\nu,\beta^\nu}(x^0,\xi^{\nu+1};\Phi_\nu,\phi_{\nu+1})\\ &\times q_{j_1,\dots,j_{\nu-1}}^{(\alpha^\nu)}(x^0,\Xi_\nu^\nu)p_{\nu+1(\beta^\nu)}(X_\nu^\nu,\xi^{\nu+1})\qquad(\nu\geqq 2),\end{aligned}$$

where $\gamma_{j_\nu,\alpha^\nu,\beta^\nu}\in S_\rho^{-(2\rho-1)j_\nu+|\alpha^\nu|-(1-\rho)|\alpha^\nu+\beta^\nu|}$ are symbols defined by (2.6) of Theorem 2.3 corresponding to Φ_ν and $\phi_{\nu+1}$, and $\{X_\nu^j,\Xi_\nu^j\}_{j=1}^\nu(x^0,\xi^{\nu+1})$ is the solution of (1.4). Then, for any N and l there exist an integer l' and a constant $C_{N,l}$ such that

$$(2.53)\qquad \begin{aligned}&|q_{j_1,\dots,j_\nu}|_l^{(\bar{m}_{\nu+1}-(2\rho-1)\bar{j}_\nu)}\\ &\leqq C_{N,l}{}^\nu\sum_{j=1}^{\nu+1}|p_j|_{l'}^{(m_j)}\qquad(\bar{j}_\nu\leqq N,\ \nu=1,2,\cdots).\end{aligned}$$

Proof. I) First we prove that the right hand sides of (2.51) and (2.52) can be written as the sum of terms of the form:

$$(2.54)\qquad \begin{aligned}I_\nu(x^0,\xi^{\nu+1})\equiv&\gamma_{j_1,\alpha^1,\beta^1}^{(\zeta^1)}(x^0,\Xi_\nu^2)\gamma_{j_2,\alpha^2,\beta^2}^{(\zeta^2)}(x^0,\Xi_\nu^3)\cdots\\ &\times\gamma_{j_{\nu-1},\alpha^{\nu-1},\beta^{\nu-1}}^{(\zeta^{\nu-1})}(x^0,\Xi_\nu^\nu)\gamma_{j_\nu,\alpha^\nu,\beta^\nu}(x^0,\xi^{\nu+1})\\ &\times p_1^{(\alpha^1+\mu^1)}(x^0,\Xi_\nu^1)p_{2(\beta^1+\delta^1)}^{(\mu^2)}(X_\nu^1,\Xi_\nu^2)\cdots\\ &\times p_{\nu(\beta^{\nu-1}+\delta^{\nu-1})}^{(\mu^\nu)}(X_\nu^{\nu-1},\Xi_\nu^\nu)p_{\nu+1(\beta^\nu)}(X_\nu^\nu,\Xi_\nu^{\nu+1})\\ &\times\Xi_{\kappa_1,\sigma_1}^{r_1(\theta^1)}\cdots\Xi_{\kappa_s,\sigma_s}^{r_s(\theta^s)}X_{\kappa_1',\sigma_1'}^{r_1'(\omega^1)}\cdots X_{\kappa_t',\sigma_t'}^{r_t'(\omega^t)}\end{aligned}$$

$$(\kappa_j,\ \kappa_j'\in\{1,\cdots,\nu-1\},\ 1\leqq\gamma_j\leqq\kappa_j,\ 1\leqq\gamma_j'\leqq\kappa_j',\ 1\leqq\sigma_j\leqq n,\ 1\leqq\sigma_j'\leqq n),$$

where

$$
(2.55)\quad \begin{cases} \text{i)} & |\alpha^k+\beta^k|\leqq 2j_k \qquad (k=1,\cdots,\nu), \\ \text{ii)} & |\zeta^1+\cdots+\zeta^{\nu-1}|+|\mu^1+\cdots+\mu^\nu|+|\delta^1+\cdots+\delta^{\nu-1}| \\ & \qquad +(|\theta^1+\cdots+\theta^s|-s)+(|\omega^1+\cdots+\omega^t|-t) \\ & \qquad =|\alpha^2+\cdots+\alpha^\nu|, \\ \text{iii)} & |\theta^k|\geqq 1\ (k=1,\cdots,s),\ |\omega^l|\geqq 1\ (l=1,\cdots,t), \\ \text{iv)} & s\leqq|\alpha^2+\cdots+\alpha^\nu|,\ t=|\delta^1+\cdots+\delta^{\nu-1}|\leqq|\alpha^2+\cdots+\alpha^\nu|. \end{cases}
$$

Let $\nu=1$. Then, (2.54) has the form

$$
I_1(x^0,\xi^2)=\gamma_{j_1,\alpha^1,\beta^1}(x^0,\xi^2)p_1^{(\alpha^1+\mu^1)}(x^0,\Xi_1^1)p_{2(\beta^1)}(X_1^1,\xi^2),
$$

and $|\mu^1|=0$ by (2.55)-ii). Hence, we see that the right hand side of (2.51) can be written as the sum of terms of (2.54) for $\nu=1$.

Now, assume that the statement is true for ν. Then, for $\nu+1$ the right hand side of (2.52) can be written as the sum of terms

$$
J_{\nu+1}(x^0,\xi^{\nu+1})\equiv\gamma_{j_{\nu+1},\alpha^{\nu+1},\beta^{\nu+1}}(x^0,\xi^{\nu+1})I^{(\alpha^{\nu+1})}(x^0,\tilde{\Xi}_{\nu+1})p_{\nu+2(\beta^{\nu+1})}(\tilde{X}_{\nu+1},\xi^{\nu+2}),
$$

where $\{\tilde{X}_{\nu+1},\tilde{\Xi}_{\nu+1}\}(x^0,\xi^{\nu+2})$ is defined by

$$
(2.56)\quad \begin{cases} \tilde{X}_{\nu+1}=\nabla_\xi\Phi_{\nu+1}(x^0,\tilde{\Xi}_{\nu+1}), \\ \tilde{\Xi}_{\nu+1}=\nabla_x\phi_{\nu+2}(\tilde{X}_{\nu+1},\xi^{\nu+2}). \end{cases}
$$

Then, by Theorem 1.6 we have

$$
(2.57)\quad \begin{cases} \{\tilde{X}_{\nu+1},\tilde{\Xi}_{\nu+1}\}(x^0,\xi^{\nu+2})=\{X_{\nu+1}^{\nu+1},\Xi_{\nu+1}^{\nu+1}\}(x^0,\xi^{\nu+2}), \\ \{X_\nu^j,\Xi_\nu^j\}_{j=1}^\nu(x^0,\tilde{\Xi}_{\nu+1}(x^0,\xi^{\nu+2}))=\{X_{\nu+1}^j,\Xi_{\nu+1}^j\}_{j=1}^\nu(x^0,\xi^{\nu+2}). \end{cases}
$$

By (2.54) we write

$$
(2.58)\quad \begin{aligned} & J_{\nu+1}(x^0,\xi^{\nu+2}) \\ & \equiv\gamma_{j_{\nu+1},\alpha^{\nu+1},\beta^{\nu+1}}(x^0,\xi^{\nu+2})\partial_{\xi^{\nu+1}}^{\alpha^{\nu+1}}\{\gamma_{j_1,\alpha^1,\beta^1}^{(\zeta^1)}(x^0,\Xi_\nu^2)\cdots \\ & \quad \times\gamma_{j_{\nu-1},\alpha^{\nu-1},\beta^{\nu-1}}^{(\zeta^{\nu-1})}(x^0,\Xi_\nu^\nu)\gamma_{j_\nu,\alpha^\nu,\beta^\nu}(x^0,\xi^{\nu+1})p_1^{(\alpha^1+\mu^1)}(x^0,\Xi_\nu^1)\cdots \\ & \quad \times p_{\nu(\beta^{\nu-1}+\delta^{\nu-1})}^{(\mu^\nu)}(X_\nu^{\nu-1},\Xi_\nu^\nu)p_{\nu+1(\beta^\nu)}(X_\nu^\nu,\xi^{\nu+1})\Xi_{\kappa_1,\sigma_1}^{r_1(\theta^1)}\cdots \\ & \quad \times\Xi_{\kappa_s,\sigma_s}^{r_s(\theta^s)}X_{\kappa',\sigma_1'}^{r_1'(\omega^1)}\cdots X_{\kappa_t',\sigma_t'}^{r_t'(\omega^t)}\}|_{\xi^{\nu+1}=\tilde{\Xi}_{\nu+1}(x^0,\xi^{\nu+2})} \\ & \quad \times p_{\nu+2(\beta^{\nu+1})}(\tilde{X}_{\nu+1},\xi^{\nu+2}). \end{aligned}
$$

Since, for example, we have

$$
\begin{aligned} & \nabla_{\xi^{\nu+1}}p_{k(\beta^{k-1}+\delta^{k-1})}^{(\mu^k)}(X_\nu^{k-1},\Xi_\nu^k) \\ & \quad =(\nabla_\xi p_{k(\beta^{k-1}+\delta^{k-1})}^{(\mu^k)})(X_\nu^{k-1},\Xi_\nu^k)\nabla_{\xi^{\nu+1}}\Xi_\nu^{k-1} \\ & \qquad +(\nabla_x p_{k(\beta^{k-1}+\delta^{k-1})}^{(\mu^k)})(X_\nu^{k-1},\Xi_\nu^k)\nabla_{\xi^{\nu+1}}X_\nu^{k-1}, \end{aligned}
$$

using (2.57) we see that $J_{\nu+1}(x^0, \xi^{\nu+2})$ can be written as the sum of terms (2.54) with the relation (2.55) for $\nu+1$. Hence, we see that the same statement holds for $q_{j_1,\dots,j_{\nu+1}}(x^0, \xi^{\nu+2})$.

