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Degenerate Quasilinear Hyperbolic Equation with Strong Damping

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§ 1. Introduction.

Let $a(\cdot)$ be a real valued $C^1[0, \infty)$-function satisfying

$$a(s) \geq 0 \qquad \text{for } s \in [0, \infty). \tag{1.1}$$

We consider the initial-boundary value problem (*IBVP*) for degenerate quasilinear hyperbolic equations

$$u_{tt}(x, t) - a\left(\int_\Omega |\nabla u(y, t)|^2 dy\right) \Delta u(x, t) - \Delta u_t(x, t) = 0 \tag{1.2}$$

on $\Omega \times (0, \infty)$ where Ω is a bounded domain with its boundary $\partial\Omega$ in the n-dimensional Euclidean space $\boldsymbol{R}^n$. When

$$a(s) \geq a_0 > 0 \qquad \text{for } s \in [0, \infty) \tag{1.3}$$

for some constant a_0 instead of (1.1), Eq. (1.2) has a classical solution with null Dirichlet boundary condition and suitable initial data, which is unique and decays exponentially as $t \to \infty$.

Our aim in this paper is to show the existence, uniqueness, regularity and asymptotic behaviour of the solution to (1.2) under the condition (1.1). We shall remark about the asymptotic behaviour: in general the decay property of the solution to (1.2) cannot be expected for an arbitrary a. In fact, let s_0 be a positive number such that $a(s_0) = 0$ and let $u_0(x)$ satisfy $u_0(x) \in H^\infty(\Omega)$, $u_0(x)|_{\partial\Omega} = 0$ and $\int_\Omega |\nabla u_0(x)|^2 dx = s_0$. Then $u(x, t) \equiv u_0(x)$ is just a solution to (1.2) with initial conditions

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = 0$$

and boundary condition

$$u(x, t)|_{\partial\Omega} = 0.$$

We note that such $u_0(x)$'s are infinitely many, and that the above example is applicable to (1.4), (1.5) appeared below. However, we can prove the decay properties of the solution to (1.2) with $a(s)=s$.

We mention about related results. The equation (1.2) arises in the study of the motion of an elastic string. The equation related to (1.2) are

$$u_{tt}-a\left(\int_{\Omega}|\nabla u(y, t)|^2 dy\right)\Delta u=0 \tag{1.4}$$

$$u_{tt}-a\left(\int_{\Omega}|\nabla u(y, t)|^2 dy\right)\Delta u+u_t=0 \tag{1.5}$$

$$u_{tt}-a\left(\int_{\Omega}|\nabla u(y, t)|^2 dy\right)\Delta u+\Delta^2 u=0 \tag{1.6}$$

and etc., which are studied together with (1.3) by many authors; for (1.4) Dickey [5], [8], [9], Nishida [22], Menzala [17], Rodriguez [24], Greenberg and Hu [12], Pohozaev [23], Nishihara [20] (see also Lions [15]); for (1.5) Dickey [6], Brito [3], Yamada [27], Nishihara [21]; for (1.6) Ball [2], Dickey [7], Medeiros [16], Menzala [17], Yamaguchi [28] and others. The degenerate case is proposed in Lions [14], [15]. However, in our results the strongly damping term is necessary.

The $(IBVP)$'s for the other (degenerate) quasilinear hyperbolic equations, such as

$$u_{tt}-\sigma(u_x)_x-u_{xxt}=f, \tag{1.7}$$

are investigated by many authors, Tsutsumi [26], Andrews [1], Greenberg, MacCamy and Mizel [11], Greenberg [10] and etc., while the periodic problems to those are also studied by Clement [4], Kakita [13], Sowunmi [25], Narazaki [19] and etc.. In a final section, we shall show the existence of periodic solutions to (1.2) with the periodic forcing term when $a(s)=s$.

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§ 2. Formulation of the problem and results.

Let H be a real, separable Hilbert space with norm $\|\cdot\|$ and scalar product $(\cdot, \cdot)$. Let A be a linear operator in H with domain $D(A)$ dense in H. We assume that A satisfies the following conditions:

(H1) A is a self-adjoint, positive definite operator with discrete spectrum.

Then the linear operator A^r $(r>0)$ with domain $D(A^r)$ is well-defined and the condition

(H2) the inject i on $D(A^r)$ into $D(A^{r'})$, $r>r'\geq 0$, is compact
is assumed. It is also assumed that

(H3) $a\in C^1[0,\infty)$ with $a(s)\geq 0$ for $s\geq 0$.

In the space H we now consider the initial value problem to the equation

$$u''(t)+a(\|A^{1/2}u(t)\|^2)Au(t)+Au'(t)=0 \tag{2.1}$$

with 'boundary' condition

$$u(t),\ u'(t)\in D(A) \qquad \text{for any } t\in[0,\infty) \tag{2.2}$$

and initial data

$$u(0)=u_0, \qquad u'(0)=u_1, \tag{2.3}$$

which is the abstract form of (*IBVP*) to (1.2). Here we denote $'=d/dt$ and $''=d^2/dt^2$.

Then we have the followings:

Theorem 1 (Existence and uniqueness). *We assume the conditions* (H1)–(H3) *and*

(H4) $u_0\in D(A)$ *and* $u_1\in D(A^{1/2})$.

Then there exists a unique solution $u\in C^1([0,\infty);H)$ *of* (2.1)–(2.3) *satisfying* Au, $A^{1/2}u'\in L^\infty(0,\infty;H)$ *and* Au', $u''\in L^2(0,\infty;H)$.

Theorem 2 (Regularity). *We assume*

(H5) *both* u_0 *and* u_1 *are belonged to* $D(A^{3/2})$.

Then the unique solution u in Theorem 1 *satisfies* $u\in C^2([0,\infty);H)$, $A^{3/2}u$, Au', $A^{1/2}u''\in L^\infty(0,\infty;H)$ *and* $A^{3/2}u'$, Au'', $u'''\in L^2(0,\infty;H)$.

Theorem 3 (Regularity). *If we assume*

$(\mathrm{H6})_k$ $u_0\in D(A^k)$ *and* $u_1\in D(A^{k-1/2})$, $k\geq 2$,

then the solution u in Theorem 2 *satisfies* $A^k u$, $A^{k-1/2}u'\in L^\infty(0,\infty;H)$ *and* $A^k u'\in L^2(0,\infty;H)$.

The bounded solution of (2.1)–(2.3) obtained above cannot be expected the decay properties, as indicated in Introduction, provided that there exists $s_0>0$ such that $a(s_0)=0$. But when $a(s)=s$ we have

Theorem 4 (Decay). *Let* (H1), (H2) *and* (H5) *be assumed and let u be the unique* *bounded solution to the equation*

$$u''(t)+\|A^{1/2}u(t)\|^2Au(t)+Au'(t)=0 \tag{2.4}$$

with (2.2) *and* (2.3). *Then* $u(t)$ *satisfies the following decay properties*:

$$(1+t)^{1/2-\varepsilon}\|A^{1/2}u(t)\|\le C_\varepsilon \tag{2.5}$$

$$(1+t)^{1/4-\varepsilon}\|Au(t)\|\le C_\varepsilon \tag{2.6}$$

$$(1+t)^{3/2-\varepsilon}\|A^{1/2}u'(t)\|\le C_\varepsilon \tag{2.7}$$

$$(1+t)^{2-\varepsilon}\|u''(t)\|\le C_\varepsilon \tag{2.8}$$

$$\int_0^\infty (1+\tau)^{4-\varepsilon}\|A^{1/2}u''(\tau)\|^2 d\tau\le C_\varepsilon \tag{2.9}$$

for each positive constant ε *and some constant* C_ε *depending upon* ε *such as* $C_\varepsilon\to\infty$ *as* $\varepsilon\to 0$.

