Funkcialaj Ekvacioj, **35** (1992) 19–30

Energy Decay Rates in Linear Thermoelasticity

By

Jaime E. Muñoz Rivera

(National Laboratory of Scientific Computation and IMUFRJ, Brazil)

1. Introduction

C. Defermos [2] studied the classical linear thermoelastic system for inhomogeneous anisotropic materials and proved that there exists only one solution which is differentiable and asymptotically stable as $t \to +\infty$. Dassios and Grillakis [3] studied the decay of energy for an isotropic model in $\boldsymbol{R}^3$, and for which the authors divided into three parts, kinetic energy, strain energy and thermal energy. They concluded that whenever the initial data are smooth with compact support, then the three parts of the energy decay to zero as $t \to +\infty$ at the rate $t^{-(m+3/2)}$, for a suitable positive number m which depends on the initial data. Recently D. Henry, O. Lopes and A. Perisinotto [4], (see also [5]), showed that the three parts of the energy decay exponentially to zero in the unidimensional case but not for $n > 1$. The authors proved the asymptotic behaviour studing the essencial spectrum of the semigroup associated to the thermoelastic system. In special situations, that is, when the restoring force is proportional to the vector velocity of the displacement field *D.* Carvalho and G. P. Menzala [8] proved that in a bounded, isotropic and inhomogeneous medium the kinetic energy, the strain energy and the thermal energy approach exponentially to zero as $t \to +\infty$,

In this paper we study the one dimensional linear thermoelastic system and present a new proof, much elementary than the one given in [4]. We use basically the energy method, regularity results and some technical ideas to show that the total energy decays exponentially as $t \to +\infty$. The one dimensional linear thermoelastic system is given by

$$u_{tt} - u_{xx} + \alpha\theta_x = 0\,, \qquad 0 < x < L\,, \quad 0 < t < T\,, \tag{1.1}$$

$$\theta_t - \theta_{xx} + \beta u_{xt} = 0\,, \qquad 0 < x < L\,, \quad 0 < t < T\,, \tag{1.2}$$

with initial conditions

$$u(x, 0) = u_0(x)\,, \qquad u_t(x, 0) = u_1(x)\,, \qquad \theta(x, 0) = \theta_0(x) \qquad \text{in }]0, L[\tag{1.3}$$

and boundary conditions

$$u(0, t) = u(L, t) = \theta(0, t) = \theta(L, t) = 0 \qquad \forall t \in]0, T[\tag{1.4}$$

In (1.1) and (1.2) α and β are constants. The existence, uniqueness and regularity of the solution of the above system is well known by now and was due to the pioneer work of C. Dafermos [2]. We sketch the proof just for matter cf completeness. In section 3 we will prove that the energy of this system decay to zero at the rate $e^{-\gamma t}$ for a suitable positive constant γ as $t \to +\infty$. Our notations are standard and follow the terminology given in the book of J. L. Lions [6].

2. Existence, uniqueness and regularity

Let's denote by A the operator of $L^2(\Omega)$ defined by $Aw = -w_{xx}$, with domain

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega)$$

where $\Omega =]0, L[$. It's well known that A is a positive self adjoint operator in the Hilbert space $L^2(\Omega)$. Let's denote by B the operator defined by $Bw = w_x$, with domain $D(B) = H_0^1(\Omega)$. We denote by $\mathscr{H}$ the space $\mathscr{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, with norm

$$\|V\|_{\mathscr{H}}^2 = \int_0^L u_x^2 dx + \int_0^L v^2 dx + \int_0^L \theta^2 dx ,$$

where $V = (u, v, \theta)^t$. Let's define the operator $\mathscr{A} : D(\mathscr{A}) \subseteq \mathscr{H} \to \mathscr{H}$

$$\mathscr{A} = -\begin{bmatrix} 0 & -1 & 0 \\ A & 0 & \alpha B \\ 0 & \beta B & A \end{bmatrix}$$

with domain:

$$D(\mathscr{A}) = H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega)$$

Let $\mathscr{A}$ as above, then we have

Theorem 2.1. (Existence and uniqueness) *Let* $(u_0, u_1, \theta_0) \in D(\mathscr{A})$ *and* $T > 0$, *then for any* α, $\beta \in \boldsymbol{R}$, *there exist only one strong solution of system* (1.1), ..., (1.4) *satisfying*

