Funkcialaj Ekvacioj, 38 (1995) 545–568

On a Local Energy Decay of Solutions of a Dissipative Wave Equation

By

Wakako DAN and Yoshihiro SHIBATA
(University of Tsukuba, Japan)

Dedicated to Professor Kôji Kubota on his sixtieth birthday

§ 1. Introduction

This paper is concerned with a local energy decay property of solutions to the initial boundary value problem of the dissipative wave equation:

$$\text{(D)}\quad \begin{cases} u_{tt} + u_t - \Delta u = 0 & \text{in } \Omega \text{ and } t > 0\,, \\ u = 0 & \text{on } \Gamma \text{ and } t > 0\,, \\ u(0, x) = u_0(x),\ u_t(0, x) = u_1(x) & \text{in } \Omega\,, \end{cases}$$

where Ω is an exterior domain in an n-dimensional Euclidean space $\boldsymbol{R}^n$, whose boundary Γ is a C^∞ and compact hypersurface. Below, $r_0 > 0$ is a fixed constant such that $\Omega^c \subset B_{r_0} = \{x \in \boldsymbol{R}^n \,|\, |x| < r_0\}$ (Ω^c is the complement of Ω).

In the case of usual wave equation, the local energy decays exponentially fast if n is odd and polynomially fast if n is even at least under the condition that Ω is nontrapping (cf. [9], [10], [11], [16]). This is reasonable from a physical point of view because the energy propagates along the wave fronts, so that the motion stops after time passes unless the wave front is trapped in a bounded set.

In the case of dissipative wave equation, the energy propagates again along the wave front. But, the trapped energy also decreases by virtue of the dissipative term u_t, so that we can expect to get a local energy decay result without any geometrical condition on Ω. In fact, Shibata [14] proved the following theorem.

Theorem 1.1. *Assume that $n \geq 3$. Let $R > r_0$ and let $u(t, x)$ be a smooth solution of* (D) *such that* $\operatorname{supp} u(0, x)$, $\operatorname{supp} u_t(0, x) \subset \Omega_R = \{x \in \Omega \,|\, |x| < R\}$. *Then, there exists a constant $C > 0$ depending on n and R such that*

$$\int_{\Omega_R} \Big\{ |u_t(t, x)|^2 + \sum_{|\alpha| \leq 1} |\partial_x^\alpha u(t, x)|^2 \Big\} dx$$
$$\leq C(1 + t)^{-n} \Big\{ \sum_{|\alpha| \leq 3} \int_\Omega |\partial_x^\alpha u_t(0, x)|^2 dx + \sum_{|\alpha| \leq 4} \int_\Omega |\partial_x^\alpha u(0, x)|^2 dx \Big\}\,,$$

where $\partial_x^\alpha v = \partial^{|\alpha|} v / \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The purpose of this paper is to extend and improve the above result as follows.

Theorem 1.2. *Assume that* $n \geq 2$. *Let* $R > r_0$ *and* $u_0 \in H^1_{0,R}(\Omega)$ *and* $u_1 \in L^2_R(\Omega)$, *where*

$$L^2_R(\Omega) = \{f \in L^2(\Omega) | \operatorname{supp} f \subset \Omega_R\},$$
$$H^1_{0,R}(\Omega) = \{f \in H^1(\Omega) | \operatorname{supp} f \subset \Omega_R, f = 0 \text{ on } \Gamma\}.$$

Let $u(t, x)$ *be a weak solution of* (D). *Then, there exists a constant* C *depending on* n *and* R *such that*

$$\int_{\Omega_R} \left\{ |u_t(t, x)|^2 + \sum_{|\alpha| \leq 1} |\partial_x^\alpha u(t, x)|^2 \right\} dx$$
$$\leq C(1 + t)^{-n} \left\{ \int_\Omega |u_1(x)|^2 dx + \sum_{|\alpha| \leq 1} \int_\Omega |\partial_x^\alpha u_0(x)|^2 dx \right\}.$$

Compared with Theorem 1.1, our result removes the smoothness assumption on solutions of (D) and includes the case $n = 2$ as well as the case $n \geq 3$.

For the Cauchy problem of the dissipative wave equation (i.e. $\Omega = \boldsymbol{R}^n$), A. Matsumura [8] studied the decay rate of solutions. His argument was based on the concrete representation of solutions by use of the Fourier transform. When Ω is bounded it is well-known that the energy of solutions decays exponentially fast. Indeed, this fact is easily proved by a standard energy method combined with Poincaré's inequality. Since Ω is unbounded in our case, we cannot use Poincaré's inequality. And also, because of the boundary, we cannot use the Fourier transform. Our method is based on a spectral analysis to the corresponding stationary problem with parameter λ:

$$(\lambda^2 + \lambda - \Delta)u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \Gamma.$$

This paper is organized as follows. In §2, we introduce the space $H_D(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to Dirichlet norm and give several properties of the space $H_D(\Omega)$. In §3, we shall prove that $A = \begin{bmatrix} 0 & 1 \\ \Delta & -1 \end{bmatrix}$ generates a C_0 semigroup on $H_D(\Omega) \times L^2(\Omega)$. In §4, we formulate the problem in the abstract manner and prove Theorem 1.2 under a suitable assumption on the behavior of $(\lambda - A)^{-1}$ near $\lambda = 0$. In §5, we investigate the behavior of $(\lambda - A)^{-1}$ near $\lambda = 0$ and complete the proof.