Remark. If $a(s)\ge a_0>0$ for some constant a_0, then it is easily seen that the solution $u(t)$ to (2.1)–(2.3) with its 'higher derivatives' decays exponentially to zero as t tends to infinity (cf. Nishihara [21], Yamada [27]).

§ 3. A priori estimates.

We employ Galerkin's procedure for the proofs.

The condition (H1) implies that A has an infinite sequence of eigenvalues $\{\lambda_j^2\}$ with

$$0<\lambda_1^2\le\lambda_2^2\le\cdots\le\lambda_j^2\le\cdots,\ \lim_{j\to\infty}\lambda_j^2=\infty$$

and there exists a complete orthonormal system $\{w_j\}$ in H, wach w_j being an eigenvector to λ_j^2. For each $u\in H$, we have an expantion:

$$u=\sum_{j=1}^\infty u_jw_j,\qquad u_j=(u, w_j)$$

with $\|u\|=\left(\sum_{j=1}^\infty u_j^2\right)^{1/2}$. If $u\in D(A^k)$, there holds

$$A^ku=\sum_{j=1}^\infty \lambda_j^{2k}u_jw_j\qquad \text{with } \|A^ku\|=\left(\sum_{j=1}^\infty \lambda_j^{4k}u_j^2\right)^{1/2}.$$

We now define Galerkin's approximation $u_m(t)=\sum_{j=1}^m g_{jm}(t)w_j$ as a solution to the initial value problem for the system:

$$(u_m''(t)+a(\|A^{1/2}u_m(t)\|^2)Au_m(t)+Au_m'(t), w)=0\qquad \text{for any fixed } w\in V^m, \tag{3.1}$$

with data

$$
\text{(3.2)} \qquad
\begin{aligned}
u_m(0) &= \sum_{j=1}^{m} \alpha_j w_j \longrightarrow u_0 \qquad \text{strongly as } m \to \infty \\
u'_m(0) &= \sum_{j=1}^{m} \beta_j w_j \longrightarrow u_1 \qquad \text{strongly as } m \to \infty,
\end{aligned}
$$

where V^m is an m-dimensional vector space spanned by $\{w_1, \cdots, w_m\}$. Then we obtain certain nonlinear ordinary differential system for the g_{jm}'s, which has a unique C^3-solution defined on some interval $[0, t_m)$ since $a \in C^1[0, \infty)$.

We estimate the u_m under the condition (H4).

(I) Putting $w = 2u'_m(t)$ in (3.1) we have

$$
\frac{d}{dt}[\|u'_m(t)\|^2 + \hat{a}(\|A^{1/2}u_m(t)\|^2)] + 2\|A^{1/2}u'_m(t)\|^2 = 0 \tag{3.3}
$$

where $\hat{a}(s) = \int_0^s a(\tau)d\tau$. Integrating both sides of (3.3) from 0 to t ($t < t_m$) and using (3.2), we have

$$
\|u'_m(t)\|^2 + \hat{a}(\|A^{1/2}u_m(t)\|^2) + 2\int_0^t \|A^{1/2}u'_m(\tau)\|^2 d\tau \le \|u_1\|^2 + \hat{a}(\|A^{1/2}u_0\|^2)
$$

from which we obtain

$$
\|u'_m(t)\| \le C_1, \tag{3.4}
$$

$$
\int_0^t \|A^{1/2}u'_m(\tau)\|^2 d\tau \le C. \tag{3.5}
$$

(From now on we denote by C or C_i ($i = 1, 2, \cdots$) various constants independent of t and m.)

(II) Putting $w = u_m(t)$ in (3.1), we have

$$
\text{(3.6)} \qquad
\begin{aligned}
&\frac{d}{dt}(u'_m(t), u_m(t)) - \|u'_m(t)\|^2 + a(\|A^{1/2}u_m(t)\|^2)\|A^{1/2}u_m(t)\|^2 \\
&\quad + \frac{1}{2}\frac{d}{dt}\|A^{1/2}u_m(t)\|^2 = 0
\end{aligned}
$$

and integrating (3.6) over $[0, t]$, $t < t_m$, we get by (3.4)

$$
\begin{aligned}
&\int_0^t a(\|A^{1/2}u_m(\tau)\|^2)\|A^{1/2}u_m(\tau)\|^2 d\tau + \frac{1}{2}\|A^{1/2}u_m(t)\|^2 \\
&= \frac{1}{2}\|A^{1/2}u_m(0)\|^2 + \int_0^t \|u'_m(\tau)\|^2 d\tau + (u'_m(0), u_m(0)) - (u'_m(t), u_m(t)) \\
&\le \frac{1}{2}\|A^{1/2}u_0\|^2 + \|u_1\|\,\|u_0\| + \frac{1}{\lambda_1^2}\int_0^t \|A^{1/2}u'_m(\tau)\|^2 d\tau + \frac{1}{\lambda_1}\|u'_m(t)\|\,\|A^{1/2}u_m(t)\| \\
&\le C + (C_1/\lambda_1)\|A^{1/2}u_m(t)\|,
\end{aligned}
$$

from which it follows that

$$\|A^{1/2}u_m(t)\|^2 \leq C_2 \tag{3.7}$$

and

$$\int_0^t a(\|A^{1/2}u_m(\tau)\|^2)\|A^{1/2}u_m(\tau)\|^2 d\tau \leq C. \tag{3.8}$$

From (3.4) and (3.7) we conclude that all the intervals $[0, t_m)$ are extended to the whole interval $[0, \infty)$.

(III) Taking $w = Au_m(t)$ in (3.1) and integrating it over $[0, t)$, we have

$$\begin{aligned} &a(\|A^{1/2}u_m(t)\|^2)\|Au_m(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|Au_m(t)\|^2 \\ &= -\frac{d}{dt}(A^{1/2}u'_m(t), A^{1/2}u_m(t)) + \|A^{1/2}u'_m(t)\|^2 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\int_0^t a(\|A^{1/2}u_m(\tau)\|^2)\|Au_m(\tau)\|^2 d\tau + \frac{1}{2}\|Au_m(t)\|^2 \\ &\leq \frac{1}{2}\|Au_0\|^2 + \|u_1\|\|Au_0\| + \|u'_m(t)\|\|Au_m(t)\| + \int_0^t \|A^{1/2}u'_m(\tau)\|^2 d\tau \\ &\leq C + C_1\|Au_m(t)\|, \end{aligned}$$

the latter of which means

$$\|Au_m(t)\| \leq C \tag{3.10}$$

and

$$\int_0^\infty a(\|A^{1/2}u_m(\tau)\|^2)\|Au_m(\tau)\|^2 d\tau \leq C. \tag{3.11}$$