II) By Theorem 1.4′ and Theorem 2.3 we have

$$(2.59)\qquad \begin{cases} \nabla_{\xi^{\nu+1}} X_\nu^k(x^0, \xi^{\nu+1}) \in S_\rho^{-1}, \qquad \nabla_{\xi^{\nu+1}} \Xi_\nu^k(x^0, \xi^{\nu+1}) \in S_\rho^0, \\ \gamma_{j_k,\alpha^k,\beta^k}^{(\zeta^k)}(x^0, \Xi_\nu^k(x^0, \xi^{\nu+1})) \in S_\rho^{-(2\rho-1)j_k+|\alpha^k|-(1-\rho)|\alpha^k+\beta^k|-\rho\zeta^k}, \\ p_{k(\beta^{k-1}+\delta^{k-1})}^{(\mu^k)}(X_\nu^{k-1}(x^0, \xi^{\nu+1}), \Xi_\nu^k(x^0, \xi^{\nu+1})) \in S_\rho^{m_k-\rho|\mu^k|+(1-\rho)|\beta^{k-1}+\delta^{k-1}|}, \end{cases}$$

and these symbols are bounded in the corresponding spaces with respect to ν, $k \leqq \nu$ and $\bar{j}_\nu \leqq N$ for any fixed N. Hence, together with the relation (2.55) we see that

$$(2.60)\qquad I_\nu(x^0, \xi^{\nu+1}) \in S_\rho^{\bar{m}_{\nu+1}-(2\rho-1)\bar{j}_\nu}$$

and for any integer N and l we have for an integer l' and a constant $C_{N,l}$ independent of ν

$$(2.61)\qquad |I_\nu|_l^{(\bar{m}_{\nu+1}-(2\rho-1)\bar{j}_\nu)} \leqq C_{N,l}^\nu \prod_{j=1}^{\nu+1} |p_j|_{l'}^{(m_j)} \qquad (\bar{j}_\nu \leqq N).$$

III) We fix an integer $N>0$ and consider the number of the terms $I_\nu(x^0, \xi^{\nu+1})$ of (2.54) for $q_{j_1,\dots,j_\nu}(x^0, \xi^{\nu+1})$ when $\bar{j}_\nu \leqq N$. We note that in (2.54)

$$(2.62)\qquad \begin{aligned} &\text{“the number of } \{\gamma_{j_k,\alpha^k,\beta^k}, p_k, \Xi_\nu^k, X_\nu^k\}\text{”} \\ &\leqq \nu+(\nu+1)+s+t \leqq 2\nu+|\alpha^2+\cdots+\alpha^\nu|+1 \\ &\leqq 2\nu+2\bar{j}_\nu+1 \leqq 2(\nu+N)+1. \end{aligned}$$

In (2.51), using $|\alpha^1+\beta^1| \leqq 2j_1$, we see that

$$\begin{aligned} &\text{“the number of terms } I_1(x^0, \xi^2) \text{ of (2.54) for } q_{j_1}(x^0, \xi^2)\text{”} \\ &\leqq (n+1)^{2j_1} \times (n+1)^{2j_1} = (n+1)^{4j_1}. \end{aligned}$$

Then, in (2.52) for $\nu=2$, using $|\alpha^2+\beta^2| \leqq 2j_2$ and (2.62) we have

$$\begin{aligned} &\text{“the number of terms } I_2(x^0, \xi^3) \text{ of (2.55) for } q_{j_1,j_2}(x^0, \xi^3)\text{”} \\ &\leqq (n+1)^{4j_1} \times (n+1)^{4j_2} \times (2(\nu+N)+1)^{2j_2} \\ &\leqq 4^{\bar{j}_2}(n+1)^{4\bar{j}_2}(\nu+N+1)^{2\bar{j}_2}. \end{aligned}$$

Finally, in (2.52) for the general ν we have

$$(2.63)\qquad \begin{aligned} &\text{“the number of terms } I_\nu(x^0, \xi^{\nu+1}) \text{ of (2.55) for } q_{j_1,\dots,j_\nu}(x^0, \xi^{\nu+1})\text{”} \\ &\leqq 4^{\bar{j}_\nu}(n+1)^{4\bar{j}_\nu}(\nu+N+1)^{2\bar{j}_\nu} \\ &\leqq 4^N(n+1)^{4N}(\nu+N+1)^{2N} \leqq M_N C_N^\nu \quad (\nu=1, 2, \cdots) \text{ for } \bar{j}_\nu \leqq N \end{aligned}$$

for constants M_N and C_N. Consequently, from (2.61) and (2.63) we get (2.53). Q.E.D.

Now, let $\psi(\xi)$ be a C^∞-function such that $\psi(\xi)=0$ ($|\xi|\leqq 1$), $=1$ ($|\xi|\geqq 2$), and set

$$q_{\nu,j,\varepsilon}(x,\xi)=\psi(\varepsilon^\nu\xi)\sum_{j_1+\cdots+j_\nu=j} q_{j_1,\cdots,j_\nu}(x,\xi) \qquad (0<\varepsilon\leqq 1). \tag{2.64}$$

Then, by Theorem 2.4 we have for constants $C_{j,\alpha,\beta}$ independent of ν and ε

$$\begin{aligned} &|q^{(\alpha)}_{\nu,j,\varepsilon(\beta)}(x,\xi)| \\ &\leqq C_{j,\alpha,\beta} \max_{j_1+\cdots+j_\nu=j} |q_{j_1,\cdots,j_\nu}|^{(\overline{m}_{\nu+1}-(2\rho-1)j)}_{|\alpha+\beta|} \\ &\qquad \times\langle\xi\rangle^{\overline{m}_{\nu+1}-(2\rho-1)j-|\alpha|+(1-\rho)|\alpha+\beta|}. \end{aligned} \tag{2.65}$$

Take $0<\varepsilon_j\leqq 1$ ($\varepsilon_j\to 0$) such that $C_{j,\alpha,\beta}A^0_j\varepsilon_j\leqq 2^{-j}$ for $|\alpha+\beta|\leqq j$ and for a large fixed $A^0_j>0$ determined later, and define $\tilde{q}_\nu(x,\xi)$ by

$$\tilde{q}_\nu(x,\xi)=\sum_{j=0}^\infty q_{\nu,j,\varepsilon_j}(x,\xi). \tag{2.66}$$

Then, we have

Theorem 2.5. *Let $p_j\in S^{m_j}_\rho$ and $\phi_j\in\mathscr{P}_\rho(\tau_j)$ ($j=1,2,\cdots,\nu+1,\cdots$). Assume that*

$$\overline{M}_\infty\equiv\sum_{j=1}^\infty |m_j|<\infty, \qquad \overline{\tau}_\infty\equiv\sum_{j=1}^\infty \tau_j\leqq\tau_0/(8C_0)$$

with a constant τ_0 of Theorem 1.4 and C_0 of Theorem 1.5, and $\{J_j/\tau_j\}_{j=1}^\infty$ is a bounded set in $S^1_\rho((2))$.

Moreover, assume that for any l there exists a constant A_l such that

$$|p_j|^{(m_j)}_l\leqq A_l \qquad (j=1,2,\cdots,\nu+1,\cdots). \tag{2.67}$$

Then we have for a constant C_l independent of ν

$$|\tilde{q}_\nu|^{(\overline{m}_{\nu+1})}_l\leqq C^\nu_l \qquad (\nu=1,2,\cdots), \tag{2.68}$$

where $\tilde{q}_\nu(x,\xi)$ are defined by (2.66) for A^0_j determined by the finite number of $\{A_l\}$ of (2.67). Furthermore, set

$$R_\nu=P_{1,\phi_1}P_{2,\phi_2}\cdots P_{\nu+1,\phi_{\nu+1}}-\tilde{Q}_{\nu,\Phi_{\nu+1}}. \tag{2.69}$$

Then, R_ν: $H_{-\infty}\to H_{-\infty}$ ($H_{-\infty}=\bigcup_\sigma H_\sigma$) is a smoothing operator in the sense: For any σ and N we have for a constant $C_{\sigma,N}$ independent of ν

$$\|R_\nu\|_{H_\sigma\to H_{\sigma+N}}\leqq C^\nu_{\sigma,N}, \tag{2.70}$$

where $\|\cdot\|_{H_\sigma\to H_{\sigma+N}}$ *denotes the operator norm of the mapping*: $H_\sigma\to H_{\sigma+N}$.