$$u \in C(0, T; D(A)) \cap C^1(0, T; H_0^1(\Omega)) \cap C^2(0, T; L^2(\Omega))$$

$$\theta \in C(0, T; D(A)) \cap C^1(0, T; L^2(\Omega))$$

Proof. System (1.1), ..., (1.4) is equivalent to

$$\frac{d}{dt} U = \mathscr{A} U , \qquad U(0) = U_0 , \qquad U \in D(\mathscr{A})$$

where

$$U = \begin{bmatrix} u \\ u_t \\ \theta \end{bmatrix}, \qquad U_0 = \begin{bmatrix} u_0 \\ u_1 \\ \theta_0 \end{bmatrix}$$

It is sufficient to prove that $\mathscr{A}$ is the infinitesimal generator of a strongly continuous semigroup. In order to show this, let's define the operator $\mathscr{B} = -(\alpha - \beta)^2 I + \mathscr{A}$, with $D(\mathscr{B}) = D(\mathscr{A})$, where I denotes the identity operator in $\mathscr{H}$. First of all, note that $\mathscr{B}$ is dissipative, in fact, set $V = (u, v, \theta)^t$, then we have

$$\begin{aligned}(\mathscr{A}V, V)_{\mathscr{H}} &= (\alpha - \beta) \int_0^L v\theta_x dx - \int_0^L \theta_x^2 dx \\ &\leq \frac{1}{2}(\alpha - \beta)^2 \int_0^L v^2 dx - \frac{1}{2}\int_0^L \theta_x^2 dx \\ &\leq \frac{1}{2}(\alpha - \beta)^2 \|V\|_{\mathscr{H}}^2 - \frac{1}{2}\int_0^L \theta_x^2 dx\end{aligned}$$

Consequently

$$(\mathscr{B}V, V)_{\mathscr{H}} \leq -\frac{1}{2}(\alpha - \beta)^2 \|V\|_{\mathscr{H}}^2 - \frac{1}{2}\int_0^L \theta_x^2 dx \tag{2.1}$$

Our next goal is to prove that $\mathrm{Im}[(I - \mathscr{B}] = \mathscr{H}$. That is, for all $(F_1, F_2, F_3) \in \mathscr{H}$, there exist $V = (u, v, \theta) \in D(\mathscr{A}) = D(I - \mathscr{B})$ satisfying

$$\mu u - v = F_1 \tag{2.2}$$

$$\mu v - u_{xx} + \alpha\theta_x = F_2 \tag{2.3}$$

$$\mu\theta - \theta_{xx} + \beta v_x = F_3 \tag{2.4}$$

where $\mu = 1 + (\alpha + \beta)^2$. In fact, let's denote by $w_1, w_2, \ldots, w_m$ and by $\lambda_1, \lambda_2, \ldots, \lambda_m$ the first m eigenfunctions and eigenvalues of the operator A respectively. Set V_m to be the finite dimensional space generated by the first m eigenfunctions, and put $\mathscr{V}_m = V_m \times V_m \times V_m$. Then, the approximated problem is given by:

$$([I - \mathscr{B}]U^{(m)}, W_j)_{\mathscr{H}} = (F, W_j)_{\mathscr{H}}, \qquad j = 1, \ldots, 3m \tag{2.5}$$

where

$$U^{(m)} = (u^{(m)}, v^{(m)}, \theta^{(m)})^t, \qquad F = (F_1, F_2, F_3)^t \qquad \text{and} \qquad U^{(m)} = \sum_{i=1}^{3m} c_{i,m} W_i$$

and $\{W_i;\ i = 1, \dots, 3m\}$ is a basis of $\mathscr{V}_m$. By (2.1) we have that the matrix $(([I - \mathscr{B}]W_i, W_j)_{\mathscr{H}})_{3m\times 3m}$ is positive definite, thus system (2.5) has only one solution. Multiplying by $c_{j,m}$ and adding up in k we have that:

$$([I - \mathscr{B}]U^{(m)}, U^{(m)})_{\mathscr{H}} = (F, U^{(m)})_{\mathscr{H}}\,.$$