(IV) Taking $w = 2Au'_m(t)$ in (3.1), we have

$$\begin{aligned} \frac{d}{dt}\|A^{1/2}u'_m(t)\|^2 + 2\|Au'_m(t)\|^2 &= -2a(\|A^{1/2}u_m(t)\|^2)(Au_m(t), Au'_m(t)) \\ &\leq a(\|A^{1/2}u_m(t)\|^2)\left(a_1\|Au_m(t)\|^2 + \frac{1}{a_1}\|Au'_m(t)\|^2\right) \\ &\leq a_1 \cdot a(\|A^{1/2}u_m(t)\|^2)\|Au_m(t)\|^2 + \|Au'_m(t)\|^2, \end{aligned}$$

that is,

(3.12) $$\frac{d}{dt}\|A^{1/2}u'_m(t)\|^2+\|Au'_m(t)\|^2\leq a_1\cdot a(\|A^{1/2}u_m(t)\|^2)\|Au_m(t)\|^2$$

where $a_1=\max\{a(s);\, 0\leq s\leq C_2\}$. Then, integrating (3.12) over $[0, t]$, we have

$$\|A^{1/2}u'_m(t)\|^2+\int_0^t\|Au'_m(\tau)\|^2d\tau\leq\|A^{1/2}u_1\|^2+a_1\int_0^\infty a(\|A^{1/2}u_m(\tau)\|^2)\|Au_m(\tau)\|^2d\tau$$

which gives by (3.11)

(3.13) $$\|A^{1/2}u'_m(t)\|\leq C$$

and

(3.14) $$\int_0^\infty\|Au'_m(\tau)\|^2d\tau\leq C.$$

(V) Putting $w=u''_m(t)$ in (3.1), we have

(3.15) $$\begin{aligned}\|u''_m(t)\|^2+\frac{1}{2}\frac{d}{dt}\|A^{1/2}u'_m(t)\|^2&=-\frac{d}{dt}\{a(\|A^{1/2}u_m(t)\|^2)(A^{1/2}u_m(t),\, A^{1/2}u'_m(t))\}\\&+2a'(\|A^{1/2}u_m(t)\|^2)(A^{1/2}u_m(t),\, A^{1/2}u'_m(t))^2+a(\|A^{1/2}u_m(t)\|^2)\|A^{1/2}u'_m(t)\|^2\end{aligned}$$

and

$$\begin{aligned}&\int_0^t\|u''_m(\tau)\|^2d\tau+\frac{1}{2}\|A^{1/2}u'_m(t)\|^2\\&\leq\frac{1}{2}\|A^{1/2}u_1\|^2+a_1\|A^{1/2}u_0\|\|A^{1/2}u_1\|+a_1\|A^{1/2}u_m(t)\|\|A^{1/2}u'_m(t)\|\\&\quad+2a_2\int_0^t\|A^{1/2}u_m(\tau)\|^2\|A^{1/2}u'_m(\tau)\|^2d\tau+a_1\int_0^t\|A^{1/2}u'_m(\tau)\|^2d\tau,\end{aligned}$$

where $a_2=\max\{|a'(s)|;\, 0\leq s\leq C_2\}$. Hence, by (3.5), (3.7) and (3.13),

(3.16) $$\int_0^\infty\|u''_m(\tau)\|^2d\tau\leq C.$$

Assuming (H5) we proceed a priori estimates.

(VI) Similarly to (III), taking $w=A^2u_m(t)$ in (3.1) derives

(3.17) $$\|A^{3/2}u_m(t)\|\leq C$$

and

(3.18) $$\int_0^\infty a(\|A^{1/2}u_m(\tau)\|^2)\|A^{3/2}u_m(\tau)\|^2d\tau\leq C.$$

(VII) Also, similar estimates to (IV) yield

$$\|Au'_m(t)\| \leq C \tag{3.19}$$

and

$$\int_0^\infty \|A^{3/2}u'_m(\tau)\|^2 d\tau \leq C \tag{3.20}$$

by virtue of (3.18).

Now we differentiate (3.1) with respect to t and have

$$\begin{aligned}(u'''_m(t) + a(\|A^{1/2}u_m(t)\|^2)Au'_m(t) \\ + 2a'(\|A^{1/2}u_m(t)\|^2)(A^{1/2}u_m(t), A^{1/2}u'_m(t))Au_m(t), w) = 0.\end{aligned} \tag{3.21}$$

(VIII) Putting $w = 2Au''_m(t)$ in (3.21) we have

$$\begin{aligned}&\frac{d}{dt}[\|A^{1/2}u''_m(t)\|^2 + a(\|A^{1/2}u_m(t)\|^2)\|Au'_m(t)\|^2] + 2\|Au''_m(t)\|^2 \\ &= 2a'(\|A^{1/2}u_m(t)\|^2)(A^{1/2}u_m(t), A^{1/2}u'_m(t))(\|Au'_m(t)\|^2 - 2(Au_m(t), Au''_m(t)))\end{aligned}$$

or

$$\begin{aligned}&\frac{d}{dt}[\|A^{1/2}u''_m(t)\|^2 + a(\|A^{1/2}u_m(t)\|^2)\|Au'_m(t)\|^2] + \|Au''_m(t)\|^2 \\ &\leq C(\|Au'_m(t)\|^2 + \|A^{1/2}u'_m(t)\|^2)\end{aligned} \tag{3.22}$$

by (3.7), (3.10) and (3.13). Integrating (3.22) over $[0, t]$ leads to

$$\|A^{1/2}u''_m(t)\| \leq C \tag{3.23}$$

and

$$\int_0^\infty \|Au''_m(\tau)\|^2 d\tau \leq C. \tag{3.24}$$

Here $\|A^{1/2}u''_m(0)\|^2 \leq C$ follows from (3.1) and (H5).

(IX) (3.21) with (3.5), (3.7), (3.10), (3.14) and (3.24) yields

$$\int_0^\infty \|u'''_m(\tau)\|^2 d\tau \leq C. \tag{3.25}$$

§ 4. Proof of Theorem 1.

By virtue of a priori estimates (3.10), (3.13), (3.14) and (3.25), we may extract a subsequence $\{u_\mu\}$ of $\{u_m\}$ with the properties

(4.1) $\quad u_\mu \to u$ in $L^\infty(0, \infty; D(A))$ weakly*

(4.2) $\quad u'_\mu \to u'$ in $L^\infty(0, \infty; D(A^{1/2})) \cap L^2(0, \infty; D(A))$ weakly*

(4.3) $\quad u''_\mu \to u''$ in $L^2(0, \infty; H)$ weakly

and

(4.4) $$a(\|A^{1/2}u_\mu(\cdot)\|^2)Au_\mu \longrightarrow \text{ in } L^\infty(0, \infty; H) \text{ weakly*}$$

for some u and χ.

We must show $\chi = a(\|A^{1/2}u(\cdot)\|^2)Au$. For any $\phi \in C_0(0, \infty; H)$ and $T>0$ we have

$$\begin{aligned}\int_0^T (\chi - a(\|A^{1/2}u(\tau)\|^2)Au(\tau), \phi)dt &= \int_0^T (\chi - a(\|A^{1/2}u_\mu(\tau)\|^2)Au_\mu(\tau), \phi)d\tau \\ &+ \int_0^T a(\|A^{1/2}u(\tau)\|^2)(Au_\mu(\tau) - Au(\tau), \phi)d\tau \\ &+ \int_0^T (a(\|A^{1/2}u_\mu(\tau)\|^2) - a(\|A^{1/2}u(\tau)\|^2))(Au_\mu(\tau), \phi)d\tau.\end{aligned} \tag{4.5}$$

But, by the mean value theorem and by (4.1) and (H2),

$$\begin{aligned}&|\text{the last term in (4.5)}| \\ &\le C\int_0^T a_2|(A^{1/2}u_\mu(\tau) - A^{1/2}u(\tau), A^{1/2}u_\mu(\tau) + A^{1/2}u(\tau))|d\tau \\ &\le C\int_0^T \|A^{1/2}u_\mu(\tau) - A^{1/2}u(\tau)\|d\tau \to 0 \qquad \text{as } \mu \to \infty.\end{aligned}$$

The other integrals in right-hand side in (4.5) also tend to zero. The arbitrariness of ϕ and T implies $\chi = a(\|A^{1/2}u(\cdot)\|^2)Au$. Hence we get (2.1) for almost all t. The initial conditions (2.3) follow from (4.3) and the next Lemma 1.