Proof. First we note that, if $\phi\in\mathscr{P}_\rho(\tau)$ and $p\in S^m_\rho$ $(1/2<\rho\leqq 1)$, we have for any σ

$$\|P_\phi u\|_\sigma\leqq C|p|_l^{(m)}\|u\|_{\sigma+m}\qquad(u\in H_{\sigma+m}),\tag{2.71}$$

where C and l are constants depending on σ, m and $\|J\|_{l'}$ for some l'. This fact is proved by the same method with the proof of Theorem 2.5 in [8].

For a fixed N set $\tilde{q}_{\nu,N}(x,\xi)=\sum_{j=N}^\infty q_{\nu,j,\varepsilon_j}(x,\xi)$. Then, using (2.53) of Theorem 2.4 and (2.67), we have by (2.65)

$$|\tilde{q}_{\nu,N}|_l^{(\overline{m}_{\nu+1}-(2\rho-1)N)}\leqq C^\nu_{N,l}\tag{2.72}$$

for a constant $C_{N,l}$ independent of ν, if we choose A^0_j sufficiently large according to $\{A_l\}$ of (2.67).

For a given N_0 we take an integer N such that $\overline{m}_{\nu+1}-(2\rho-1)N\leqq -N_0$, and write R_ν in the form

$$\begin{aligned}R_\nu=&\Big(P_{1,\phi_1}P_{2,\phi_2}-\sum_{j_1=0}^{N-1}Q_{j_1;\,\Phi_2}\Big)P_{3,\phi_3}\cdots P_{\nu+1,\phi_{\nu+1}}\\&+\sum_{k=1}^{\nu-2}\Big\{\sum_{\bar{j}_k<N}\Big(Q_{j_1,\cdots,j_k;\,\Phi_{k+1}}P_{k+2,\phi_{k+2}}\\&-\sum_{j_{k+1}=0}^{N-\bar{j}_k-1}Q_{j_1,\cdots,j_{k+1};\,\Phi_{k+2}}\Big)P_{k+3,\phi_{k+3}}\cdots P_{\nu+1,\phi_{\nu+1}}\Big\}\\&+\sum_{\bar{j}_{\nu-1}<N}\Big(Q_{j_1,\cdots,j_{\nu-1};\,\Phi_\nu}P_{\nu+1,\phi_{\nu+1}}\\&-\sum_{j_\nu=0}^{N-\bar{j}_{\nu-1}-1}Q_{j_1,\cdots,j_\nu;\,\Phi_{\nu+1}}\Big)+\Big(\sum_{\bar{j}_\nu<N}Q_{j_1,\cdots,j_\nu;\,\Phi_{\nu+1}}-\tilde{Q}_{\nu,\Phi_{\nu+1}}\Big)\\\equiv& R_{\nu,0}+\sum_{k=1}^{\nu-2}R_{\nu,k}+R_{\nu,\nu-1}+\tilde{R}_{\nu,\nu}\qquad(\bar{j}_k=j_1+\cdots+j_k).\end{aligned}\tag{2.73}$$

We first note that for any fixed σ the operator-norms for

$$P_{k,\phi_k}\colon H_{\sigma-\overline{\overline{m}}_{k+1}}\to H_{\sigma-\overline{\overline{m}}_k}$$
$$(\overline{\overline{m}}_k=m_k+\cdots+m_{\nu+1},\ \overline{\overline{m}}_{\nu+2}=0,\ k=\nu+1,\cdots,2)$$

are bounded with respect to $k=\nu+1,\cdots,2$, since $\overline{M}_\infty\equiv\sum_{j=1}^\infty|m_j|<\infty$ and $\{|p_j|_l^{(m_j)}\}_{j=1}^\infty$ is bounded for any l. Hence, we have for a constant C_1

$$\begin{aligned}&\|R_{\nu,0}\|_{H_\sigma\to H_{\sigma+N_0}}\leqq C_1^\nu\Big\|P_{1,\phi_1}P_{2,\phi_2}-\sum_{j_1=0}^{N-1}Q_{j_1;\,\Phi_2}\Big\|_{H_{\sigma-\overline{\overline{m}}_3}\to H_{\sigma+N_0}},\\&\|R_{\nu,k}\|_{H_\sigma\to H_{\sigma+N_0}}\end{aligned}\tag{2.74}$$

$$\leq C_1^{\nu-k-1} \sum_{\bar{j}_k<N} \Big\| Q_{j_1,\cdots,j_k;\,\Phi_{k+1}} P_{k+2,\phi_{k+2}} - \sum_{j_{k+1}=0}^{N-\bar{j}_k-1} Q_{j_1,\cdots,j_{k+1};\,\Phi_{k+2}} \Big\|_{H_{\sigma-\bar{\bar{m}}_{k+3}}\to H_{\sigma+N_0}} \tag{2.74}$$

$$(k=1,\cdots,\nu-1,\ \bar{\bar{m}}_{\nu+2}=0).$$

Hence, if we prove for a constant $C_{N,\sigma}$

$$\text{(2.75)}\quad \begin{cases} \text{i)} & \Big\| P_{1,\phi_1} P_{2,\phi_2} - \sum_{j_1=0}^{N-1} Q_{j_1;\,\Phi_2} \Big\|_{H_{\sigma-\bar{\bar{m}}_3}\to H_{\sigma+N_0}} \leq C_{N,\sigma}, \\ \text{ii)} & \Big\| Q_{j_1,\cdots,j_k;\,\Phi_{k+1}} P_{k+2,\phi_{k+2}} - \sum_{j_{k+1}=0}^{N-\bar{j}_k-1} Q_{j_1,\cdots,j_{k+1};\,\Phi_{k+2}} \Big\|_{H_{\sigma-\bar{\bar{m}}_{k+3}}\to H_{\sigma+N_0}} \\ & \leq C_{N,\sigma}^{\nu} \qquad (\nu=2,3,\cdots,\ k=1,\cdots,\nu-1), \\ \text{iii)} & \Big\| \sum_{\bar{j}_\nu<N} Q_{j_1,\cdots,j_\nu;\,\Phi_{\nu+1}} - \tilde{Q}_{\nu;\,\Phi_{\nu+1}} \Big\|_{H_\sigma\to H_{\sigma+N_0}} \leq C_{N,\sigma}^{\nu} \qquad (\nu=1,2,\cdots), \end{cases}$$

then, we get (2.70).

Using Theorem 2.3, (2.53) of Theorem 2.4 and (2.71), and noting $\bar{\bar{m}}_{\nu+1}-(2\rho-1)N \leq -N_0$, we get (2.75)-i), ii). Now, we write

$$\begin{aligned} &\sum_{\bar{j}_\nu<N} Q_{j_1,\cdots,j_\nu;\,\Phi_{\nu+1}} - \tilde{Q}_{\nu;\,\Phi_{\nu+1}} \\ &= \Big\{ \sum_{\bar{j}_\nu<N} Q_{j_1,\cdots,j_\nu;\,\Phi_{\nu+1}} - (\tilde{Q}_{\nu;\Phi_{\nu+1}} - \tilde{Q}_{\nu,N;\Phi_{\nu+1}}) \Big\} + \tilde{Q}_{\nu,N;\Phi_{\nu+1}} \\ &\equiv \tilde{R}_{\nu,N,\Phi_{\nu+1}} + \tilde{R}'_{\nu,N,\Phi_{\nu+1}}. \end{aligned}$$

Then, since

$$\sigma(\tilde{Q}_{\nu;\Phi_{\nu+1}}) - \sigma(\tilde{Q}_{\nu,N;\Phi_{\nu+1}}) = \sum_{\bar{j}_\nu<N} \psi(\varepsilon_{\bar{j}_\nu}^{\nu}\xi) q_{j_1,\cdots,j_\nu}(x,\xi),$$

we have

$$\sigma(\tilde{R}_{\nu,N,\Phi_{\nu+1}}) = \sum_{\bar{j}_\nu<N} (1-\psi(\varepsilon_{\bar{j}_\nu}^{\nu}\xi)) q_{j_1,\cdots,j_\nu}(x,\xi).$$

Then, we see that for a constant M_N independent of ν

$$\sigma(\tilde{R}_{\nu,N,\Phi_{\nu+1}}) = 0 \qquad (|\xi| \geq M_N^{\nu}).$$

Hence, there exists a constant $C'_{N,l}$ such that

$$|\sigma(\tilde{R}_{\nu,N,\Phi_{\nu+1}})|_l^{(\bar{\bar{m}}_{\nu+1}-(2\rho-1)N)} \leq C_{N,l}'^{\nu}.$$

So, by (2.71) we have for a constant $C''_{N,\sigma}$

$$\|\tilde{R}_{\nu,N,\Phi_{\nu+1}} u\|_{\sigma-(\bar{\bar{m}}_{\nu+1}-(2\rho-1)N)} \leq C_{N,\sigma}''^{\nu} \|u\|_\sigma. \tag{2.76}$$

On the other hand by (2.72) we have (2.76) for $\tilde{R}'_{\nu,N,\Phi_{\nu+1}}$. Hence, noting $\bar{m}_{\nu+1}-(2\rho-1)N\leqq -N_0$, we get (2.75)-iii). Thus, from (2.74) and (2.75) we get (2.70).