§2. The properties of $H_D(\Omega)$

For any open set $\mathcal{O} \subset \boldsymbol{R}^n$, $C_0^\infty(\mathcal{O})$ denotes the space of all C^∞ functions on $\boldsymbol{R}^n$ whose support is compact and lies in $\mathcal{O}$ (in particular, such functions vanish near the boundary of $(\mathcal{O})$, $L^2(\mathcal{O})$ a usual L^2 space on $\mathcal{O}$ with norm $\|\cdot\|_{\mathcal{O}}$ and inner product $(\ ,\ )_{\mathcal{O}}$, and $H^s(\mathcal{O})$ a usual Sobolev space of order s on $\mathcal{O}$ with norm $\|\cdot\|_{s,\mathcal{O}}$. $\|\cdot\|_{k,\Omega}$ will be denoted simply by $\|\cdot\|_k$. Likewise for $\|\cdot\|_\Omega$ and $(\ ,\ )_\Omega$. Let us define the space $H_D(\Omega)$ by

$$H_D(\Omega) = \left\{u \in H^1_{loc}(\Omega) | \nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) \in L^2(\Omega), \quad u = 0 \text{ on } \Gamma, \right.$$

$$\left. \exists \text{ a sequence } \{u_n\} \subset C_0^\infty(\Omega) \text{ s.t. } \|\nabla(u_n - u)\| \to 0 \text{ as } n \to \infty \right\},$$

where $H^1_{loc}(\Omega) = \{u \in \mathcal{D}'(\Omega) | u \in H^1(\Omega_R)\ \forall R > r_0\}$. As we mentioned in §1, $H_D(\Omega)$ will play an important role since A will be dissipative in this space. Although $H_D(\Omega)$ coincides with the completion of $C_0^\infty(\Omega)$ with respect to $\|\nabla\cdot\|$ we prefer to adopt the above definition to make some properties clearer.

Lemma 2.1. (1) *For any $R \geq r_0$, there exists a constant $C = C(R)$ such that*

$$\|\varphi\|_{0,\Omega_R} \leq C\|\nabla\varphi\|_{0,\Omega_R} \qquad \forall\varphi \in C_0^\infty(\Omega). \tag{2.1}$$

Here and hereafter, the letter C is used to denote various constants and $C(A, B, \dots)$ denotes a constant depending on A, B, ... in the parenthesis.

(2) *There exists a constant C such that*

$$\int_\Omega \frac{|\varphi(x)|^2}{d(x)^2} dx \leq C\|\nabla\varphi\|^2 \qquad \forall\varphi \in C_0^\infty(\Omega), \tag{2.2}$$

where

$$d(x) = \begin{cases} |x| & \text{if } n \geq 3, \\ |x| \log(B|x|) & \text{if } n = 2 \end{cases}$$

and B is a constant such that $B|x| \geq 2$ as $x \in \Omega$.

Proof. (2.1) is well-known and the proof is omitted. (2.2) is also well-known (cf. [6], [12], [15, Lemma 1.3]). For the completeness, however, we shall give a proof for the case $n = 2$. We fix $R \geq r_0$. Noting that

$$\int_{|x|\geq R} |\varphi(x)|^2 (\log(B|x|)|x|)^{-2} dx = \int_{|\omega|=1} \int_R^\infty \varphi(r\omega)^2 (\log(Br))^{-2} r^{-1} dr d\omega$$

and

$$\int_R^\infty \varphi(r\omega)^2 (\log(Br))^{-2} r^{-1} dr = \varphi(R\omega)^2 (\log(BR))^{-1}$$

$$+ 2\int_R^\infty \varphi(r\omega)\omega\cdot\nabla\varphi(r\omega)(\log(Br))^{-1} dr,$$

we have

$$\int_{|x|\geq R}|\varphi(x)|^2(\log B|x|)|x|)^{-2}dx\leq C(R)\int_{|\omega|=1}|\varphi(R\omega)|^2d\omega$$
$$+2\left(\int_{|x|\geq R}|\varphi(x)|^2(\log(B|x|)|x|)^{-2}dx\right)^{1/2}\left(\int_{|x|\geq R}|\nabla\varphi(x)|^2dx\right)^{1/2}.$$

To calculate the first term of the right-hand side, we take a function $\rho(x)\in C_0^\infty(\boldsymbol{R}^n)$ such that $\rho(\omega)=1$ for any $|\omega|\leq R$ and $\operatorname{supp}\rho\subset B_{2R}$. From

$$\varphi(R\omega)^2=-\int_R^\infty\frac{\partial}{\partial r}\{\varphi(r\omega)^2\rho(r\omega)\}dr\,,$$

it follows that

$$|\varphi(R\omega)|^2\leq\frac{2}{R}\int_R^\infty|\varphi(r\omega)||\nabla\varphi(r\omega)||\rho(r\omega)|rdr+\frac{1}{R}\int_R^\infty|\varphi(r\omega)|^2|\nabla\rho(r\omega)|rdr\,.$$