Lemma 1 (Ball [2]). *Let X be a Banach space. If $f \in L^2(0, T; X)$ and $f' \in L^2(0, T; X)$, then f, possibly after redefinition on a set of measure zero, is continuous from $[0, T]$ to X.*

Thus the existence part of Theorem 1 is completed.

In order to prove the uniqueness part, let u and v be two solutions of (2.1)–(2.3). Then $w \equiv u - v$ satisfies

$$\begin{aligned}&w''(t) + a(\|A^{1/2}u(t)\|^2)Aw(t) + Aw'(t) \\ &\quad = -[a(\|A^{1/2}u(t)\|^2) - a(\|A^{1/2}v(t)\|^2)]Av(t)\end{aligned} \tag{4.6}$$

and

$$(4.7) \qquad w(0)=0, \qquad w'(0)=0.$$

Applying $2w'(t)+\gamma w(t)$, $0<\gamma<\lambda_1^2$, to (4.6)–(4.7) and integrating it over $[0, t]$, we have

$$\begin{aligned}
&[\|w'(t)\|^2+a(\|A^{1/2}u(t)\|^2)\|A^{1/2}w\|^2+\frac{\gamma}{2}\|A^{1/2}w(t)\|^2+\gamma(w'(t), w(t))] \\
&\quad+\int_0^t [2\|A^{1/2}w'(\tau)\|^2-\gamma\|w'(\tau)\|^2+a(\|A^{1/2}u(\tau)\|^2)\|A^{1/2}w(\tau)\|^2]d\tau \\
(4.8) \quad &=\int_0^t [2a'(\|A^{1/2}u(\tau)\|^2)(A^{1/2}u(\tau), A^{1/2}u'(\tau))\|A^{1/2}w(\tau)\|^2 \\
&\quad-\{a(\|A^{1/2}u(\tau)\|^2)-a(\|A^{1/2}v(\tau)\|^2)\}\{2(Av(\tau), w'(\tau))+(A^{1/2}v(\tau), A^{1/2}w(\tau))\}]d\tau \\
&\leq C\int_0^t [\|w'(\tau)\|^2+\|A^{1/2}w(\tau)\|^2]d\tau
\end{aligned}$$

since

$$|a(\|A^{1/2}u(\tau)\|^2)-a(\|A^{1/2}v(\tau)\|^2)|\leq C(\|A^{1/2}u(\tau)\|+\|A^{1/2}v(\tau)\|)\|A^{1/2}w(\tau)\|.$$

We estimate the left-hand side of (4.8) from below:

$$\begin{aligned}
(\text{first term}) &\geq \|w'(t)\|^2+\frac{\gamma}{2}\|A^{1/2}w(t)\|^2-\frac{\gamma}{2}\left(\frac{2}{\lambda_1^2}\|w'(t)\|^2+\frac{\lambda_1^2}{2}\|w(t)\|^2\right) \\
&\geq c_0[\|w'(t)\|^2+\|A^{1/2}w(t)\|^2], \qquad c_0=\min(1-\gamma/\lambda_1^2, \gamma/4)
\end{aligned}$$

and

$$(\text{second term})\geq\int_0^t (2\lambda_1^2-\gamma)\|w'(\tau)\|^2 d\tau\geq 0.$$

Therefore (4.8) yields

$$c_0[\|w'(t)\|^2+\|A^{1/2}w(t)\|^2]\leq C\int_0^t [\|w'(\tau)\|^2+\|A^{1/2}w(\tau)\|^2]d\tau$$

which concludes $w\equiv 0$.

Thus Theorem 1 is completed.

§ 5. Proofs of Theorem 2 and 3.

Proof of Theorem 2. The unique solution $u(t)$ in Theorem 1 satisfies

(5.1) $\quad u \in L^\infty(0, \infty; D(A^{3/2}))$

(5.2) $\quad u' \in L^\infty(0, \infty; D(A))\cap L^2(0, \infty; D(A^{3/2}))$

(5.3) $\quad u'' \in L^\infty(0, \infty; D(A^{1/2}))\cap L^2(0, \infty; D(A))$

and

$$(5.4) \qquad u''' \in L^2(0, \infty; H)$$

by (3.16)–(3.19) and (3.22)–(3.24). Combining Lemma 1 and (5.1)–(5.4) implies $u \in C^2([0, \infty); H)$. Q.E.D.

Proof of Theorem 3. Putting $w = A^{2k-1}u_m(t)$, $k=2, 3, \cdots$, in (3.1) and integrating it from 0 to t, we obtain

$$(5.5) \qquad \|A^k u_m(t)\| \leq C$$

and

$$(5.6) \qquad \int_0^\infty a(\|A^{1/2}u_m(\tau)\|^2)\|A^k u_m(\tau)\|^2 d\tau \leq C$$

by the similar way to (III). Then, similarly to (IV), we get

$$(5.7) \qquad \|A^{k-1/2}u_m'(t)\|^2 \leq C$$

and

$$(5.8) \qquad \int_0^\infty \|A^k u_m'(\tau)\|^2 d\tau \leq C,$$

using (5.6). Hence Theorem 3 follows from (5.5), (5.7) and (5.8) Q.E.D.

§ 6. Proof of Theorem 4.

From Theorem 2 it suffices to obtain the apriori estimates (2.5)–(2.9) to $u_m(t)$, which satisfies

$$(6.1) \qquad (u_m''(t) + \|A^{1/2}u_m(t)\|^2 A u_m(t) + A u_m'(t), w) = 0 \qquad \text{for any } w \in V^m$$

together with (3.2). Since $a(s)=s$, (3.5) and (3.8) give

$$(6.2) \qquad \int_0^t \|A^{1/2}u'(\tau)\|^2 d\tau \leq C$$

and

$$(6.3) \qquad \int_0^t \|A^{1/2}u(\tau)\|^4 d\tau \leq C.$$

(From now on we abbreviate the suffix m for simplicity.)