Q.E.D.

§ 3. Fundamental solution of a hyperbolic system

Consider a hyperbolic system

$$L=D_t+\mathscr{D}(t)+B(t) \qquad \text{on } [0, T]\ (T>0), \tag{3.1}$$

where

$$\mathscr{D}(t)=\begin{pmatrix} \lambda_1(t, X, D_x) & & 0 \\ & \ddots & \\ 0 & & \lambda_l(t, X, D_x) \end{pmatrix} \tag{3.2}$$

$$(\lambda_j(t, x, \xi) \in \mathscr{B}^\infty([0, T]; S^1), \text{ real valued}),$$

and

$$B(t)=(b_{jk}(t, X, D_x))_{k\to 1,\dots,l}^{j\downarrow 1,\dots,l} \tag{3.3}$$

$$(b_{jk}(t, x, \xi) \in \mathscr{B}^\infty([0, T]; S^0)).$$

We also assume that for a constant $M>0$ we have

$$\lambda_j(t, x, \delta\xi)=\delta\lambda_j(t, x, \xi) \qquad (|\xi|\geqq M,\ \delta\geqq 1). \tag{3.4}$$

Let $\phi_j(t, s)=\phi_j(t, s; x, \xi)$ be the solutions of the eiconal equations

$$\begin{cases} \partial_t\phi_j+\lambda_j(t, x, \nabla_x\phi_j)=0 \text{ on } [0, T], \\ \phi_j|_{t=s}=x\cdot\xi, \end{cases} \tag{3.5}$$

and set

$$\boldsymbol{I}_\phi(t, s)=\begin{pmatrix} I_{\phi_1}(t, s) & & 0 \\ & \ddots & \\ 0 & & I_{\phi_l}(t, s) \end{pmatrix}, \tag{3.6}$$

where $I_{\phi_j}(t, s)$ are Fourier integral operators with phase functions $\phi_j(t, s; x, \xi)$ and symbol 1. Then, we have by [8]

$$L\boldsymbol{I}_\phi(t, s)=R_\phi(t, s), \tag{3.7}$$

where $R_\phi(t, s)=\sum_{j=1}^{l} R_{j,\phi_j}(t, s)$ is a matrix of Fourier integral operators with phase functions $\phi_j(t, s)$ and symbols $r_j(t, s; x\cdot\xi)$ of class $\mathscr{B}^\infty(\Delta; S^0)$ $(\Delta=\{0\leqq s\leqq t\leqq T\})$. Hence, the fundamental solution $\boldsymbol{E}(t, s)$ for L, as the continuous operator from the Sobolev space H_σ into itself for any fixed real σ, is constructed in the form:

$$\boldsymbol{E}(t, s)=\boldsymbol{I}_\phi(t, s)+\int_s^t \boldsymbol{I}_\phi(t, \theta) \sum_{\nu=1}^{\infty} W_\nu(\theta, s)d\theta. \tag{3.8}$$

Here, $\{W_\nu(t, s)\}_{\nu=1}^{\infty}$ are defined by

$$\begin{cases} W_1(t, s)=-iR_\phi(t, s), \\ W_{\nu+1}(t, s)=\int_s^t W_1(t, \theta)W_\nu(\theta, s)d\theta \end{cases} \tag{3.9}$$

$$(\nu=1, 2, \cdots, \text{c.f., [8] and [9]}).$$

We note that $W_{\nu+1}(t, s)$ can be written in the form

$$W_{\nu+1}(t, s)=\int_s^t\int_s^{t_1}\cdots\int_s^{t_{\nu-1}} W^{(\nu+1)}(t, t_1, \cdots, t_\nu, s)dt_1dt_2\cdots dt_\nu \tag{3.10}$$

$$(W^{(\nu+1)}=W_1(t, t_1)W_1(t_1, t_2)\cdots W_1(t_\nu, s),\ t_0=t),$$

and that $W^{(\nu+1)}(t, t_1, \cdots, t_\nu, s)$ has the form

$$\begin{aligned} &W^{(\nu+1)}(t, t_1, \cdots, t_\nu, s) \\ &=\sum_{j_1,\cdots,j_{\nu+1}=1}^{l}(-i)^{\nu+1}R_{j_1,\phi_{j_1}}(t, t_1)\cdots R_{j_{\nu+1},\phi_{j_{\nu+1}}}(t_\nu, s). \end{aligned} \tag{3.11}$$

By Proposition 1.7 and Theorem 1.5 we have

$$\begin{cases} \phi_{j_k}(t_{k-1}, t_k) \in \mathscr{P}_1(c_0(t_{k-1}-t_k)) \qquad (t_0=t,\ t_{\nu+1}=s), \\ \Phi_{j_1,\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s) \\ \qquad =\phi_{j_1}(t, t_1)\#\cdots\#\phi_{j_{\nu+1}}(t_\nu, s) \in \mathscr{P}_1(c_1(t-s)) \qquad (0\leqq s\leqq t\leqq T) \end{cases} \tag{3.12}$$

for a small constant $T>0$ and constants $c_0>0$, $c_1>0$ (see also Theorem 1.5′). Then, by Theorem 2.5 we can find the Fourier integral operators

$$W_{j_1,j_2,\cdots,j_{\nu+1},\Phi_{j_1,j_2,\cdots,j_{\nu+1}}}(t, t_1, \cdots, t_\nu, s)$$

with phase functions

$$\Phi_{j_1,j_2,\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)=\phi_{j_1}(t, t_1)\#\phi_{j_2}(t_1, t_2)\#\cdots\#\phi_{j_{\nu+1}}(t_\nu, s)$$

and symbols $W_{j_1,j_2,\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)$ of class $\mathscr{B}^\infty(\Delta_\nu; S^0)$ ($\Delta_\nu=\{0\leqq s\leqq t_\nu\leqq\cdots\leqq t_1\leqq t\leqq T\}$), such that for any $\boldsymbol{k}$ and l we have semi-norm estimates

$$\|\partial_{\boldsymbol{t}}^{\boldsymbol{k}} W_{j_1,\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)\|_l^{(0)}\leqq C_{\boldsymbol{k},l}^\nu \tag{3.13}$$

$$(\boldsymbol{t}=(t, t_1, \cdots, t_\nu),\ \boldsymbol{k}=(k_0, k_1, \cdots, k_\nu),\ \nu=1, 2, \cdots)$$

for a constant $C_{\boldsymbol{k},l}$ independent of ν, and for any $\boldsymbol{k}$, real σ and integer $N>0$ we have

$$\begin{aligned}(3.14)\qquad &\|\partial_t^k\{R_{j_1,\phi_1}(t, t_1)R_{j_2,\phi_2}(t_1, t_2)\cdots R_{j_{\nu+1},\phi_{j_{\nu+1}}}(t_\nu, s)\\ &\quad -W_{j_1,\cdots,j_{\nu+1},\Phi_{j_1},\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)\}\|_{H_\sigma\to H_{\sigma+N}}\\ &\leqq C^\nu_{k,\sigma,N}\end{aligned}$$

for a constant $C_{k,\sigma,N}$ independent of ν. Set

$$\begin{aligned}(3.15)\qquad &W_{\nu+1,\Phi_{\nu+1}}(t, t_1, \cdots, t_\nu, s)\\ &=\sum_{j_1,\cdots,j_{\nu+1}=1}^{l}(-i)^{\nu+1}W_{j_1,\cdots,j_{\nu+1};\,\Phi_{j_1},\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)\end{aligned}$$

and

$$\begin{aligned}(3.16)\qquad &W_{-\infty}(t, s)\\ &=\int_s^t I_\phi(t, \theta)\Big[\sum_{\nu=1}^{\infty}\int_s^\theta\int_s^{t_1}\cdots\\ &\quad\cdots\int_s^{t_{\nu-1}}\{W^{(\nu+1)}-W_{\nu+1,\Phi_{\nu+1}}\}(\theta, t_1, \cdots, t_\nu, s)dt_1\cdots dt_\nu\Big]d\theta \qquad (t_0=\theta).\end{aligned}$$

Then, we have

Theorem 3.1. *The fundamental solution $\boldsymbol{E}(t, s)$ for L can be represented in the form*

$$\begin{aligned}(3.17)\qquad \boldsymbol{E}(t, s)&=\boldsymbol{I}_\phi(t, s)+\int_s^t \boldsymbol{I}_\phi(t, \theta)\Big\{W_1(\theta, s)\\ &\quad+\sum_{\nu=1}^{\infty}\int_s^\theta\int_s^{t_1}\cdots\int_s^{t_{\nu-1}}W_{\nu+1,\Phi_{\nu+1}}(\theta, t_1, \cdots, t_\nu, s)dt_1\cdots dt_\nu\Big\}d\theta\\ &\quad+W_{-\infty}(t, s)\quad (t_0=\theta,\ W_{-\infty}(t, s)\in\mathscr{B}^\infty(\Delta; S^{-\infty}),\ \Delta=\{0\leqq s\leqq t\leqq T\}),\end{aligned}$$

where $W_{-\infty}(t, s)$ is defined by (3.16).