Applying (2.1), we have

$$\int_{|\omega|=1}|\varphi(R\omega)|^2d\omega\leq\frac{2}{R}\left(\int_{|x|\geq R}|\varphi(x)|^2|\rho(x)|dx\right)^{1/2}\left(\int_{|x|\geq R}|\nabla\varphi(x)|^2|\rho(x)|dx\right)^{1/2}$$
$$+\frac{1}{R}\left(\int_{|x|\geq R}|\varphi(x)|^2|\nabla\rho(x)|dx\right)$$
$$\leq C(R)\|\varphi\|_{1,\Omega_R}^2\leq C(R)\|\nabla\varphi\|^2\,.$$

Therefore we have proved that

$$\int_{|x|\geq R}\frac{|\varphi(x)|^2}{d(x)^2}dx\leq C(R)\|\nabla\varphi\|^2\,. \tag{2.3}$$

On the other hand, since there exists a constant $C_R>0$ such that $d(x)>C_R$ in Ω_R, from (2.1) we have

$$\int_{\Omega_R}\frac{|\varphi(x)|^2}{d(x)^2}dx\leq\frac{1}{C_R^2}\int_{\Omega_R}|\varphi(x)|^2dx\leq C(R)\|\nabla\varphi\|^2\,. \tag{2.4}$$

(2.3) and (2.4) imply (2.2). □

From the definition of $H_D(\Omega)$ we have immediately the following.

Lemma 2.2. *Let $\{v_n\}\subset C_0^\infty(\Omega)$ be a sequence such that $\|\nabla(v_n-v_m)\|\to0$ as n, $m\to\infty$. Then there exists a $v\in H_D(\Omega)$ such that $\|\nabla(v_n-v)\|\to0$ as $n\to\infty$.*

From Lemma 2.1 we have:

Lemma 2.3. *If $u \in H_D(\Omega)$, then u satisfies the following inequalities*:

$$\|u\|_{0,\Omega_R} \leq C(R)\|\nabla u\|_{0,\Omega_R}, \tag{2.5}$$

$$\int_\Omega \frac{|u(x)|^2}{d(x)^2} dx \leq C\|\nabla u\|^2. \tag{2.6}$$

Moreover, $H_D(\Omega)$ is a Hilbert space equipped with an inner product $(u, v)_D = (\nabla u, \nabla v)$.

§ 3. A construction of C_0 semigroup solving (D)

Putting $u_t = v$, we rewrite the problem (D) in the following form:

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = A\begin{bmatrix} u \\ v \end{bmatrix}.$$

An underlying space for A is

$$\mathscr{H} = \left\{\begin{bmatrix} u \\ v \end{bmatrix} \middle| u \in H_D(\Omega), v \in L^2(\Omega)\right\}.$$

From Lemma 2.3 we know that $\mathscr{H}$ is a Hilbert space equipped with the inner product

$$\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} w \\ z \end{bmatrix}\right)_{\mathscr{H}} = (u, w)_D + (v, z).$$

The domain of A is

$$D(A) = \left\{\begin{bmatrix} u \\ v \end{bmatrix} \in \mathscr{H} \middle| A\begin{bmatrix} u \\ v \end{bmatrix} \in \mathscr{H}\right\} = \left\{\begin{bmatrix} u \\ v \end{bmatrix} \in \mathscr{H} \middle| v \in H_D(\Omega), \Delta u \in L^2(\Omega)\right\}.$$

For any open subset $\mathcal{O}$ we put

$$\boldsymbol{C}_0^\infty(\mathcal{O}) = \left\{\begin{bmatrix} f \\ g \end{bmatrix} \middle| f \text{ and } g \in C_0^\infty(\mathcal{O})\right\}.$$

In order to prove that A generates a C^0 semigroup on $\mathscr{H}$ it is sufficient, in view of the Lumer and Phillips theorem [13, Chapter 1, Theorem 4.3], to prove the following proposition.