Multiplying (3.3) by $(1+t)$ and integrating it over $[0, t]$, we have

$$\frac{d}{dt}\Big[(1+t)\Big(\|u'(t)\|^2+\frac{1}{2}\|A^{1/2}u(t)\|^4\Big)\Big]+2(1+t)\|A^{1/2}u'(t)\|^2$$
$$=\|u'(t)\|^2+\frac{1}{2}\|A^{1/2}u(t)\|^4$$

and

$$(6.4)\quad \begin{aligned}&(1+t)\Big(\|u'(t)\|^2+\frac{1}{2}\|A^{1/2}u(t)\|^4\Big)+2\int_0^t(1+\tau)\|A^{1/2}u'(\tau)\|^2d\tau\\ &\le\|u_1\|^2+\frac{1}{2}\|u_0\|^4+\int_0^t\Big(\|u'(\tau)\|^2+\frac{1}{2}\|A^{1/2}u(\tau)\|^4\Big)d\tau.\end{aligned}$$

By virtue of (6.2), (6.3) and (H5), (6.4) gives

$$(6.5)\qquad (1+t)\|u'(t)\|^2\le C$$

$$(6.6)\qquad (1+t)\|A^{1/2}u(t)\|^4\le C$$

and

$$(6.7)\qquad \int_0^t(1+\tau)\|A^{1/2}u'(\tau)\|^2d\tau\leqq C.$$

Multiplying (3.6) by $(1+t)^{1/2-\varepsilon}$ for small positive constant ε, we have

$$(1+t)^{1/2-\varepsilon}\|A^{1/2}u(t)\|^4+\frac{1}{2}\frac{d}{dt}(1+t)^{1/2-\varepsilon}\|A^{1/2}u(t)\|^2$$
$$=\frac{1}{2}(1/2-\varepsilon)(1+t)^{-1/2-\varepsilon}\|A^{1/2}u(t)\|^2+(1+t)^{1/2-\varepsilon}\|u'(t)\|^2$$
$$-\frac{d}{dt}(1+t)^{1/2-\varepsilon}(u'(t),u(t))+(1/2-\varepsilon)(1+t)^{-1/2-\varepsilon}(u'(t),u(t))$$

which yields, by integration over $[0, t]$ and by (6.2)–(6.3), (6.5)–(6.7),

$$\begin{aligned}&\int_0^t(1+\tau)^{1/2-\varepsilon}\|A^{1/2}u(\tau)\|^4d\tau+\frac{1}{2}(1+t)^{1/2-\varepsilon}\|A^{1/2}u(t)\|^2\\
&\le\frac{1}{2}\|A^{1/2}u_0\|^2+\|u_1\|\|u_0\|+(1+t)^{1/2-\varepsilon}\|u'(t)\|\|u(t)\|+\frac{1}{4}\int_0^t(1+\tau)^{-1/2-\varepsilon}\|A^{1/2}u(\tau)\|^2d\tau\\
&\quad+\int_0^t(1+\tau)^{1/2-\varepsilon}\|u'(\tau)\|^2d\tau+\frac{1}{2}\int_0^t(1+\tau)^{-1/2-\varepsilon}\|u'(\tau)\|\|u(\tau)\|d\tau\\
&\le C+\{(1+t)\|u'(t)\|^2\}^{1/2}\|u(t)\|+\frac{1}{\lambda_1^2}\int_0^t(1+\tau)\|A^{1/2}u'(\tau)\|^2d\tau\\
&\quad+\frac{1}{4}\Big(\int_0^\infty(1+\tau)^{-1-2\varepsilon}d\tau\Big)^{1/2}\Big(\int_0^\infty\|A^{1/2}u(\tau)\|^4d\tau\Big)^{1/2}\\
&\quad+\frac{1}{2\lambda_1}\max_\tau\|u(\tau)\|\cdot\Big(\int_0^\infty(1+\tau)^{-1-2\varepsilon}d\tau\Big)^{1/2}\Big(\int_0^\infty\|A^{1/2}u'(\tau)\|^2d\tau\Big)^{1/2}\le C_{2\varepsilon}.\end{aligned}$$

Here we denote by C_ε the constant depending only upon

$$\int_0^\infty (1+\tau)^{-1-\varepsilon} d\tau.$$

Hence we obtain

$$\int_0^t (1+\tau)^{1/2-\varepsilon} \|A^{1/2}u(\tau)\|^4 d\tau \le C_{2\varepsilon}. \tag{6.8}$$

By induction we shall show

$$\int_0^t (1+\tau)^{1-\varepsilon} \|A^{1/2}u(\tau)\|^4 d\tau \le C_{\varepsilon/2} \tag{6.9}$$

$$(1+t)^{1-\varepsilon} \|A^{1/2}u(t)\|^2 \le C_{\varepsilon/2}, \tag{6.10}$$

the latter of which gives (2.5). For $N \ge 1$ we assume

$$\int_0^t (1+\tau)^{1-2^{-N}-\varepsilon/2} \|A^{1/2}u(\tau)\|^4 d\tau \le C_{\varepsilon/2} \tag*{(6.11)$_N$}$$

$$(1+t)^{1-2^{-N}-\varepsilon/2} \|A^{1/2}u(t)\|^2 \le C_{\varepsilon/2}. \tag*{(6.12)$_N$}$$

Note that $(6.11)_1$ and $(6.12)_1$ are valid from (6.8) and (6.6). Multiplying (3.6) by $(1+t)^{1-2^{-N-1}-\varepsilon/2}$ and integrating it over $[0, t]$, we get

$$\begin{aligned}
&(1+t)^{1-2^{-N-1}-\varepsilon/2} \|A^{1/2}u(t)\|^4 + \frac{1}{2}\frac{d}{dt}(1+t)^{1-2^{-N-1}-\varepsilon/2} \|A^{1/2}u(t)\|^2 \\
&= \frac{1}{2}(1-2^{-N-1}-\varepsilon/2)(1+t)^{-2^{-N-1}-\varepsilon/2} \|A^{1/2}u(t)\|^2 \\
&\quad + (1+t)^{1-2^{-N-1}-\varepsilon/2} \|u'(t)\|^2 - \frac{d}{dt}(1+t)^{1-2^{-N-1}-\varepsilon/2}(u'(t), u(t)) \\
&\quad + (1-2^{-N-1}-\varepsilon/2)(1+t)^{-2^{-N-1}-\varepsilon/2}(u'(t), u(t))
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t (1+\tau)^{1-2^{-N-1}-\varepsilon/2} \|A^{1/2}u(\tau)\|^4 d\tau + \frac{1}{2}(1+t)^{1-2^{-N-1}-\varepsilon/2} \|A^{1/2}u(t)\|^2 \\
&\le \frac{1}{2}\|A^{1/2}u_0\|^2 + \|u_1\|\|u_0\| + \frac{1}{2}\left(\int_0^\infty (1+\tau)^{-1-\varepsilon/2} d\tau\right)^{1/2} \\
&\quad \times \left(\int_0^\infty (1+\tau)^{-1-2^{-N}-\varepsilon/2} \|A^{1/2}u(\tau)\|^4 d\tau\right)^{1/2} + \frac{1}{\lambda_1^2}\int_0^\infty (1+\tau)\|A^{1/2}u'(\tau)\|^2 d\tau \\
&\quad + \frac{1}{\lambda_1}\{(1+t)\|u'(t)\|^2\}^{1/2}\{(1+t)^{1-2^{-N}-\varepsilon/2}\|A^{1/2}u(t)\|^2\}^{1/2} \\
&\quad + \frac{1}{\lambda_1}\sup_\tau \|u(\tau)\| \cdot \left(\int_0^\infty (1+\tau)^{-1-\varepsilon} d\tau\right)^{1/2} \left(\int_0^\infty (1+\tau)\|A^{1/2}u'(\tau)\|^2 d\tau\right)^{1/2} \le C_{\varepsilon/2}.
\end{aligned}$$

This shows $(6.11)_{N+1}$ and $(6.12)_{N+1}$. Thus we obtain (6.9) and (6.10) by selecting N so that $2^{-N}<\varepsilon/2$.