From (2.1) and the last relation we conclude that there exist a positive constant C such that:

$$\|U^{(m)}\|_{\mathscr{H}} \le C\|F\|_{\mathscr{H}} \tag{2.6}$$

$$\int_0^L |\theta_x^{(m)}|^2 dx \le C\|F\|_{\mathscr{H}}^2 \tag{2.7}$$

Denoting by

$$u^{(m)} = \sum_{i=1}^{m} a_{i,m} w_i\,, \qquad v^{(m)} = \sum_{i=1}^{m} b_{i,m} w_i\,, \qquad \theta^{(m)} = \sum_{i=1}^{m} c_{i,m} w_i$$

we conclude that system (2.5) is equivalent to

$$\mu\int_0^L u^{(m)} w_j dx - \int_0^L v^{(m)} w_j dx = \int_0^L F_1 w_j dx \tag{2.8}$$

$$\mu\int_0^L v^{(m)} w_j dx + \int_0^L u_{xx}^{(m)} w_j dx + \alpha\int_0^L \theta_x^{(m)} w_j dx = \int_0^L F_2 w_j dx \tag{2.9}$$

$$\mu\int_0^L \theta^{(m)} w_j dx + \int_0^L \theta_{xx}^{(m)} w_j dx + \beta\int_0^L v_x^{(m)} w_j dx = \int_0^L F_3 w_j dx \tag{2.10}$$

for $j = 1, \dots, m$. Let's multiply equation (2.8), (2.9) and (2.10) by $-b_j\lambda_j$, $a_j\lambda_j$ and $c_j\lambda_j$ respectively and add up in j, to obtain

$$-\mu\int_0^L u_x^{(m)} v_x^{(m)} dx + \int_0^L |v_x^{(m)}|^2 dx = -\int_0^L (F_1)_x v_x^{(m)} dx \tag{2.11}$$

$$\mu\int_0^L v^{(m)} u_{xx}^{(m)} dx + \int_0^L |u_{xx}^{(m)}|^2 dx + \alpha\int_0^L \theta_x^{(m)} u_{xx}^{(m)} dx = \int_0^L F_2 u_{xx}^{(m)} dx \tag{2.12}$$

$$\mu\int_0^L \theta^{(m)} \theta_{xx}^{(m)} dx + \int_0^L |\theta_{xx}^{(m)}|^2 dx + \beta\int_0^L v_x^{(m)} \theta_{xx}^{(m)} dx = \int_0^L F_3 \theta_{xx}^{(m)} dx \tag{2.13}$$

From (2.6) and (2.11) we conclude that there exists a positive constant C such that

$$\int_0^L |v_x^{(m)}|^2 dx \le C\|F\|_{\mathscr{H}}^2\,. \tag{2.14}$$

From (2.7), (2.12), (2.13) and (2.14) we conclude that there exists a positive constant C satisfying

$$\int_0^L |u_{xx}^{(m)}|^2 dx + \int_0^L |\theta_{xx}^{(m)}|^2 dx \leq C\|F\|_{\mathscr{H}}^2 . \tag{2.15}$$

From (2.15) it follows that there exist a subsequence of $(u^{(m)})_{m\in N}$ and $(\theta^{(m)})_{m\in N}$, (which we still denote in the same way) such that

$$u^{(m)} \to u \qquad \text{weakly in } D(A)$$

$$\theta^{(m)} \to \theta \qquad \text{weakly in } D(A)$$

From (2.8), ..., (2.10) and the above two convergences we conclude that u, v and θ satisfy

$$\mu\int_0^L uw_j dx - \int_0^L vw_j dx = \int_0^L F_1 w_j dx$$

$$\mu\int_0^L vw_j dx + \int_0^L u_{xx} w_j dx + \alpha\int_0^L \theta_x w_j dx = \int_0^L F_2 w_j dx$$

$$\mu\int_0^L \theta w_j dx + \int_0^L \theta_{xx} w_j dx + \beta\int_0^L v_x w_j dx = \int_0^L F_3 w_j dx$$

for all j. Since finite linear combinations of the eigenfunctions are dense in $L^2(\Omega)$, we conclude that u, v and θ satisfy (2.2), (2.3) and (2.4), that is $D(I - \mathscr{B}) = \mathscr{H}$. Finally since $D(\mathscr{B})$ is dense in $\mathscr{H}$, from Lummer-Phillips's theorem we conclude that $\mathscr{B}$ is the infinitesimal generator of a strongly continuous semigroup, so is $\mathscr{A}$, hence the result follows □