Proposition 3.1. (1) *A is a closed operator.*
(2) *A is a dissipative operator.*
(3) *$\mathscr{R}(I - A) = \mathscr{H}$.*
(4) *$D(A)$ is dense in $\mathscr{H}$.*

Proof. To prove (1), let us assume that $D(A) \ni \begin{bmatrix} u_n \\ v_n \end{bmatrix} \to \begin{bmatrix} u \\ v \end{bmatrix}$ in $\mathscr{H}$ and $A\begin{bmatrix} u_n \\ v_n \end{bmatrix} \to \begin{bmatrix} f \\ g \end{bmatrix}$ in $\mathscr{H}$. Then we have $v_n \to v$ and $\nabla v_n \to \nabla f$ in $L^2(\Omega)$, which implies that $\nabla v = \nabla f$ and $v_n \to v$ in $H^1(\Omega)$. Since $v = f = 0$ on Γ we see $v = f$ and $v \in H_D(\Omega) \cap L^2(\Omega)$. Since $\Delta u_n \to g + v$ in $L^2(\Omega)$ and $\Delta u_n \to \Delta u$ in $\mathscr{D}'$, $\Delta u = g + v$ in $L^2(\Omega)$, which implies that A is closed. To prove (2) we calculate $\left(A\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}\right)_{\mathscr{H}} = \left(\begin{bmatrix} v \\ \Delta u - v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}\right)_{\mathscr{H}}$. Since ∇u, Δu, v and $\nabla v \in L^2(\Omega)$ and $v = 0$ on Γ,

$$(\Delta u, v) = \lim_{R\to\infty} \int_\Omega \rho_R \Delta u v dx$$

$$= -\lim_{R\to\infty} \int_\Omega \nabla \rho_R \cdot \nabla u v dx - \lim_{R\to\infty} \int_\Omega \rho_R \nabla u \cdot \nabla v dx = -(\nabla u, \nabla v),$$

where $\rho(x) \in C_0^\infty(\boldsymbol{R}^n)$ such that $\rho(x) = 1$ if $|x| \leq 1$ and $= 0$ if $|x| \geq 2$ and $\rho_R(x) = \rho(x/R)$. Therefore we have

$$\text{(3.1)} \qquad \mathrm{Re}\left(A\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}\right)_{\mathscr{H}} = -\|v\|^2 \leq 0,$$

which implies that A is dissipative. Moreover, it follows from (3.1) that

$$\text{(3.2)} \qquad \left\|(\lambda I - A)\begin{bmatrix} u \\ v \end{bmatrix}\right\|_{\mathscr{H}} \geq \lambda \left\|\begin{bmatrix} u \\ v \end{bmatrix}\right\|_{\mathscr{H}} \qquad \text{for } \lambda > 0 \text{ and } \begin{bmatrix} u \\ v \end{bmatrix} \in D(A)$$

(cf. [13, Chapter 1, Theorem 4.2]). We shall prove (3). Since $C_0^\infty(\Omega)$ is dense both in $H_D(\Omega)$ and in $L^2(\Omega)$, in view of (1) and (3.2) it is sufficient to prove that for any $\begin{bmatrix} f \\ g \end{bmatrix} \in C_0^\infty(\Omega)$, there exists a $\begin{bmatrix} u \\ v \end{bmatrix} \in D(A)$ such that

$$\text{(3.3)} \qquad (I - A)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}.$$

Substitute the relation: $u - v = f$ of the first component into the second component in (3.3), and the problem is reduced to finding a solution $u \in H^2(\Omega)$ of the equation:

$$2u - \Delta u = g + 2f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \Gamma.$$

Since it is well-known that $\{2u - \Delta u | u \in H^2(\Omega) \text{ and } u = 0 \text{ on } \Gamma\} = L^2(\Omega)$ (cf. [12, Chapter 3]), there exists a $\begin{bmatrix} u \\ v \end{bmatrix} \in D(A)$ satisfying (3.3) for any $\begin{bmatrix} f \\ g \end{bmatrix} \in C_0^\infty(\Omega)$.

Last of all we shall prove (4). Assume that there exists a $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathscr{H}$ such that $\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix}\right)_{\mathscr{H}} = 0$ for any $\begin{bmatrix} f \\ g \end{bmatrix} \in D(A)$. Since there exists a $\begin{bmatrix} p \\ q \end{bmatrix} \in D(A)$ such that $(I - A)\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$,

$$0 = \left((I - A)\begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} p \\ q \end{bmatrix}\right)_{\mathscr{H}} = \left\|\begin{bmatrix} p \\ q \end{bmatrix}\right\|_{\mathscr{H}}^2 - \left(A\begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} p \\ q \end{bmatrix}\right)_{\mathscr{H}}.$$

Therefore, by (2) we have $\begin{bmatrix} p \\ q \end{bmatrix} = 0$, that is, $\begin{bmatrix} u \\ v \end{bmatrix} = 0$. □

The Lumer and Phillips theorem implies the following theorem:

Theorem 3.2. *A generates a C^0 semigroup $\{T(t)\}$ on $\mathscr{H}$.*

Put the region D as follows:

$$D = D_i \cup D_r ,$$

where

$$D_i = \{\lambda \in \boldsymbol{C} | 2 \operatorname{Re} \lambda + 1 > 0, \operatorname{Im} \lambda \neq 0\} \quad \text{and} \quad D_r = \{\lambda \in \boldsymbol{R} | \lambda > 0\} .$$

In view of the Lumer and Phillips theorem and [13, Chapter 1, Corollary 3.6] we know that

$$\rho(A) \supset \{\lambda \in \boldsymbol{C} | \operatorname{Re} \lambda > 0\} \quad \text{and} \quad \|(\lambda I - A)^{-1}\| \leq \frac{1}{\operatorname{Re} \lambda} \quad \text{for } \operatorname{Re} \lambda > 0 . \tag{3.4}$$

Lemma 3.3. *For $\lambda \in D \cap \rho(A)$ we have*

$$\left\|\begin{bmatrix} u \\ v \end{bmatrix}\right\|_{\mathscr{H}} \leq C(\lambda) \left\|(\lambda I - A)\begin{bmatrix} u \\ v \end{bmatrix}\right\|_{\mathscr{H}} \quad \forall \begin{bmatrix} u \\ v \end{bmatrix} \in D(A) , \tag{3.5}$$

where $C(\lambda)$ is a constant depending on λ continuously.