Also multiplying (3.3) by $(1+t)^{2-\varepsilon}$ and integrating it over $[0, t]$ yields

$$\begin{aligned}&(1+t)^{2-\varepsilon}\left(\|u'(t)\|^2+\frac{1}{2}\|A^{1/2}u(t)\|^4\right)+2\int_0^t(1+\tau)^{2-\varepsilon}\|A^{1/2}u'(\tau)\|^2d\tau\\&\quad\leq\|u_1\|^2+\frac{1}{2}\|A^{1/2}u_0\|^4+2\int_0^\infty(1+\tau)^{1-\varepsilon}\left(\|u'(\tau)\|^2+\frac{1}{2}\|A^{1/2}u(\tau)\|^4\right)d\tau\end{aligned}$$

which, combined with (6.7), (6.9), gives

$$(1+t)^{1-\varepsilon}\|u'(t)\|\leq C_{\varepsilon/4} \tag{6.13}$$

$$\int_0^t(1+\tau)^{2-\varepsilon}\|A^{1/2}u'(\tau)\|^2d\tau\leq C_{\varepsilon/2}. \tag{6.14}$$

Multiplying (3.14) by $(1+t)^{3-\varepsilon}$ and integrating it over $[0, t]$ we have

$$\begin{aligned}&(1+t)^{3-\varepsilon}\|u''(t)\|^2+\frac{1}{2}\frac{d}{dt}(1+t)^{3-\varepsilon}\|A^{1/2}u'(t)\|^2\\&=\frac{1}{2}(3-\varepsilon)(1+t)^{2-\varepsilon}\|A^{1/2}u'(t)\|^2-\frac{d}{dt}\{(1+t)^{3-\varepsilon}\|A^{1/2}u(t)\|^2(Au(t), u'(t))\}\\&\quad+(3-\varepsilon)(1+t)^{2-\varepsilon}\|A^{1/2}u(t)\|^2(A^{1/2}u(t), A^{1/2}u'(t))\\&\quad+(1+t)^{3-\varepsilon}\{2(A^{1/2}u(t), A^{1/2}u'(t))^2+\|A^{1/2}u(t)\|^2\|A^{1/2}u'(t)\|^2\}\end{aligned}$$

and, by virtue of (6.9), (6.10), (6.13), (6.14),

$$\begin{aligned}&\int_0^t(1+\tau)^{3-\varepsilon}\|u''(\tau)\|^2d\tau+\frac{1}{2}(1+t)^{3-\varepsilon}\|A^{1/2}u'(t)\|^2\\&\quad\leq\frac{1}{2}\|A^{1/2}u_1\|^2+\|A^{1/2}u_0\|^3\|A^{1/2}u_1\|+\frac{3}{2}\int_0^\infty(1+\tau)^{2-\varepsilon}\|A^{1/2}u'(\tau)\|^2d\tau\\&\qquad+\{(1+t)^{1-\varepsilon/3}\|A^{1/2}u(t)\|^2\}^{3/2}(1+t)^{3/2-\varepsilon/2}\|A^{1/2}u'(t)\|\\&\qquad+3\{\sup_\tau(1+\tau)^{1-2\varepsilon/3}\|A^{1/2}u(\tau)\|^2\}^{1/2}\left(\int_0^\infty(1+\tau)^{1-2\varepsilon/3}\|A^{1/2}u(\tau)\|^4d\tau\right)^{1/2}\\&\qquad\times\left(\int_0^\infty(1+\tau)^{2-2\varepsilon/3}\|A^{1/2}u'(\tau)\|^2d\tau\right)^{1/2}+3\sup_\tau\{(1+\tau)^{1-\varepsilon/2}\|A^{1/2}u(\tau)\|^2\}\\&\qquad\times\int_0^\infty(1+\tau)^{2-\varepsilon/2}\|A^{1/2}u'(\tau)\|^2d\tau\\&\quad\leq C_{\varepsilon/4}+C_{\varepsilon/6}[\sup_t\{(1+t)^{3-\varepsilon}\|A^{1/2}u'(t)\|^2\}]^{1/2}.\end{aligned}$$

This implies

$$\int_0^\infty(1+\tau)^{3-\varepsilon}\|u''(\tau)\|^2d\tau\leq C_{3/6} \tag{6.15}$$

(6.16) $$(1+t)^{3/2-\varepsilon}\|A^{1/2}u'(t)\|\leq C_{\varepsilon/12}.$$

(6.16) gives (2.7).

In (3.21), putting $w=2(1+t)^{4-\varepsilon}u''(t)$ and integrating it from 0 to t, we obtain

$$\frac{d}{dt}[(1+t)^{4-\varepsilon}(\|u''(t)\|^2+\|A^{1/2}u(t)\|^2\|A^{1/2}u'(t)\|^2)]+2(1+t)^{4-\varepsilon}\|A^{1/2}u''(t)\|^2$$
$$=(4-\varepsilon)(1+t)^{3-\varepsilon}(\|u''(t)\|^2+\|A^{1/2}u(t)\|^2\|A^{1/2}u'(t)\|^2)$$
$$+2(1+t)^{4-\varepsilon}(Au(t),u'(t))(\|A^{1/2}u'(t)\|^2-2(Au(t),u''(t)))$$

and

$$(1+t)^{4-\varepsilon}[\|u''(t)\|^2+\|A^{1/2}u(t)\|^2\|A^{1/2}u'(t)\|^2]+2\int_0^t(1+\tau)^{4-\varepsilon}\|A^{1/2}u''(\tau)\|^2d\tau$$
$$\leq\|u''(0)\|^2+\|A^{1/2}u_0\|^2\|A^{1/2}u_1\|^2+4\int_0^\infty(1+\tau)^{3-\varepsilon}\|u''(\tau)\|^2d\tau$$
$$+4\sup_\tau\{(1+\tau)^{1-\varepsilon/2}\|A^{1/2}u(\tau)\|^2\}\int_0^\infty(1+\tau)^{2-\varepsilon/2}\|A^{1/2}u'(\tau)\|^2d\tau$$
$$+2[\sup_\tau\{(1+\tau)^{1-2\varepsilon/3}\|A^{1/2}u'(\tau)\|^2\}]^{1/2}\sup_\tau\{(1+\tau)^{3/2-\varepsilon/3}\|A^{1/2}u'(\tau)\|\}$$
$$\times\int_0^\infty(1+\tau)^{2-\varepsilon/3}\|A^{1/2}u'(\tau)\|^2d\tau+4\sup_\tau\{(1+\tau)^{1-\varepsilon/6}\|A^{1/2}u(\tau)\|^2\}$$
$$\times\sup_\tau\{(1+\tau)^{3/2-\varepsilon/6}\|A^{1/2}u'(\tau)\|\}\left(\int_0^\infty(1+\tau)^{-1-\varepsilon/3}d\tau\right)^{1/2}$$
$$\times\left(\int_0^t(1+\tau)^{4-\varepsilon}\|A^{1/2}u''(\tau)\|^2d\tau\right)^{1/2}$$
$$\leq C_{\varepsilon/9}+C_{\varepsilon/18}\left(\int_0^t(1+\tau)^{4-\varepsilon}\|A^{1/2}u''(\tau)\|^2d\tau\right)^{1/2}$$

which gives (2.8), (2.9).

Finally we show (2.6). In fact, by the Cauchy-Schwarz inequality

$$\|Au\|^4=(\sum\lambda_j^4g_{jm}^2)^2\leq(\sum\lambda_j^2g_{jm}^2)(\sum\lambda_j^6g_{jm}^2)=\|A^{1/2}u\|^2\|A^{3/2}u\|^2$$

and hence, by (6.9) and (3.17),

$$(1+t)^{1-\varepsilon}\|Au(t)\|^4\leq(1+t)^{1-\varepsilon}\|A^{1/2}u(t)\|^2\cdot\|A^{3/2}u(t)\|^2\leq C_{\varepsilon/2}.$$

Thus the proof of Theorem 4 is completed. Q.E.D.