Remark 2.1. If we define $D(\mathscr{A}^2)$ and $D(\mathscr{A}^3)$ as:

$$D(\mathscr{A}^2) = \{V \in D(\mathscr{A});\ \mathscr{A}V \in D(\mathscr{A})\} ,$$

$$D(\mathscr{A}^3) = \{V \in D(\mathscr{A}^2);\ \mathscr{A}V \in D(\mathscr{A}^2)\} ,$$

then it is easy to see that whenever $U_0 = (u_0, u_1, \theta_0)^t \in D(\mathscr{A}^3)$, we have

$$U = (u, u_t, \theta)^t \in C^2(0, T; D(\mathscr{A}))$$

or, in particular

$$u \in C^3(0, T; D(H_0^1(\Omega))) , \qquad \theta \in C^2(0, T; D(A)) .$$

Remark 2.2. $D(\mathscr{A}^3)$ is dense in $\mathscr{H}$. In fact, take $F \in \mathscr{H}$ such that

$$(F, v)_{\mathscr{H}} = 0 \qquad \forall v \in D(\mathscr{A}) . \tag{2.16}$$

Since $\mathscr{A}$ is the infinitesimal generator of a strongly continuous semigroup, there exist $U \in \mathscr{H}$, such that $(\alpha-\beta)^2 U - \mathscr{A}U = F$, taking $V = U$ in (2.16) we conclude that $U = 0$, that is $F = 0$. With the same reasoning we can prove that $D(\mathscr{A}^2)$ is dense in $D(\mathscr{A})$, and $D(\mathscr{A}^3)$ is dense in $D(\mathscr{A}^2)$, with the corresponding graph norm. From here our claim follows □

3. Asymptotic behaviour

Before we prove the main result of this paper let's recall the simple lemma

Lemma 3.1. *Let v $(= v(x, t))$ be the solution of the inhomogeneous scalar wave equation*

$$v_{tt} - v_{xx} = f(x,t) \quad \text{for } 0 < x < L\,, \quad 0 < t < T\,, \tag{3.1}$$

$$v(x,0) = v_0(x)\,, \qquad v_t(x,0) = v_1(x) \qquad \text{for } 0 < x < L\,,$$

$$v(0,t) = v(L,t) = 0 \qquad \text{for } 0 < t < T\,,$$

where v_0, v_1 and f belong to $H_0^1(\Omega)\cap H^2(\Omega)$, $H_0^1(\Omega)$ and $H^1(0, T; L^2(\Omega))$ respectively. Then, the identity

$$\begin{aligned}\frac{1}{4}L[v_x^2(L,t) + v_x^2(0,t)] = \frac{d}{dt}\int_0^L \left(x - \frac{L}{2}\right) v_t(x,t) v_x(x,t)dx \\ + \frac{1}{2}\int_0^L [v_x^2(x,t) + v_t^2(x,t)]dx \\ - \int_0^L \left(x - \frac{L}{2}\right) f(x,t) v_x(x,t)dx\end{aligned}$$

holds.

Proof. Multiply (3.1) by $(x - (L/2))v_x$ and integrate in x to obtain

$$\int_0^L \left(x - \frac{L}{2}\right) v_{tt}v_x dx - \int_0^L \left(x - \frac{L}{2}\right) v_{xx}v_x dx = \int_0^L \left(x - \frac{L}{2}\right) f v_x dx\,. \tag{3.2}$$

Since $v_t(0,t) = v_t(L,t) = 0$, direct calculation give us the identities

$$\begin{aligned}\int_0^L \left(x - \frac{L}{2}\right) v_{tt}v_x dx &= \frac{d}{dt}\int_0^L \left(x - \frac{L}{2}\right) v_t v_x dx - \int_0^L \left(x - \frac{L}{2}\right) v_t v_{xt} dx \\ &= \frac{d}{dt}\int_0^L \left(x - \frac{L}{2}\right) v_t v_x dx - \frac{1}{2}\int_0^L \left(x - \frac{L}{2}\right)(v_t^2)_x dx \\ &= \frac{d}{dt}\int_0^L \left(x - \frac{L}{2}\right) v_t v_x dx + \frac{1}{2}\int_0^L v_t^2 dx\,.\end{aligned} \tag{3.3}$$