For the proof we have only to prove that the operator $W_{-\infty}(t, s)$ of (3.16) belongs to $\mathscr{B}^\infty(\Delta; S^{-\infty})$. For that purpose we prepare the following

Proposition 3.2. *Let $P=p(X, D_x)$ be of class $S^m_{0,\delta}$ (δ may be bigger than 1), and assume that P is a smoothing operator: $H_{-\infty}\to H_\infty$ in the sense: For any $\sigma\geqq 0$ and $\sigma'\geqq 0$ there exists a constant $C_{\sigma,\sigma'}$ such that*

$$(3.18)\qquad \|Pu\|_\sigma\leqq C_{\sigma,\sigma'}\|u\|_{-\sigma'}\qquad (u\in\mathscr{S}).$$

Then, $P=p(X, D_x)$ belongs to $S^{-\infty}$ and for any $m'\leqq m$ and l we have

$$(3.19)\qquad |p|_l^{(m')}\leqq C_{m',l}|p|_{l+1}^{(m)}$$

for a constant $C_{m',l}$ depending on $C_{\sigma,\sigma'}$.

Proof of Proposition 3.2. Choosing $\sigma>n/2$, $\sigma'=1, 2, \cdots$ and using Sobolev's lemma, we have

$$|Pu(x)|\leqq C_k\|u\|_{-k} \qquad (u\in\mathscr{S}),\ k=1, 2, \cdots.$$

Hence, we have

$$\left|\int e^{ix\cdot\xi}p(x, \xi)\hat{u}(\xi)d\xi\right|\leqq C_k\left\{\int\langle\xi\rangle^{-2k}|\hat{u}(\xi)|^2 d\xi\right\}^{1/2} \tag{3.20}$$

for the Fourier transform $\hat{u}(\xi)$ of $u\in\mathscr{S}$.

Now, for fixed (x^0, ξ^0) $(|\xi^0|\geqq 1)$ and $l>0$ set $\Omega=\{\xi; |\xi-\xi^0|\leqq\langle\xi^0\rangle^{-l}\}$ and choose $\hat{u}_j(\xi)\in C_0^\infty$ in $\{\xi; |\xi-\xi^0|<2\langle\xi^0\rangle^{-l}\}$ such that

$$\hat{u}_j(\xi)\to\chi_\Omega(\xi)\frac{|p|}{p}e^{-ix^0\cdot\xi} \quad \text{in } L_2(R^n_\xi) \quad (j\to\infty),$$

where $\chi_\Omega(\xi)$ is the characteristic function of the set Ω. Then, by (3.20) we have

$$\int_\Omega|p(x^0, \xi)|d\xi\leqq C_k\left\{\int_\Omega\langle\xi\rangle^{-2k}d\xi\right\}^{1/2}\leqq C'_k\langle\xi^0\rangle^{-k-nl/2}. \tag{3.21}$$

On the other hand since $p(x, \xi)\in S^m_{0,\delta}$, we have

$$|p(x^0, \xi^0)|\leqq|p(x^0, \xi)|+C\langle\xi^0\rangle^{-l+m} \qquad \text{for } \xi\in\Omega.$$

So, using (3.21) we have

$$|p(x^0, \xi^0)|\langle\xi^0\rangle^{-nl}\leqq C''_k\{\langle\xi^0\rangle^{-(1+n)l+m}+\langle\xi^0\rangle^{-k-nl/2}\}.$$

Hence, we have

$$|p(x^0, \xi^0)|\leqq C''_k\{\langle\xi^0\rangle^{-l+m}+\langle\xi^0\rangle^{-k+nl/2}\},$$

and setting $k=(1+n/2)l$ we have

$$|p(x^0, \xi^0)|\leqq 2C''_k\langle\xi^0\rangle^{-l+|m|} \qquad \text{for any } l\ (k=(1+n/2)l). \tag{3.22}$$

For $p^{(j)}=\partial_{\xi_j}p$ and $p_{(j)}=D_{x_j}p$ we use the interpolation inequalities:

$$|p^{(j)}(x^0, \xi^0)|^2\leqq C_0\max_{|\xi-\xi^0|\leqq 1}|p(x^0, \xi)|\left\{\max_{|\xi-\xi^0|\leqq 1}|p(x^0, \xi)|+\max_{|\xi-\xi^0|\leqq 1}\max_{|\alpha|=2}|p^{(\alpha)}(x^0, \xi)|\right\},$$

and

$$|p_{(j)}(x^0, \xi^0)|^2\leqq C_0\max_{|x-x^0|\leqq 1}|p(x, \xi^0)|\left\{\max_{|x-x^0|\leqq 1}|p(x, \xi^0)|+\max_{|x-x^0|\leqq 1}\max_{|\beta|=2}|p_{(\beta)}(x, \xi^0)|\right\}$$

for a constant C_0. Then, using (3.22) and noting $p(x, \xi)\in S^m_{0,\delta}$ we have

$$|p^{(j)}(x^0, \xi^0)|+|p_{(j)}(x^0, \xi^0)|\leqq C''\langle\xi^0\rangle^{-l}, \qquad j=1, \cdots, n.$$

Repeating this we get (3.19). Q.E.D.

Proof of Theorem 3.1. Set

$$\tag{3.23} \begin{aligned}\tilde{E}(t, s)=\boldsymbol{I}_\phi(t, s)+\int_s^t \boldsymbol{I}_\phi(t, \theta)\Big\{W_1(\theta, s) \\ +\sum_{\nu=1}^{\infty}\int_s^\theta\int_s^{t_1}\cdots\int_s^{t_{\nu-1}} W_{\nu+1,\Phi_{\nu+1}}(\theta, t_1, \cdots, t_\nu, s)dt_1\cdots dt_\nu\Big\}.\end{aligned}$$

Then, from (3.8)–(3.15) and $LE(t, s)=0$ we see that $L\tilde{E}(t, s)$ is a smoothing operator which satisfies (3.18) for constants $C_{\sigma,\sigma'}$ independent of $0\leqq s\leqq\leqq t\leqq T$. On the other hand we write

$$\begin{aligned}&\phi_j(t, s)=x\cdot\xi+J_j(t, s; x, \xi), \\ &\Phi_{j_1,\dots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s) \\ &\quad=x\cdot\xi+\boldsymbol{J}_{j_1,\dots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s; x, \xi).\end{aligned}$$

Then, noting $e^{iJ_j(t,s)}$ and $\exp\{i\boldsymbol{J}_{j_1,\dots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)\}$ are bounded in $S^0_{0,1}$, we see that $L\tilde{E}(t, s)$ belongs to $S^1_{0,1}$ and $\{\sigma(L\tilde{E}(t, s))(x, \xi)\}_{0\leqq s\leqq t\leqq T}$ is bounded in $S^1_{0,1}$. Hence, by Proposition 3.2 we get

$$\tag{3.24} \tilde{R}(t, s)\equiv L\tilde{E}(t, s)\in\mathscr{B}^\infty(\Delta; S^{-\infty}) \qquad (\Delta=\{0\leqq s\leqq t\leqq T\}).$$

Then, setting

$$\begin{cases}\tilde{W}_1(t, s)=-i\tilde{R}(t, s), \\ \tilde{W}_\nu(t, s)=\displaystyle\int_s^t \tilde{W}_1(t, \theta)\tilde{W}_\nu(\theta, s)d\theta \qquad (\nu=1, 2, \cdots)\end{cases}$$

and using the uniqueness of the fundamental solution $\boldsymbol{E}(t, s)$, we can represent $\boldsymbol{E}(t, s)$ in the form

$$\tag{3.25} \boldsymbol{E}(t, s)=\tilde{E}(t, s)+\int_s^t \tilde{E}(t, \theta)\sum_{\nu=1}^{\infty}\tilde{W}_\nu(\theta, s)d\theta.$$

Finally, using the fundamental theorem on the theory of the pseudo-differential operators of multiple symbol in [7], we see that $\sum_{\nu=1}^{\infty}\tilde{W}_\nu(t, s)\in\mathscr{B}^\infty(\Delta; S^{-\infty})$, and consequently we see that

$$W_{-\infty}(t, s)\equiv\int_s^t \tilde{E}(t, \theta)\sum_{\nu=1}^{\infty}\tilde{W}_\nu(\theta, s)d\theta\in\mathscr{B}^\infty(\Delta; S^{-\infty}).$$

Q.E.D.