§ 7. Periodic solution.

In this section we seek the ω-periodic solution $u(t)$ to the equation

(7.1) $$u''(t)+\|A^{1/2}u(t)\|^2Au(t)+Au'(t)=f(t)\qquad\text{in } H$$

where $f:(-\infty, \infty)\rightarrow H$ is an ω-periodic function. When X is a Banach space with norm $\|\cdot\|_X$, $L^p(\omega; X)$ means the space of functions $f(t)\in X$ for each $t\in(-\infty, \infty)$ and periodic, with period ω, such that $\int_0^\omega \|f(t)\|_X^p\,dt<\infty$, equipped with norm

$$\|f\|_{L^p(\omega;\,X)}=\left(\int_0^\omega \|f(t)\|_X^p\,dt\right)^{1/p}\qquad \text{for } 1\leq p<\infty.$$

In the case $p=\infty$, the norm should be

$$\|f\|_{L^\infty(\omega;\,X)}=\operatorname*{ess.\,sup}_{0\leq t\leq\omega}\|f(t)\|_X.$$

Then our theorem is stated as follow.

Theorem 5. *The conditions* (H1) *and* (H2) *are assumed. If* $f\in L^\infty(\omega; H)\cap L^2(\omega; D(A))$, *then there exists an ω-periodic solution $u(t)$ to* (7.1) *satisfying*

$$u\in C^1(\omega; D(A)) \quad \textit{and} \quad u''\in L^\infty(\omega; H).$$

Proof. We employ Galerkin's method combined with the fixed point theorem. Leray-Schauder's degree theorem yields that for each constant ε, $0<\varepsilon<1$, there exists a solution

$$u_{m,\varepsilon}(t)=\sum_{j=1}^m g_{jm,\varepsilon}(t)w_j \qquad \text{in } V^m$$

of the ordinary differential system

$$(u''_{m,\varepsilon}(t)+\varepsilon Au_{m,\varepsilon}(t)+\|A^{1/2}u_{m,\varepsilon}(t)\|^2Au_{m,\varepsilon}(t)+Au'_{m,\varepsilon}(t),\, w)=(f(t),\, w) \quad \text{for any } w\in V^m \tag{7.2}$$

with periodic conditions

$$u_{m,\varepsilon}(t)=u_{m,\varepsilon}(t+\omega), \qquad u'_{m,\varepsilon}(t)=u'_{m,\varepsilon}(t+\omega). \tag{7.3}$$

(For the details see Clement [4], Kakita [13]).

We require a priori estimates to $u_{m,\varepsilon}$ independent of m and ε, and briefly denote $u_{m,\varepsilon}$ by u_m without confusions.

Putting $w=2u'_m$, in (7.2) we have

$$\begin{aligned}&\frac{d}{dt}\left(\|u'_m(t)\|^2+\varepsilon\|A^{1/2}u_m(t)\|^2+\frac{1}{2}\|A^{1/2}u_m(t)\|^4\right)+2\|A^{1/2}u'_m(t)\|^2\\&=2(f(t), u'_m(t))\leq\frac{1}{\lambda_1^2}\|f(t)\|^2+\|A^{1/2}u'_m.(t)\|^2\end{aligned}\tag{7.4}$$

and

$$\int_{\omega} \|A^{1/2}u'_m(\tau)\|^2 d\tau \leq \frac{1}{\lambda_1^2}\int_{\omega} \|f(\tau)\|^2 d\tau \equiv C_3 \tag{7.5}$$

by integration of (7.4) over $[t, t+\omega]$ and (7.3), where we denote $\int_t^{t+\omega} = \int_{\omega}$ for simplicity. (In this section several constants C and C_i are independent of t, m and ε.) If we put $w = u_m$ in (7.2), then we have

$$\frac{d}{dt}(u'_m(t), u_m(t)) - \|u'_m(t)\|^2 + \varepsilon\|A^{1/2}u_m(t)\|^2 + \|A^{1/2}u_m(t)\|^4$$
$$+\frac{1}{2}\frac{d}{dt}\|A^{1/2}u_m(t)\|^2 = (f(t), u_m(t))$$

and

$$\int_{\omega} \|A^{1/2}u_m(\tau)\|^4 d\tau \leq C_3/\lambda_1^2 + \frac{1}{2\lambda_1}\int_{\omega} \|f(\tau)\|^2 d\tau + \frac{\sqrt{\omega}}{2\lambda_1}\left(\int_{\omega} \|A^{1/2}u_m(\tau)\|^4 d\tau\right)^{1/2}$$

and hence

$$\int_{\omega} \|A^{1/2}u_m(\tau)\|^4 d\tau \leq C. \tag{7.6}$$

By virtue of (7.5) and (7.6) there exists $t_0 \in [0, \omega]$ such that

$$\|u'_m(t_0)\|, \qquad \|A^{1/2}u_m(t_0)\| \leq C$$

from which and (7.4) it follows that, for $t \in [t_0, t_0+\omega]$,

$$\|u'_m(t)\|^2 + \varepsilon\|A^{1/2}u_m(t)\|^2 + \frac{1}{2}\|A^{1/2}u_m(t)\|^4 + \int_{t_0}^{t} \|A^{1/2}u'_m(\tau)\|^2 d\tau$$
$$\leq \|u'_m(t_0)\|^2 + \varepsilon\|A^{1/2}u_m(t_0)\|^2 + \frac{1}{2}\|A^{1/2}u_m(t_0)\|^4 + \frac{1}{\lambda_1^2}\int_{t_0}^{t} \|f(\tau)\|^2 d\tau,$$

that is,

$$\|u'_m(t)\| \leq C \quad \text{and} \quad \|A^{1/2}u_m(t)\|^2 \leq C_4. \tag{7.7}$$

Taking $w = Au_m$ in (7.2) and integrating it over $[t, t+\omega]$, we have

$$\begin{aligned}&\varepsilon\|Au_m(t)\|^2 + \|A^{1/2}u_m(t)\|^2\|Au_m(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|Au_m(t)\|^2 \\ &= (A^{1/2}f(t), A^{1/2}u_m(t)) + \|A^{1/2}u'_m(t)\|^2 - \frac{d}{dt}(A^{1/2}u'_m(t), A^{1/2}u_m(t))\end{aligned} \tag{7.8}$$

and

$$(7.9)\quad \int_\omega \|A^{1/2}u_m(\tau)\|^2\|Au_m(\tau)\|^2 d\tau \leq \int_\omega \|A^{1/2}u'_m(\tau)\|^2 d\tau + \int_\omega (A^{1/2}f(\tau), A^{1/2}u_m(\tau))d\tau \leq C$$

by (7.5) and (7.6). Putting $w=2Au'_m$ in (7.2) we have

$$\begin{aligned}&\frac{d}{dt}(\|A^{1/2}u'_m(t)\|^2+\varepsilon\|Au_m(t)\|^2)+2\|Au'_m(t)\|^2\\ &\quad=2(f(t), Au'_m(t))-2\|A^{1/2}u_m(t)\|^2(Au_m(t), Au'_m(t))\\ &\quad\leq \|A^{1/2}f(t)\|^2+\|A^{1/2}u'_m(t)\|^2+\|A^{1/2}u_m(t)\|^2\Big(C_4\|Au_m(t)\|^2+\frac{1}{C_4}\|Au'_m(t)\|^2\Big)\end{aligned}$$

and

$$(7.10)\quad \begin{aligned}&\frac{d}{dt}(\|A^{1/2}u'_m(t)\|^2+\varepsilon\|Au_m(t)\|^2)+\|Au'_m(t)\|^2\\ &\quad\leq \|A^{1/2}f(t)\|^2+\|A^{1/2}u'_m(t)\|^2+C_4\|A^{1/2}u_m(t)\|^2\|Au_m(t)\|^2.\end{aligned}$$