On the other hand,

$$\int_0^L \left(x - \frac{L}{2}\right) v_{xx} v_x dx = \frac{1}{2}\int_0^L \left(x - \frac{L}{2}\right)(v_x^2)_x dx \tag{3.4}$$

$$= \frac{1}{4}L[v_x^2(L, t) + v_x^2(0, t)] - \frac{1}{2}\int_0^L v_x^2 dx .$$

From (3.2), (3.3) and (3.4) the result follows □

Let's define the following auxiliary functions:

$$E_1(t) = \frac{1}{2}\int_0^L \left[u_t^2(x, t) + u_x^2(x, t) + \frac{\alpha}{\beta}\theta^2(x, t)\right] dx ,$$

$$E_2(t) = \frac{1}{2}\int_0^L \left[u_{tt}^2(x, t) + u_{xx}^2(x, t) + \frac{\alpha}{\beta}\theta_t^2(x, t)\right] dx ,$$

$$E_3(t) = \frac{1}{2}\int_0^L \left[u_{xt}^2(x, t) + u_{xx}^2(x, t) + \frac{\alpha}{\beta}\theta_x^2(x, t)\right] dx .$$

Theorem 3.1. *Let $(u_0, u_1, \theta_0) \in D(\mathscr{A})$ and $\alpha\beta > 0$. Then there exist positive constants C and γ such that the solution of system (1.1), ..., (1.4) satisfies*

$$E_1(t) + E_2(t) + E_3(t) \leq Ce^{-\gamma t} .$$

Proof. Let's take (u_0, u_1, θ_0) in $D(\mathscr{A}^3)$. Multiply equations (1.1) and (1.2) by u_t and θ respectively, then we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L [u_t^2(x, t) + u_x^2(x, t)]dx = -\alpha\int_0^L \theta_x(x, t)u_t(x, t)dx , \tag{3.5}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L \theta^2(x, t)dx + \int_0^L \theta_x^2(x, t)dx = -\beta\int_0^L \theta(x, t)u_{xt}(x, t)dx . \tag{3.6}$$

From (3.6) it follows that

$$\int_0^L \theta_x(x, t)u_t(x, t)dx = \frac{1}{\beta}\left\{\frac{1}{2}\frac{d}{dt}\int_0^L \theta^2(x, t)dx + \int_0^L \theta_x^2(x, t)dx\right\}$$

and the substitution of this identity in (3.5) implies that

$$\frac{d}{dt}E_1(t) = -\frac{\alpha}{\beta}\int_0^L \theta_x^2(x, t)dx . \tag{3.7}$$

Next we find the derivative of E_2. Differenciate equations (1.1) and (1.2) with respect to t (this is possible from Remark 2.1), multiply by u_{tt} and θ_t respectively and apply a similar idea as above to obtain

$$\frac{d}{dt}E_2(t) = -\frac{\alpha}{\beta}\int_0^L \theta_{xt}^2(x,t)dx\,. \tag{3.8}$$

Let's multiply equations (1.1) and (1.2) by u_{xxt} and θ_{xx} respectively to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L \{u_{xt}^2(x,t) + u_{xx}^2(x,t)\}dx &= -\alpha\int_0^L \theta_x(x,t)u_{txx}(x,t)dx\\ &= -\alpha\theta_x(x,t)u_{tx}(x,t)]_{x=0}^{x=L}\\ &\quad + \alpha\int_0^L \theta_{xx}(x,t)u_{tx}(x,t)dx\end{aligned}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L \theta_x^2(x,t)dx + \int_0^L \theta_{xx}^2(x,t)dx = -\beta\int_0^L \theta_{xx}(x,t)u_{xt}(x,t)dx\,.$$

The above two identities imply that

$$\frac{d}{dt}E_3(t) = -\frac{\alpha}{\beta}\int_0^L \theta_{xx}^2(x,t)dx - \alpha\theta_x(x,t)u_{tx}(x,t)|_{x=0}^{x=L}\,. \tag{3.9}$$