Using Theorem 3.1 we get a generalization of the results obtained by Ludwig-

Granoff [11] and Hata [4] (see also [3])

Theorem 3.3. *Assume that the Poisson brackets*

$$\begin{aligned}(3.26)\qquad &\{\tau+\lambda_j, \tau+\lambda_k\}\\ &\equiv\partial_t\lambda_j-\partial_t\lambda_k+\nabla_x\lambda_j\cdot\nabla_\xi\lambda_k-\nabla_x\lambda_k\cdot\nabla_\xi\lambda_j=0 \qquad (j, k=1, \cdots, l).\end{aligned}$$

Then, we can represent $\boldsymbol{E}(t, s)$ in the form

$$\begin{aligned}(3.27)\qquad \boldsymbol{E}(t, s)=&\sum_{j=1}^{l}\tilde{W}_{j,\phi_j}(t, s)\\ &+\sum_{1\leqq j_1<\cdots<j_{k+1}\leqq l}\int_s^t\cdots\int_s^{t_{k-1}}\tilde{W}_{j_1,\cdots,j_{k+1};\,\phi_{j_1},\cdots,j_{k+1}}(t, t_1, \cdots, t_k, s)dt_1\cdots dt_k\\ &\qquad (t_0=t).\end{aligned}$$

Here, $W_{j_1,\cdots,j_k;\,\Phi_{j_1},\cdots,j_k}(t, t_1, \cdots, t_{k-1}, s)$ are matrices of Fourier integral operators with phase functions $\Phi_{j_1,\cdots,j_k}(t, t_1, \cdots, t_{k-1}, s)$ and symbols of class $\mathscr{B}^\infty(\Delta_{k-1}; S^{-\infty})$ $(\Phi_{j_1}=\phi_{j_1}, \Delta_0=\Delta, t_0=t)$.

Proof. Let $W\equiv W_{j_1,\cdots,j_{\nu+1};\,\Phi_{j_1},\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)$ be a Fourier integral operator with phase function

$$\Phi_{j_1,\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)=\phi_{j_1}(t, t_1)\#\cdots\#\phi_{j_{\nu+1}}(t_\nu, s)$$

and symbol

$$\begin{gathered}W_{j_1,\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)\in\mathscr{B}^\infty(\Delta_\nu; S^0)\\ (\Delta_\nu=\{0\leqq s\leqq t_\nu\leqq\cdots\leqq t_1\leqq t=T\}),\end{gathered}$$

and let $\tilde{W}\equiv\tilde{W}_{j_1,\cdots,j_{\nu+1};\,\tilde{\Phi}_{j_1},\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)$ be a Fourier integral operator with phase function

$$\begin{aligned}&\Phi_{j_1,\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)\\ &\qquad=\tilde{\Phi}_{j_1,\cdots,j_{k+1},j_k,\cdots,j_\nu}(t, t_1, \cdots, t_{k-1}, t_k, t_{k+1}, \cdots, t_\nu, s)\end{aligned}$$

and symbol

$$\begin{aligned}&\tilde{W}_{j_1,\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)\\ &\qquad=W_{j_1,\cdots,j_{\nu+1}}(t, t_1, \cdots, t_{k-1}, t_{k-1}-t_k+t_{k+1}, t_{k+1}, \cdots, t_\nu, s).\end{aligned}$$

If we set

$$I(t, t_1, \cdots, t_{k+2}, s)=\int_s^{t_{k+2}}\cdots\int_s^{t_{\nu-1}}W dt_{k+3}\cdots dt_\nu,$$

we have

$$J\equiv\int_s^{t_{k-1}}\left\{\int_s^{t_k}\left(\int_s^{t_{k+1}}I dt_{k+2}\right)dt_{k+1}\right\}dt_k$$

$$=\int_s^{t_{k-1}}\left\{\int_{t_{k+1}}^{t_{k-1}}\left(\int_s^{t_{k+1}} Idt_{k+2}\right)dt_k\right\}dt_{k+1}$$
$$=\int_s^{t_{k-1}}\left\{\int_s^{t_{k+1}}\left(\int_{t_{k+1}}^{t_{k-1}} Idt_k\right)dt_{k+2}\right\}dt_{k+1}$$

and

$$\int_{t_{k+1}}^{t_{k-1}} Idt_k=\int_s^{k+2}\cdots\int_s^{t_{\nu-1}}\left(\int_{t_{k+1}}^{t_{k-1}} Wdt_k\right)dt_{k+2}\cdots dt_\nu.$$

Then, by (1.36) of Theorem 1.10 we have

$$\int_{t_{k+1}}^{t_{k-1}} Wdt_k=\int_{t_{k+1}}^{t_{k-1}} \tilde{W}dt_k.$$

Hence, we have

$$\int_s^t\cdots\int_s^{t_{\nu-1}} Wdt_1\cdots dt_\nu=\int_s^t\cdots\int_s^{t_{\nu-1}} \tilde{W}dt_1\cdots dt_\nu.$$

Consequently, using $\phi_j(t_{k-1},t_k)\#\phi_j(t_k,t_{k+1})=\phi_j(t_{k-1},t_{k+1})$ $(j=1,\cdots,l)$, we get the expression (3.27). Q.E.D.

Now, we define the trajectry $\{Q_{j_1,\dots,j_{\nu+1}}, P_{j_1,\dots,j_{\nu+1}}\}(t,t_1,\cdots,t_\nu;y,\xi)$ $(\nu=0,1,\cdots)$ for fixed $\lambda_{j_1},\cdots,\lambda_{j_{\nu+1}}$, $0=s\leqq t_{\nu+1}\leqq t_\nu\leqq\cdots\leqq t_0=t$ and $(y,\eta)\in R^{2n}$ as follows: First define $\{Q_{j_{\nu+1}}, P_{j_{\nu+1}}\}(t;y,\eta)$ as the solution of

$$\text{(3.28)}\qquad\begin{cases}\dfrac{dq}{dt}=\nabla_\xi\lambda_{j_{\nu+1}}(t,q,p), & \dfrac{dp}{dt}=-\nabla_x\lambda_{j_{\nu+1}}(t,q,p),\\ (q,p)|_{t=0}=(y,\eta).\end{cases}$$

Next, for $1\leqq k\leqq\nu$ we define $\{Q_{j_k,\dots,j_{\nu+1}}, P_{j_k,\dots,j_{\nu+1}}\}$ $(t,t_k,\cdots,t_\nu;y,\eta)$ as the solution of

$$\text{(3.29)}\qquad\begin{cases}\dfrac{dq}{dt}=\nabla_\xi\lambda_{j_k}(t,q,p), \quad \dfrac{dq}{dt}=-\nabla_k\lambda_{j_k}(t,q,p),\\ (q,p)|_{t=t_k}=\{Q_{j_{k+1},\dots,j_{\nu+1}}, P_{j_{k+1},\dots,j_{\nu+1}}\}(t_k,\cdots,t_\nu;y,\eta) \qquad (k=1,\cdots,\nu).\end{cases}$$

For $\lambda_{j_1},\cdots,\lambda_{j_{\nu+1}},(y,\eta)$ and a fixed $0\leqq\varepsilon<1$ we define an ε-station-chain $\{t_1,\cdots,t_\nu\}$ as the points $t=t_0>t_1>\cdots>t_\nu>0$ such that

$$\text{(3.30)}\qquad\begin{aligned}&|\lambda_{j_k}(t_k,x^k,\xi^k)-\lambda_{j_{k+1}}(t_k,x^k,\xi^k)|\leqq\varepsilon\langle\xi^k\rangle\\ &\text{at } (x^k,\xi^k)=\{Q_{j_{k+1},\dots,j_{\nu+1}}, P_{j_{k+1},\dots,j_{\nu+1}}\}(t_k,\cdots,t_\nu;y,\eta) \qquad (k=1,\cdots,\nu),\end{aligned}$$

and define the ε-station-set $\Lambda_{\varepsilon,j_1,\dots,j_{\nu+1}}(y,\eta)$ by the set of all ε-station-chains $\{t_1,\cdots,t_\nu\}$. When $\varepsilon=0$, we often use the words 'station-chain' and 'station-set' simply, and denote $\Lambda_{0,j_1,\dots,j_{\nu+1}}(y,\eta)$ by $\Lambda_{j_1,\dots,j_{\nu+1}}(y,\eta)$.