Proof. For $\lambda \in D_i \cap \rho(A)$, let $\begin{bmatrix} u \\ v \end{bmatrix} \in D(A)$ be a couple of functions satisfying that

$$(\lambda I - A)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \quad \text{for } \begin{bmatrix} f \\ g \end{bmatrix} \in \boldsymbol{C}_0^\infty(\Omega) . \tag{3.6}$$

Then from (3.6) it follows that

$$\lambda(\lambda + 1)u - \Delta u = g + (\lambda + 1)f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \Gamma. \tag{3.7}$$

Since $u = \lambda^{-1}(v + f) \in L^2(\Omega)$ and $\Delta u \in L^2(\Omega)$ we can multiply (3.7) by Δu to obtain

$$\text{(3.8)} \quad \{(\mathrm{Re}\,\lambda)^2 - (\mathrm{Im}\,\lambda)^2 + \mathrm{Re}\,\lambda\} \|\nabla u\|^2 + \|\Delta u\|^2 + i(2\,\mathrm{Re}\,\lambda + 1)\,\mathrm{Im}\,\lambda \|\nabla u\|^2 = -(g, \Delta u) + (\lambda + 1)(\nabla f, \nabla u).$$

Taking the imaginary part of (3.8), we have

$$\text{(3.9)} \qquad \|\nabla u\|^2 \le \frac{|\lambda + 1|^2}{(|2\,\mathrm{Re}\,\lambda + 1||\mathrm{Im}\,\lambda|)^2} \|\nabla f\|^2 + \frac{2}{|2\,\mathrm{Re}\,\lambda + 1||\mathrm{Im}\,\lambda|} \|g\| \|\Delta u\|.$$

On the other hand by (3.8)

$$\text{(3.10)} \qquad \|\Delta u\|^2 \le |\lambda||\lambda + 1| \|\nabla u\|^2 + |\lambda + 1| \|\nabla u\| \|\nabla f\| + \|g\| \|\Delta u\|.$$

Combining (3.9) and (3.10), we have

$$\|\nabla u\|^2 + \|\Delta u\|^2 \le \left\{\frac{|\lambda + 1|^2}{l^2} + \frac{3}{2}\left(\frac{3|\lambda||\lambda + 1|^3}{l^2} + \frac{|\lambda + 1|}{|\lambda|}\right)\right\} \|\nabla f\|^2 + \left\{\frac{2}{l^2} + \frac{3}{2}\left(\frac{3|\lambda||\lambda + 1|}{l} + 1\right)^2\right\} \|g\|^2,$$

where $l = |2\,\mathrm{Re}\,\lambda + 1||\mathrm{Im}\,\lambda|$. Since $v = (\lambda + 1)^{-1}(g + \Delta u)$ and $\lambda \in D_i$, it follows that

$$\|v\|^2 \le 8(\|\Delta u\|^2 + \|g\|^2).$$

Therefore we obtain that

$$\text{(3.11)} \qquad \|\nabla u\|^2 + \|v\|^2 \le C_1(\lambda)(\|\nabla f\|^2 + \|g\|^2) \qquad \text{for } \lambda \in D_i \cap \rho(A) \text{ and } \begin{bmatrix} f \\ g \end{bmatrix} \in C_0^\infty(\Omega).$$

Here $C_1(\lambda)$ is continuous in D_i. Since $C_0^\infty(\Omega)$ is dense in both $H_D(\Omega)$ and $L^2(\Omega)$, (3.11) holds for any $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathscr{H}$ and $\lambda \in D_i \cap \rho(A)$. Combining the above argument and (3.4) implies the lemma. □

Lemma 3.4.

$$\text{(1)} \qquad D \subset \rho(A).$$

$$\text{(2)} \qquad \|(\lambda I - A)^{-1}\| \le 7 + \frac{2}{|\lambda|} \qquad \textit{if } \mathrm{Re}\,\lambda = 0 \textit{ and } \mathrm{Im}\,\lambda \neq 0.$$

Proof. (1) Put $E = D \cap \rho(A)$. Since D is a connected set, it is sufficient to prove that E is non-empty, open and closed. It is clear that $E \neq \phi$ and E

is open. Our task is to prove that E is closed in D. Let $\{\lambda_n\} \subset E$ such that $\lambda_n \to \lambda$ in D as $n \to \infty$. By Lemma 3.3 there exists an M such that