On the other hand, for any $\varepsilon > 0$ and any pair of numbers we have $\pm ab \le a^2/\varepsilon + \varepsilon b^2/4$, therefore

$$\begin{aligned}\theta_x(x,t)u_{tx}(x,t)|_{x=0}^{x=L} &\le \frac{1}{\varepsilon}[\theta_x^2(L,t) + \theta_x^2(0,t)] + \frac{\varepsilon}{4}[u_{xt}^2(L,t) + u_{xt}^2(0,t)]\\ &\le \frac{2}{\varepsilon}\sup\{\theta_x^2(\xi,t); \xi\in\overline{\Omega}\} + \frac{\varepsilon}{4}[u_{xt}^2(L,t) + u_{xt}^2(0,t)]\,.\end{aligned} \tag{3.10}$$

From Gagliardo-Nirenberg's inequality (see Adams [1]) we know that there exists a positive constant c satisfying:

(3.11)

$$\begin{aligned}\sup(\theta_x^2\{\xi,t); \xi\in\overline{\Omega}\} &\le c\left\{\int_0^L \theta_x^2(x,t)dx\right\}^{1/2}\left\{\int_0^L \theta_x^2(x,t)dx + \int_0^L \theta_{xx}^2(x,t)dx\right\}^{1/2}\\ &\le \left[c + \left(\frac{c}{\varepsilon}\right)^2\right]\int_0^L \theta_x^2(x,t)dx + \frac{\varepsilon^2}{4}\int_0^L \theta_{xx}^2(x,t)dx\,.\end{aligned}$$

From (3.9), (3.10) and (3.11) we obtain

$$\begin{aligned}\frac{d}{dt}E_3(t) &= -\left(\frac{\alpha}{\beta} - \alpha\frac{\varepsilon}{2}\right)\int_0^L \theta_{xx}^2(x,t)dx\\ &\frac{\varepsilon}{4}\alpha[u_{xt}^2(L,t) + u_{xt}^2(0,t)] + C(\varepsilon)\int_0^L \theta_x^2(x,t)dx\,,\end{aligned} \tag{3.12}$$

where $C(\varepsilon)$ is a constant approaching $+\infty$ when $\varepsilon \to 0$. On the other hand, let's differentiate equation (1.1) respect to t and set $v = u_t$ and $f = -\alpha\theta_{xt}$. Since $(u_0, u_1, \theta_0) \in D(\mathscr{A}^3)$ we conclude that $v(0) = u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $v_t(0) = u_{0,xx} - \theta_{0,x} \in H_0^1(\Omega)$. From Remark 2.1 it follows that $f \in C^1(0, T; H^1(\Omega))$, and from Lemma 3.1 we have that the following identity is valid:

$$\frac{1}{4}L\{u_{xt}^2(L, t) + u_{xt}^2(0, t)\} = \frac{d}{dt}\int_0^L \left(x - \frac{L}{2}\right) u_{tt}(x, t) u_{xt}(x, t)dx + \frac{1}{2}\int_0^L \left\{u_{xt}^2(x, t) + u_{tt}^2(x, t)\right\}dx + \alpha\int_0^L \left(x - \frac{L}{2}\right)\theta_{xt}(x, t) u_{xt}(x, t)dx \,. \tag{3.13}$$

Let's define the auxiliary functions:

$$H(t) = -\int_0^L \left(x - \frac{L}{2}\right) u_{tt}(x, t) u_{xt}(x, t)dx \,,$$

$$I(t) = \int_0^L \theta(x, t) u_{xt}(x, t)dx \,,$$

$$J(t) = \int_0^L u_x(x, t) u_{xt}(x, t)dx \,.$$

Clearly

$$|H(t)| \le \frac{L}{2}\int_0^L [u_{xx}^2(x, t) + u_{xt}^2(x, t) + \alpha^2\theta_x^2(x, t)]dx \,, \tag{3.14}$$

$$|I(t)| \le \frac{1}{2}\int_0^L [\theta_x^2(x, t) + u_t^2(x, t)]dx \,, \tag{3.15}$$

$$|J(t)| \le \frac{1}{2}\int_0^L [u_x^2(x, t) + u_{xt}^2(x, t)]dx \,. \tag{3.16}$$