Furthermore, set

$$\begin{aligned}(3.31)\qquad &\Lambda_\varepsilon(t;y,\eta)\\ &=\{\{Q_{j_1,\dots,j_{\nu+1}}, P_{j_1,\dots,j_{\nu+1}}\}(t,t_1,\cdots,t_\nu;y,\eta);\{t_1,\cdots,t_\nu\}\\ &\qquad \in \Lambda_{\varepsilon,j_1,\dots,j_{\nu+1}}(y,\eta), j_1,\cdots,j_{\nu+1}\in\{1,\cdots,l\}, \nu=0,1,\cdots\}\end{aligned}$$

Then we have

Theorem 3.4. *The solution of the Cauchy problem*

$$(3.32)\qquad \begin{cases} LU=0 & on\ [0,T],\\ U|_{t=0}=U_0\end{cases}$$

for $U_0={}^t(u_{01},\cdots,u_{0l})\in H_{-\infty}\ (=\bigcup_\sigma H_\sigma)$ *is given by*

$$U(t,x)={}^t(u_1(t,x),\cdots,u_l(t,x))=\boldsymbol{E}(t,0)U_0\in\mathscr{B}^\infty([0,T];H_{-\infty}).$$

Furthermore, if we set

$$(3.33)\qquad \begin{aligned}&\Gamma_{t,\varepsilon}=\{\delta\Lambda_\varepsilon(t;y,\eta);(y,\eta)\in WF_\varepsilon(U_0),\delta>0,|\eta|\geqq M_0\}\\ &(WF_\varepsilon(U_0)=\{(y,\eta);\text{dis}\{(y,|\eta|^{-1}\eta),WF(U_0)\}\leqq\varepsilon\})\end{aligned}$$

for large $M_0(>0)$ *depending on* M *of* (3.4), *then,* $\Gamma_t=\bigcap_{0<\varepsilon<1}\Gamma_{t,\varepsilon}$ *is closed and we have for the wave front set* $WF(U(t))$

$$(3.34)\qquad WF(U(t))\subset\Gamma_t.$$

If $\lambda_1,\cdots,\lambda_l$ *satisfies the condition* (3.26) *of Theorem* 3.3, *we can replace* (3.34) *by*

$$(3.34)'\qquad WF(U(t))\subset\Gamma'_t=\{\Lambda'(t;y,\eta);(y,\eta)\in WF(U_0)\},$$

where $\Lambda'(t;y,\eta)$ *is defined by*

$$\begin{aligned}(3.31)'\qquad &\Lambda'(t;y,\eta)\\ &=\{\{Q_{j_1,\dots,j_{\nu+1}}, P_{j_1,\dots,j_{\nu+1}}\}(t,t_1,\cdots,t_\nu;y,\eta);\\ &\qquad \{t_1,\cdots,t_\nu\}\in\Lambda_{j_1,\dots,j_{\nu+1}}(y,\eta), 1\leqq j_1<\cdots<j_{\nu+1}\leqq l\}.\end{aligned}$$

Proof. It is easy to see that $\boldsymbol{E}(t,0)U_0$ is the solution of (3.32). For any $\lambda_{j_1},\cdots,\lambda_{j_{\nu+1}}$ and (y,η) we consider

$$\begin{aligned}&(X_k,\Xi_k)(t;y,\eta)\\ &=\{Q_{j_{k+1},\dots,j_{\nu+1}}, P_{j_{k+1},\dots,j_{\nu+1}}\}(t,t_k,\cdots,t_\nu;y,\eta)\qquad (k=0,1,\cdots,\nu).\end{aligned}$$

Then, using (3.28) and (3.29), we see that for any $\delta>0$ there exists $\delta'>0$ such that

$$(3.35)\quad \begin{aligned}&|(t;y,\langle\eta\rangle^{-1}\eta)-(t';y',\langle\eta'\rangle^{-1}\eta')|<\delta'\\ &\Rightarrow|(X_k,\langle\eta\rangle^{-1}\Xi_k)(t;y,\eta)-(X_k,\langle\eta'\rangle^{-1}\Xi_k)(t':y',\eta')|<\delta.\end{aligned}$$

Now, we prove that Γ_t is closed. Let $(x^0,\xi^0)\in\bar{\Gamma}_{t_0}$ for a fixed $t_0>0$. Then, we can choose $(x^m,\xi^m)\in\Gamma_{t_0}$, $m=1,2,\cdots$ such that $(x^m,\xi^m)\to(x^0,\xi^0)$ as $m\to\infty$. By the definition we can choose for $\varepsilon_m=2^{-m}$

$$\{t_1^m,\cdots,t_{\nu m}^m\}\in\Lambda_{\varepsilon_m,j_1^m,\cdots,j_{\nu_m+1}^m}(y^m,\eta^m)$$

for some $\lambda_{j_1^m},\cdots,\lambda_{j_{\nu_m+1}^m}$ and $(y^m,\eta^m)\in WF_{\varepsilon_m}(U_0)$

such that

$$(3.36)\quad (x^m,\xi^m)=\{Q_{j_1^m,\cdots,j_{\nu_m+1}^m},P_{j_1^m,\cdots,j_{\nu_m+1}^m}\}(t_0,t_1^m,\cdots,t_{\nu m}^m;y^m,\eta^m)\qquad(m=1,2,\cdots).$$

Using (3.35), we see that there exists a subsequence $\{\gamma=m_\mu\}_{\mu=1}^\infty$ such that (y^γ,η^γ) converges to some $(y^\infty,\eta^\infty)\in WF(U_0)$ and the corresponding trajectries converge uniformly to some continuous curve. Now consider the trajectries defined by

$$(x^0,\xi^0)=\{Q_{j_1^\gamma,\cdots,j_{\nu_\gamma+1}^\gamma},P_{j_1^\gamma,\cdots,j_{\nu_\gamma+1}^\gamma}\}(t_0,t_1^\gamma,\cdots,t_{\nu_\gamma}^\gamma;\tilde{y}^\gamma,\tilde{\eta}^\gamma).$$

Then, again using (3.35), we see that $(\tilde{y}^\gamma,\tilde{\eta}^\gamma)\to(y^\infty,\eta^\infty)\in WF(U_0)$ as $\gamma\to\infty$, and $\{t_1^\gamma,\cdots,t_{\nu_\gamma}^\gamma\}\in\Lambda_{\varepsilon'_\gamma,j_1^\gamma,\cdots,j_{\nu\gamma+1}^\gamma}(\tilde{y}^\gamma,\tilde{\eta}^\gamma)$ for some $\{\varepsilon'_\gamma\}$ such that $\varepsilon'_\gamma\to0$ as $\gamma\to\infty$, which means that $(x^0,\xi^0)\in\Gamma_{t_0}$. Hence, Γ_{t_0} is closed. From this proof it is clear that $\Gamma'_t=\Gamma_t$, if we consider $1\leqq j_1<\cdots<j_{\nu+1}\leqq l$.

II) Take a fixed point $(x^0,\xi^0)\notin\Gamma_{t_0}$ for a fixed $t_0>0$. For a fixed (y^0,η^0) choose $a(x),a_1(x),b(\xi),b_1(\eta)$ which have small supports in (conic-) neighborhoods of x^0,y^0,ξ^0,η^0, respectively.

Consider $a(X)b(D_x)\boldsymbol{E}(t_0,0)b_1(D_x)a_1(X)$ for the fundamental solution $\boldsymbol{E}(t,s)$ of L. Let $W_\Phi(t_0,t_1,\cdots,t_\nu,0)$ be a Fourier integral operator appearing in the expression of $\boldsymbol{E}(t,s)$ in the form (3.17). Then, from the product formulas for pseudo-differential operators and Fourier integral operators in [8] (or from Theorem 2.3 of the present paper), we have for $W'_\Phi=a(X)b(D_x)W_\Phi b_1(D_x)a_1(X)$

$$(3.37)\quad \begin{aligned}&\sigma(W'_\Phi)(t_0,t_1,\cdots,t_\nu,0;x,\eta)\\ &\sim\sum_{\zeta,\alpha,\beta,\mu,\theta}C_{\zeta,\alpha,\beta,\mu,\theta}(x,\eta)a(x)b^{(\zeta)}(\nabla_x\Phi(x,\eta))W_{(\beta)}^{(\alpha)}(x,\eta)b_1^{(\mu)}(\eta)a_{1(\theta)}(\nabla_\xi\Phi(x,\eta)).\end{aligned}$$

Hence, we see that

$$(3.38)\quad \begin{aligned}&\operatorname{supp}a(x)b(\nabla_x\Phi(x,\eta))\cap\operatorname{supp}a_1(\nabla_\xi\Phi(x,\eta))b_1(\eta)=\phi\\ &\Rightarrow W'_\Phi\in S^{-\infty}\qquad\text{at }(t_0,t_1,\cdots,t_\nu,0).\end{aligned}$$

So we see that if we define (y^0,η^0) by

$$(3.39)\qquad y^0=\nabla_\xi\Phi(x^0,\eta^0),\qquad \xi^0=\nabla_x\Phi(x^0,\eta^0),$$

we may only consider W'_Φ for such (y^0, η^0) in order to investigate $WF(U(t_0))$.