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_{\mathscr{H}} \leq M \left\| (\mu I - A) \begin{bmatrix} u \\ v \end{bmatrix} \right\|_{\mathscr{H}} \quad \forall \begin{bmatrix} u \\ v \end{bmatrix} \in D(A), \tag{3.12}$$

where $\mu = \lambda_n (n \in N)$ and λ. If we prove that $\lambda I - A$ is surjective, then (3.12) implies immediately $\lambda \in \rho(A)$, that is, E is closed. Let us show that $\lambda I - A$ is surjective. Since $\lambda_n \in \rho(A)$, for any $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathscr{H}$ there exists a $\begin{bmatrix} u_n \\ v_n \end{bmatrix} \in D(A)$ such that $(\lambda_n I - A) \begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$. If there exists a $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathscr{H}$ such that

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} \to \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{in } \mathscr{H}, \tag{3.13}$$

we can conclude that $(\lambda I - A)$ is surjective, because A is closed operator. From (3.12) it follows that

$$\left\| \begin{bmatrix} u_n \\ v_n \end{bmatrix} \right\|_{\mathscr{H}} \leq M \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathscr{H}}. \tag{3.14}$$

Observing that

$$(\lambda_n I - A)\left(\begin{bmatrix} u_n \\ v_n \end{bmatrix} - \begin{bmatrix} u_m \\ v_m \end{bmatrix} \right) = (\lambda_m - \lambda_n) \begin{bmatrix} u_m \\ v_m \end{bmatrix}$$

we have from (3.12) and (3.14)

$$\left\| \begin{bmatrix} u_n \\ v_n \end{bmatrix} - \begin{bmatrix} u_m \\ v_m \end{bmatrix} \right\|_{\mathscr{H}} \leq M |\lambda_m - \lambda_n| \left\| \begin{bmatrix} u_m \\ v_m \end{bmatrix} \right\|_{\mathscr{H}} \leq M^2 |\lambda_m - \lambda_n| \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathscr{H}} \to 0,$$

as $n, m \to \infty$, which implies (3.13).

(2) We put $\lambda = ik$, $k \neq 0$ and $(ikI - A) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$. Then we have

$$ik \|\nabla u\|^2 + (1 - ik)\|v\|^2 = (\nabla f, \nabla u) + (v, g). \tag{3.15}$$

Taking the real part of (3.15) we have

$$\|v\|^2 \leq \|g\|^2 + 2\|\nabla f\| \, \|\nabla u\|. \tag{3.16}$$

Taking the imaginary part of (3.15) and considering (3.16) we have

$$\|\nabla u\|^2 \leq \left(3 + \frac{1}{|k|^2}\right) \|g\|^2 + 2\left(9 + \frac{1}{|k|^2}\right) \|\nabla f\|^2,$$

which implies (2). □

By the resolvent equation and (3.4) we have the following lemma.

Lemma 3.5. *Put $b(a) = a/2(2 + 7a)$. Then for any $a > 0$ there exists an $M_a > 0$ such that*

$$\|(\lambda I - A)^{-1}\| \leq M_a \tag{3.17}$$

for $\lambda \in D_{a,b(a)} = \{\lambda \in \boldsymbol{C} \,|\, |\mathrm{Re}\,\lambda| \leq b(a)$ *and* $|\mathrm{Im}\,\lambda| \geq a\} \cup \{\lambda \in \boldsymbol{C} \,|\, \mathrm{Re}\,\lambda \geq b(a)\}$.

Proof. At first, we consider the case that $\lambda = \mu + ik$, $|k| \geq a$. When $\lambda = ik$, $|k| \geq a > 0$, from (2) of Lemma 3.4, we have $\|(ikI - A)^{-1}\| \leq 7 + 2/a$. By the resolvent equation we obtain the estimate:

$$\|((\mu + ik)I - A)^{-1}\| \leq 7 + \frac{2}{a} + |\mu| \left(7 + \frac{2}{a}\right) \|((\mu + ik)I - A)^{-1}\| .$$

Since $b(a) = a/2(2 + 7a)$, then we have $\|(\lambda I - A)^{-1}\| \leq 2(7 + 2/a)$ for $|\mu| \leq b(a)$ and $|k| \geq a$. Moreover from (2) of Lemma 3.4 we have $\|(\lambda I - A)^{-1}\| \leq 1/\mathrm{Re}\,\lambda \leq 1/b(a) = 2(7 + 2/a)$ for $\mathrm{Re}\,\lambda \geq b(a)$. Therefore if we put $M_a = 2(7 + 2/a)$, then (3.17) holds. □

§ 4. A proof of Theorem 1.2

In view of § 3, we know the following fact.

(f.1) Let $a > 0$. Then, $\rho(A) \supset D_{a,b(a)}$ and $\sup\{\|(\lambda I - A)^{-1}\| \,|\, \lambda \in D_{a,b(a)}\} \leq M_a$.