From (3.12), (3.13) and the definition of H we obtain

$$\frac{d}{dt}\left\{E_3(t) + \alpha\frac{\varepsilon}{L}H(t)\right\} \le -\left(\frac{\alpha}{\beta} - \alpha\frac{\varepsilon}{2}\right)\int_0^L \theta_{xx}^2(x, t)dx + C(\varepsilon)\int_0^L \theta_x^2(x, t)dx$$
$$\frac{\alpha\varepsilon}{2L}\int_0^L [u_{xt}^2(x, t) + u_{tt}^2(x, t)]dx + \alpha^2\frac{\varepsilon}{L}\int_0^L \left(x - \frac{L}{2}\right)\theta_{xt}(x, t) u_{xt}(x, t)dx \,.$$

From (1.1) and the above inequality it follows that there exists a positive constant c which does not depend of ε, such that:

$$
\frac{d}{dt}\left\{E_3(t)+\frac{\varepsilon}{L}H(t)\right\} \tag{3.17}
$$

$$
\leq -\left(\frac{\alpha}{\beta}-\alpha\frac{\varepsilon}{2}\right)\int_0^L \theta_{xx}^2(x,t)dx+\left[C(\varepsilon)+2\alpha^3\frac{\varepsilon}{L}\right]\int_0^L \theta_x^2(x,t)dx
$$

$$
+c\varepsilon\int_0^L [u_{xt}^2(x,t)+u_{xx}^2(x,t)+u_{xt}^2(x,t)+\theta_{xt}^2(x,t)]dx\,.
$$

Let's differentiate $I(t)$ and $J(t)$ and use (1.1) and (1.2) to obtain

$$
\frac{d}{dt}I(t)=\int_0^L \theta_{xx}(x,t)u_{xt}(x,t)dx-\beta\int_0^L u_{xt}^2(x,t)dx
$$

$$
-\int_0^L \theta_x(x,t)u_{xx}(x,t)dx+\alpha\int_0^L \theta_x^2(x,t)dx\,,
$$

$$
\frac{d}{dt}J(t)=\int_0^L u_{xt}^2(x,t)dx-\int_0^L u_{xx}^2(x,t)dx+\alpha\int_0^L \theta_x(x,t)u_{xx}(x,t)dx\,,
$$

from where it follows that

$$
\frac{d}{dt}I(t)\leq -\frac{1}{2}\beta\int_0^L u_{xt}^2(x,t)dx+\frac{1}{2\beta}\int_0^L \theta_{xx}^2(x,t)dx \tag{3.18}
$$

$$
+\frac{\beta}{16}\int_0^L u_{xx}^2(x,t)dx+\left(\alpha+\frac{8}{\beta}\right)\int_0^L \theta_x^2(x,t)dx\,,
$$

$$
\frac{d}{dt}J(t)\leq \int_0^L u_{xt}^2(x,t)dx-\frac{1}{2}\int_0^L u_{xx}^2(x,t)dx+\frac{\alpha^2}{2}\int_0^L \theta_x^2(x,t)dx\,. \tag{3.19}
$$

From (3.18) and (3.19) we deduce

$$
\frac{d}{dt}\left\{I(t)+\frac{\beta}{4}J(t)\right\}\leq \frac{1}{2\beta}\int_0^L \theta_{xx}^2(x,t)dx-\frac{\beta}{4}\int_0^L u_{xt}^2(x,t)dx
$$

$$
-\frac{\beta}{16}\int_0^L u_{xx}^2(x,t)dx+\left(\alpha+\frac{8}{\beta}+\frac{\beta\alpha^2}{8}\right)\int_0^L \theta_x^2(x,t)dx\,.
$$

From (3.17) and the last inequality we have

$$
\frac{d}{dt}\left\{E_3(t)+\alpha\frac{\varepsilon}{L}H(t)+\frac{32}{\beta}c\varepsilon\left[I(t)+\frac{\beta}{4}J(t)\right]\right\}\leq -\left(\frac{\alpha}{\beta}-\alpha\frac{\varepsilon}{2}-4\varepsilon c/\beta^2\right)\int_0^L \theta_{xx}^2(x,t)dx
$$

$$
-c\varepsilon\int_0^L [u_{xt}^2(x,t)+u_{xx}^2(x,t)]dx
$$

$$
+C(\varepsilon)\int_0^L [\theta_x^2(x,t)+\theta_{xt}^2(x,t)]dx\,.
$$

Let K the following function

$$K(t) = \frac{\alpha}{L} H(t) + \frac{32}{\beta} cI(t) + 8cJ(t).$$

Taking $\varepsilon > 0$ such that $\varepsilon((\alpha/2) + 4c/\beta^2) < \alpha/\beta$, we obtain

$$\frac{d}{dt}\{E_3(t) + \varepsilon K(t)\} \leq -c\varepsilon \int_0^L [u_{xt}^2(x, t) + u_{xx}^2(x, t)]dx$$
$$+ C(\varepsilon) \int_0^L [\theta_x^2(x, t) + \theta_{xt}^2(x, t)]dx .$$