III) Now, for $\Phi=\phi_{j_1}(t_0, t_1)\#\cdots\#\phi_{j_\nu+1}(t_\nu, 0)$ we write

$$W'_\Phi U_0(x)=\int e^{i\Phi}(W'_\Phi)(x, \eta)\hat{U}_0(\eta)d\eta. \tag{3.40}$$

We note that the relation (3.39) is decomposed into

$$(x^0, \xi^0)\xrightarrow{\lambda_{j_1}}(X^1_\nu, \Xi^1_\nu)\xrightarrow{\lambda_{j_2}}\cdots\xrightarrow{\lambda_{j_\nu}}(X^\nu_\nu, \Xi^\nu_\nu)\xrightarrow{\lambda_{j_\nu+1}}(y^0, \eta^0), \tag{3.41}$$

where $\{X^j_\nu, \Xi^j_\nu\}^\nu_{j=1}$ is defined by

$$\begin{cases} X^j_\nu=\nabla_\xi\phi_j(t_{j-1}, t_j; X^{j-1}_\nu, \Xi^j_\nu), \\ \Xi^j_\nu=\nabla_x\phi_{j+1}(t_j, t_{j+1}; X^j_\nu, \Xi^{j+1}_\nu), \qquad j=1, \cdots, \nu \end{cases} \tag{3.42}$$
$$(X^0=x^0,\ \xi^0=\nabla_x\Phi(x^0, \eta^0)=\nabla_x\phi_1(t_0, t_1; x^0, \Xi^1_\nu),$$
$$y^0=\nabla_\xi\Phi(x^0, \eta^0)=\nabla_\xi\phi_{\nu+1}(X^\nu_\nu, \eta^0),\ \Xi^{\nu+1}_\nu=\eta^0,\ t_{\nu+1}=0).$$

Then, if we assume $(x^0, \xi^0)\notin\Gamma_{t_0,\varepsilon_0}$ for some $\varepsilon_0>0$, we have for any $\{t^0_1, \cdots, t^0_\nu\}$

$$\begin{aligned} &(y^0, \eta^0)\in WF_{\varepsilon_0}(U_0)\Rightarrow\exists(X^k_\nu, \Xi^k_\nu) \qquad \text{such that} \\ &\varepsilon_0\langle\Xi^k_\nu\rangle\leqq|\lambda_{j_k}(t^0_k, X^k_\nu, \Xi^k_\nu)-\lambda_{j_{k+1}}(t^0_k, X^k_\nu, \Xi^k_\nu)| \\ &\left(=\left|\frac{d}{dt_k}\Phi\right| \text{ by Theorem 1.9-ii)}\right) \qquad \text{at } \{t^0_1, \cdots, t^0_\nu\}. \end{aligned} \tag{3.43}$$

Then, choosing a C^∞-function $\gamma(t_k)$ on $[0, T]$ such that $\gamma(t^0_k)\neq0$ and (3.43) holds for $\varepsilon_0/2$ on supp γ, we write

$$\begin{aligned} \int_{t_{k+1}}^{t_{k-1}}\gamma(t_k)e^{i\Phi}\sigma(W'_\Phi)dt_k &= -i\int_{t_{k+1}}^{t_{k-1}}\frac{d}{dt_k}e^{i\Phi}\cdot\left(\frac{d\Phi}{dt_k}\right)^{-1}\gamma\sigma(W'_\Phi)dt_k \\ &=\left[-ie^{i\Phi}\left(\frac{d\Phi}{dt_k}\right)^{-1}\gamma\sigma(W'_\Phi)\right]_{t_{k+1}}^{t_{k-1}}+i\int_{t_{k+1}}^{t_{k-1}}e^{i\Phi}\cdot\frac{d}{dt_k}\left\{\left(\frac{d\Phi}{dt_k}\right)^{-1}\gamma\sigma(W'_\Phi)\right\}dt_k. \end{aligned}$$

Hence, we see that $\int_{t_{k+1}}^{t_{k-1}}\gamma(t_k)W'_\Phi dt_k$ can be written as the Fourier integral operator of order -1. For any $W_{\nu+1,\Phi_{\nu+1}}$ in the expression of (3.17) for $\boldsymbol{E}(t, s)$, we consider one element

$$W''_{\nu+1,\Phi_{\nu+1}}=\gamma(t_1)\cdots\gamma(t_\nu)a(X)b(D_x)W_{\nu+1,\Phi_{\nu+1}}b_1(D_x)a_1(X).$$

Then, repeating the integration by parts, and noting that $\{\boldsymbol{J}_{\nu+1}=\Phi_{\nu+1}-x\cdot\xi\}$ and $\{\sigma(W''_{\nu+1,\Phi_{\nu+1}})/C^\nu\}$ (for a constant C independent of ν) are bounded in S^1 and S^0 with respect to ν, respectively, we see that $(x^0, \xi^0)\notin WF(U(t_0))$. Hence, we get (3.34).

Q.E.D.

Finally we consider a system of differential operators of the form

$$\mathscr{L}=D_t+\sum_{j=1}^{n} A_j(t,x)D_{x_j}+B(t,x) \qquad \text{on } [0,T]\times R_x^n, \tag{3.44}$$

where $A_j(t,x)$, $j=1,\cdots,n$, and $B(t,x)$ are $l\times l$ matrices of $\mathscr{B}^\infty$-functions on $[0,T]\times R_x^n$. Assume that the matrix $A(t,x,\xi)=\sum_{j=1}^n A_j(t,x)\xi_j$ has real eigenvalues $\lambda_1(t,x,\xi),\cdots,\lambda_l(t,x,\xi)$ of class $\mathscr{B}^\infty([0,T];S^1)$ outside of $\xi=0$. Assume, furthermore, that there exist eigenvectors $N_j(t,x,\xi)$ corresponding to $\lambda_j(t,x,\xi)$, respectively, such that for $N(t,x,\xi)=(N_1(t,x,\xi),\cdots,N_l(t,x,\xi))$

$$N(t,x,\xi),\quad N(t,x,\xi)^{-1}\in\mathscr{B}^\infty([0,T];S^0) \quad (\text{outside } \xi=0). \tag{3.45}$$

Then, modifying λ_j and N_j in a neighborhood of $\xi=0$, we can use $N(t,x,\xi)$ as the diagonalizer of $\mathscr{L}$. Hence, letting $Q(t,X,D_x)$ be the parametrix of $N(t,X,D_x)$, we see that $L=Q(t,X,D_x)\mathscr{L}N(t,X,D_x)$ has the form (3.1), and we have

Theorem 3.5. *Under the above assumptions the operator $\mathscr{L}$ of* (3.44) *has the fundamental solution $\tilde{E}(t,s)$ of the form*

$$\tilde{E}(t,s)=N(t,X,D_x)E(t,s)Q(t,X,D_x)+\tilde{W}_{-\infty}(t,s), \tag{3.46}$$

where $E(t,s)$ is the fundamental solution for L and $\tilde{W}_{-\infty}$ is a smoothing operator of class $\mathscr{B}^\infty(\Delta,S^{-\infty})$ $(\Delta=\{0\leqq s\leqq t\leqq T\})$.

Furthermore, the initial value problem

$$\begin{cases}\mathscr{L}U=0 & \text{on } [0,T],\\ U|_{t=0}=U_0\end{cases} \tag{3.47}$$

can be solved by $U(t)=\tilde{E}(t,0)U_0$, and for the solution $U(t)$ we get the statement (3.34).

References

[1] Carathéodory, C., Calculus of variations and partial differential equations of the first order, Part I, Holden-Day, Amsterdam, 1965 (translated from the German original, 1935).
[2] Fedoryuk, M. V., The stationary phase method and pseudo-differential operators, Russian Math. Surveys, **26** (1971), 65–115.
[3] Gautesen, A. K. and Ludwig, D., Tangential characteristics and coupling waves, J. Math. Anal. Appl., **38** (1972), 430–446.
[4] Hata, M., On the Cauchy problem for hyperbolic operators with characteristic roots of variable multiplicity, Master thesis, Osaka University, 1977.
[5] Hörmander, L., Pseudo-differential operators and hypoelliptic equations, Proc. Symposium on Singular Integrals, Amer. Math. Soc., **10** (1967), 138–183.
[6] Hörmander, L., Fourier integral operators I, Acta Math., **127** (1971), 79–183.
[7] Kumano-go, H., Pseudo-differential operators, Iwanami Shoten, Tokyo, 1974 (in Japanese).

[8] Kumano-go, H., A calculus of Fourier integral operators on R^n and the fundamental solution for an operator of hyperbolic type, Comm. Partial Differential Equations, **1** (1976), 1–44.
[9] Kumano-go, H., Fundamental solutions for operators of regularly hyperbolic type, J. Math. Soc. Japan, **29** (1977), 399–406.
[10] Kumano-go, H., Taniguchi, K. and Tozaki, Y., Multi-products of phase functions for Fourier integral operators with an application, Comm. Partial Differential Equations, **3** (1978), 349–380.
[11] Ludwig, D. and Granoff, B., Propagation of singularities along characteristics with non-uniform multiplicity, J. Math. Anal. Appl., **21** (1968), 556–574.

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