We know also that:

(f.2) Let $R \geq r_0$. For any $\mathbf{x} \in \mathscr{H}_R$ there exists a sequence $\{\mathbf{x}_j\} \subset C_0^\infty(\Omega_{R+1})$ such that $\mathbf{x}_j \to \mathbf{x}$ in $\mathscr{H}$.

Here and hereafter we put $\mathscr{H}_R = \left\{\begin{bmatrix} u \\ v \end{bmatrix} \in \mathscr{H} \,|\, \mathrm{supp}\, u, \mathrm{supp}\, v \subset \Omega_R\right\}$. In fact, since $C_0^\infty(\Omega)$ is dense in $\mathscr{H}$, there exists a sequence $\{\mathbf{x}_j\} \subset C_0^\infty(\Omega)$ such that $\mathbf{x}_j \to \mathbf{x}$ in $\mathscr{H}$. Below, $\varphi_R(x)$ always refers to a function in $C_0^\infty(\boldsymbol{R}^n)$ such that $\varphi_R(x) = 1$ if $|x| \leq R$ and $= 0$ if $|x| \geq R + 1$. Since $\varphi_R \mathbf{x} = \mathbf{x}$, from Lemma 2.3

$$\|\varphi_R \mathbf{x}_j - \mathbf{x}\|_{\mathscr{H}} = \|\varphi_R(\mathbf{x}_j - \mathbf{x})\|_{\mathscr{H}} \leq C(R)\|\mathbf{x}_j - \mathbf{x}\|_{\mathscr{H}} \to 0 \qquad \text{as } j \to \infty ,$$

which implies that $\{\varphi_R \mathbf{x}_j\}$ satisfies the desired property.

Now we shall introduce some function spaces. Let E be a Banach space with norm $|\cdot|_E$, $N \geq 0$ an integer and $k = N + \sigma$ with $0 < \sigma \leq 1$. Put

$$\mathscr{C}^k = \mathscr{C}^k(\boldsymbol{R}^1; E) = \{u \in C^\infty(\boldsymbol{R}^1 \backslash \{0\}; E) \,|\, \langle\!\langle u \rangle\!\rangle_{k,E} < \infty\} ,$$

where

$$\langle\!\langle u\rangle\!\rangle_{k,E} = \sum_{j=0}^{N} \int_{\boldsymbol{R}} \left|\left(\frac{d}{d\tau}\right)^{j} u(\tau)\right|_{E} d\tau$$
$$+ \sup_{h\neq 0} |h|^{-\sigma} \int_{\boldsymbol{R}} \left|\left(\frac{d}{d\tau}\right)^{N} u(\tau+h) - \left(\frac{d}{d\tau}\right)^{N} u(\tau)\right|_{E} d\tau \qquad \text{if } 0<\sigma<1\,,$$

$$\langle\!\langle u\rangle\!\rangle_{k,E} = \sum_{j=0}^{N} \int_{\boldsymbol{R}} \left|\left(\frac{d}{d\tau}\right)^{j} u(\tau)\right|_{E} d\tau$$
$$+ \sup_{h\neq 0} |h|^{-1} \int_{\boldsymbol{R}} \left|\left(\frac{d}{d\tau}\right)^{N} u(\tau+2h) - 2\left(\frac{d}{d\tau}\right)^{N} u(\tau+h) + \left(\frac{d}{d\tau}\right)^{N} u(\tau)\right|_{E} d\tau\,,$$

if $\sigma = 1$. Here, $\left(\frac{d}{d\tau}\right)^{0} = 1$. Moreover, we put

$$\mathscr{H}_{loc} = \left\{\begin{bmatrix} u \\ v \end{bmatrix} \middle| u \in H^1(\Omega_R),\, v\in L^2(\Omega_R) \quad \forall R\ge r_0\right\}\,,$$

$$\mathscr{H}_{comp} = \bigcup_{R\ge r_0} \mathscr{H}_R\,.$$

$\mathscr{L}(B_1, B_2)$ denotes the set of all bounded linear operators from B_1 into B_2 and $Anal(I, B)$ the set of all B-valued analytic functions in I. In §5, we shall show the following fact:

(f.3) Put $Q_d = \{\lambda\in \boldsymbol{C} | 0 < \mathrm{Re}\,\lambda < d, |\mathrm{Im}\,\lambda| < d\}$. Then, there exist a $d>0$ and an $R(\lambda)\in Anal(Q_d; \mathscr{L}(\mathscr{H}_{comp}, \mathscr{H}_{loc}))$ such that:

(a) $R(\lambda)\mathbf{x} = (\lambda I - A)^{-1}\mathbf{x}$ for $\mathbf{x}\in\mathscr{H}_{comp}$ and $\lambda\in Q_d$;

(b) For any $R\ge r_0$ and $\rho(s)\in C_0^\infty(\boldsymbol{R})$ such that $\rho(s)=1$ if $|s|<d/2$ and $=0$ if $|s|>d$, there exists an $M_1>0$ depending on R, ρ and d such that

$$\langle\!\langle \rho(\cdot)(\varphi_R R(\alpha+i\cdot)\mathbf{x}, \mathbf{y})_{\mathscr{H}}\rangle\!\rangle_{n/2,R} \le M_1 \|\mathbf{x}\|_{\mathscr{H}} \|\mathbf{y}\|_{\mathscr{H}}$$

for any $\mathbf{x}\in\mathscr{H}_R$, $\mathbf{y}\in\mathscr{H}$ and $0<\alpha<d$.