From (3.7), (3.9) and the last inequality we have

$$\frac{d}{dt}\left\{E_3(t) + \varepsilon K(t) + \frac{\beta}{\alpha}[C(\varepsilon) + c\varepsilon][E_1(t) + E_2(t)]\right\} \tag{3.20}$$
$$\leq -c\varepsilon \int_0^L [u_{xt}^2(x, t) + u_{xx}^2(x, t) + \theta_x^2(x, t) + \theta_t^2(x, t)]dx ,$$

where c denotes various positive constants independent of ε. From (3.14), (3.15) and (3.16) we have that there exists a positive constant, say c_1 such that:

$$E_3(t) + \varepsilon K(t) + C(\varepsilon)[E_1(t) + E_2(t)] \tag{3.21}$$
$$\leq c_1 \int_0^L [\theta_{xx}^2(x, t) + u_{xt}^2(x, t) + u_{xx}^2(x, t) + \theta_x^2(x, t) + \theta_{xt}^2(x, t)]dx ,$$

where $C(\varepsilon)$ denote various positive constants. Multiplying (3.20) by c_0 and (3.21) by c_1, adding the resulting inequalities and integrating, we conclude that there exists a positive constant γ, such that

$$E_3(t) + \varepsilon K(t) + C(\varepsilon)[E_1(t) + E_2(t)] \leq C(\varepsilon)\{E_1(0) + E_2(0) + E_3(0)\}e^{-\gamma t} .$$

Taking $0 < \varepsilon$ small enough we have that

$$E_3(t) + \varepsilon K(t) + C(\varepsilon)[E_1(t) + E_2(t)] \geq E_1(t) + E_2(t) + E_3(t)$$

From the last two inequalities we obtain:

$$E_1(t) + E_2(t) + E_3(t) \leq C(\varepsilon)\{E_1(0) + E_2(0) + E_3(0)\}e^{-\gamma t}$$

for all $t > 0$. Finally from Remark 2.2 the result follows. □

Acknowledgment. The author is very grateful to Professor Gustavo Perla Menzala for fruitful discussions concerning this paper.

References

[1] Adams, R. A., *Sobolev Spaces*, Academic Press, New York 1975.

[2] Dafermos, C. M., On the existence and the asymptotic stability of solution to the equations of linear thermoelasticity, Arch. Rational Mech. Anal., **29** (1968), 241–271.

[3] Dassios, G. and Grillakis, M., Dissipation rates and partition of energy in thermoelasticity, Arch. Rational Mech. Anal., **87** (1984), 49–91.

[4] Henry, D., Lopes, O. and Perisinotto, A., Linear thermoelasticity: asymptotic stability and essential spectrum (unpublished manuscript)

[5] Henry, D., *Topics in analysis*, Publications, secció de Matematique, vol. **31** (1), Universitat. Autonoma de Barcelona, 1987.

[6] Lions. J. L., *Quelques méthodes de resolution de problèmes aux limites non lineares.* Dunod Gauthier Villars, Paris 1969.

[7] Pazy, A., *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York 1983.

[8] Pereira, D. C. and Menzala, G. P., Exponential decay of solutions to a coupled system of equations of linear thermoelasticity, Comp. Appl. Math., **8** (1989), 193–204.

[9] Racke, R., On the time-asymptotic behaviour of solutions in thermoelasticity. Proceeding of the Royal Society of Edinburgh, **107A** (1987), 289–298.

[10] Zuazua, E., Stability and decay for a class of non linear hyperbolic problems, Asymptotic Analysis **1** (1988), 161–185.

nuna adreso:
National Laboratory of Scientific Computation
Department of Research and Development
Rua Lauro Muller 455
Botafogo, CEP. 22290
Rio de Janeiro, RJ
Brazil

(Ricevita la 17-an de januaro, 1990)