Then, integrating (7.10) over $[t, t+\omega]$ and using (7.7), (7.9), we have

$$(7.11)\quad \int_\omega \|Au'_m(\tau)\|^2 d\tau \leq C$$

which implies $\|Au'_m(t_1)\|\leq C$ for some $t_1\in[0, \omega]$. Integrating (7.10) from t_1 to t, $t\in[t_1, t_1+\omega]$, we obtain

$$(7.12)\quad \|A^{1/2}u'_m(t)\|\leq C.$$

Putting $w=A^2u_m$ in (7.2) and integrating it over $[t, t+\omega]$, we get

$$(7.13)\quad \begin{aligned}&\varepsilon\int_\omega \|A^{3/2}u_m(\tau)\|^2 d\tau+\int_\omega \|A^{1/2}u_m(\tau)\|^2\|A^{3/2}u_m(\tau)\|^2 d\tau\\ &\quad\leq \frac{1}{2}\int_\omega \|Af(\tau)\|^2 d\tau+\frac{1}{2}\int_\omega \|Au_m(\tau)\|^2 d\tau+\int_\omega \|Au'_m(\tau)\|^2 d\tau.\end{aligned}$$

By virtue of the Cauchy-Schwarz inequality it holds

$$\|Au_m\|^2\leq \|A^{1/2}u_m\|\,\|A^{3/2}u_m\|.$$

Hence

$$\int_\omega \|Au_m(\tau)\|^4 d\tau \leq \frac{1}{2}\int_\omega \|Af(\tau)\|^2 d\tau+\frac{\sqrt{\omega}}{2}\Big(\int_\omega \|Au_m(\tau)\|^4 d\tau\Big)^{1/2}+\int_\omega \|Au'_m(\tau)\|^2 d\tau$$

and

$$(7.14)\quad \int_\omega \|Au_m(\tau)\|^4 d\tau \leq C.$$

It also follows from (7.13) that

$$\int_{\omega} \varepsilon \| A^{3/2}u_m(\tau)\|^2 d\tau + \int_{\omega} \| A^{1/2}u_m(\tau)\|^2 \| A^{3/2}u_m(\tau)\|^2 d\tau \leq C. \tag{7.15}$$

From (7.14) there exists $t_2 \in [0, \omega]$ such that $\| Au_m(t_2)\| \leq C$. If we consider (7.8) as the first order differential equation in $\| Au_m(t)\|^2$ with the initial value $\| Au_m(t_2)\|^2$ at $t=t_2$, then we have for t, $t_2 \leq t \leq t_2+\omega$,

$$\begin{aligned} \| Au_m(t)\|^2 = e^{-P(t)}[\| Au_m(t_2)\|^2 - 2(A^{1/2}u'_m(t_2), A^{1/2}u_m(t_2))] \\ +2(A^{1/2}u'_m(t), A^{1/2}u_m(t)) + 2e^{-P(t)} \int_{t_2}^{t} e^{P(\tau)}\{\| A^{1/2}u'_m(\tau)\|^2 \\ -2(A^{1/2}u'_m(\tau), A^{1/2}u_m(\tau))\| A^{1/2}u_m(\tau)\|^2 + (A^{1/2}f(\tau), A^{1/2}u_m(\tau))\}d\tau \end{aligned}$$

where $P(t) = 2\int_{t_2}^{t} (\varepsilon + \| A^{1/2}u_m(\tau)\|^2)d\tau$. The estimates obtained above gives

$$\| Au_m(t)\| \leq C. \tag{7.16}$$

Next, if we replace $w = 2A^2u'_m$ in (7.2), then we have

$$\begin{aligned} \frac{d}{dt}(\| Au'_m(t)\|^2 + \varepsilon \| A^{3/2}u_m(t)\|^2) + 2\| A^{3/2}u'_m(t)\|^2 \\ = 2(Af(t), Au'_m(t)) - 2\| A^{1/2}u_m(t)\|^2 (A^{3/2}u_m(t), A^{3/2}u'_m(t)) \end{aligned} \tag{7.17}$$

and

$$2\int_{\omega} \| A^{3/2}u'_m(\tau)\|^2 d\tau \leq C_4 \int_{\omega} \| A^{1/2}u_m(\tau)\|^2 \| A^{3/2}u_m(\tau)\|^2 d\tau + \int_{\omega} (\| Af(\tau)\|^2 + \| Au'_m(\tau)\|^2) d\tau.$$

By (7.15)

$$\int_{\omega} \| A^{3/2}u'_m(\tau)\|^2 d\tau \leq C$$

which means with (7.15) that there is some $t_3 \in [0, \omega]$ such that $\varepsilon \| A^{3/2}u_m(t_3)\| + \| A^{3/2}u'_m(t_3)\| \leq C$. Then, integrating (7.17) over $[t_3, t]$, $t \in [t_3, t_3+\omega]$, we have

$$\| Au'_m(t)\| \leq C. \tag{7.18}$$

Finally putting $w = u''_m$ in (7.2) then we have

$$\| u''_m(t)\| \leq C \tag{7.19}$$

since $f \in L^\infty(\omega; H)$.

We now pass $u_{m,\varepsilon}$ to the limit. From (7.7), (7.12), (7.16), (7.18) and (7.19) we may extract a subsequence $\{u_{\nu,\delta}\}$ of $\{u_{m,\varepsilon}\}$ such that for some u and χ

$$u_{\nu,\delta} \longrightarrow u \text{ in } L^\infty(\omega; D(A)) \text{ weakly*}$$
$$u'_{\nu,\delta} \longrightarrow u' \text{ in } L^\infty(\omega; D(A)) \text{ weakly*}$$
$$u''_{\nu,\delta} \longrightarrow u'' \text{ in } L^\infty(\omega; H) \text{ weakly*}$$

and

$$\|A^{1/2}u_{\nu,\delta}(\cdot)\|^2 Au_{\nu,\delta} \longrightarrow \chi \text{ in } L^\infty(\omega; H) \text{ weakly*}$$

as $\nu\to\infty$ and $\delta\to 0$. $\chi=\|A^{1/2}u(\cdot)\|^2Au$ is shown by the compact method as in the proof of Theorem 1. Hence this u is the solution to (7.1).

Thus the proof is completed Q.E.D.

We shall state the regularity theorem without proof.

Theorem 6. *In addition to the assumptions in Theorem* 5, $f' \in L^\infty(\omega; H)$ *is assumed. Then the solution* $u(t)$ *satisfies*

$$u \in C^1(\omega; D(A)) \cap C^2(\omega; D(A^{1/2})).$$

Moreover, if we assume $f^{(j)} \in L^\infty(\omega; D(A^k))$, $j, k=0, 1, 2, \cdots$, *then we have*

$$u \in C^\infty(\omega; D(A^{k'})), \qquad k'=0, 1, 2, \cdots.$$

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