By using facts (f.1)–(f.3) we prove the following main result of this section, which will imply Theorem 1.2.

Proposition 4.1. *We have*

$$\|\varphi_R T(t)\mathbf{x}\|_{\mathscr{H}} \le C(1+t)^{-n/2}\|\mathbf{x}\|_{\mathscr{H}} \tag{4.1}$$

for $\mathbf{x}\in\mathscr{H}_R$, *where* $C = C(R, M_a, M_1)$.

Before going to a proof of Proposition 4.1, we shall prepare some lemmas, below. Since A is dissipative, $T(t)$ is a C^0 semigroup of contractions, so that

$$\|T(t)\| \le 1 \qquad \forall t\ge 0\,. \tag{4.2}$$

By a lemma due to F. Huang in [5, §1, Lemma 1] (also see [7]), we have the following lemma.

Lemma 4.2. *For any* $\alpha > 0$ *and* $\mathbf{x} \in \mathscr{H}$, *put*

$$g(\omega) = \|((\alpha + i\omega)I - A)^{-1}\mathbf{x}\|_{\mathscr{H}}.$$

Then $g(\omega) \in L^2(\mathbf{R})$ *and*

$$\lim_{|\omega|\to\infty} g(\omega) = 0\,, \tag{4.3}$$

$$\int_{-\infty}^{\infty} g(\omega)^2 d\omega \le \frac{\pi}{\alpha}\|\mathbf{x}\|_{\mathscr{H}}^2\,. \tag{4.4}$$

The following lemma is concerned with the properties of the Fourier transformation of functions belonging to $\mathscr{C}^k$, which was proved in [14, Part 1, Theorem 3.7].

Lemma 4.3. *Let E be a Banach space with norm $|\cdot|_E$. Let $N \ge 0$ be an integer and σ be a positive number ≤ 1. Assume that $f \in \mathscr{C}^{N+\sigma}(\mathbf{R}^1; E)$. Put*

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\exp(\sqrt{-1}\tau t)d\tau\,.$$

Then,

$$|F(t)|_E \le C(1 + |t|)^{-(N+\sigma)}\langle\!\langle f\rangle\!\rangle_{N+\sigma,E}\,.$$

Proof of Proposition 4.1. Let α be a fixed positive number. In view of (4.2), we have the following expression:

$$T(t)\mathbf{x} = \lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\alpha-i\omega}^{\alpha+i\omega} e^{\lambda t}(\lambda I - A)^{-1}\mathbf{x}d\lambda \tag{4.5}$$

(cf. [12, p. 295] or [13, Chapter 1, Corollary 7.5]). Let us take $\alpha < d$, $\mathbf{x} \in C_0^\infty(\Omega_{R+1})$ and $\mathbf{y} \in \mathscr{H}$. Note that $\mathbf{x} \in D(A^2) \cap \mathscr{H}_{R+1}$. Then,

$$\begin{aligned}(\varphi_R T(t)\mathbf{x}, \mathbf{y})_{\mathscr{H}} &= \frac{1}{2\pi}\lim_{|\omega|\to\infty}\int_{-\omega}^{\omega} e^{(\alpha+is)t}(\varphi_R((\alpha+is)I - A)^{-1}\mathbf{x}, \mathbf{y})_{\mathscr{H}}ds \\ &= \frac{1}{2\pi}e^{\alpha t}\int_{-\infty}^{\infty} e^{ist}\rho(s)(\varphi_R((\alpha+is)I - A)^{-1}\mathbf{x}, \mathbf{y})_{\mathscr{H}}ds \\ &\quad + \frac{1}{2\pi}e^{\alpha t}\lim_{|\omega|\to\infty}\int_{-\omega}^{\omega} e^{ist}(1-\rho(s))(\varphi_R((\alpha+is)I - A)^{-1}\mathbf{x}, \mathbf{y})_{\mathscr{H}}ds \\ &= J_1(t) + J_2(t)\,.\end{aligned}$$

Now we consider the term $J_1(t)$. (a) of (f.3) implies that $\rho(s)((\alpha+is)I-A)^{-1}\mathbf{x}=\rho(s)R(\alpha+is)\mathbf{x}$. From (b) of (f.3) and Lemma 4.3 it follows that

$$
\begin{aligned}
(4.6)\qquad |J_1(t)| &\le e^{\alpha t}C(1+|t|)^{-n/2}\langle\!\langle \rho(\cdot)(\varphi_R R(\alpha+i\cdot)\mathbf{x},\mathbf{y})_{\mathscr{H}}\rangle\!\rangle_{n/2,\boldsymbol{R}} \\
&\le e^{\alpha t}CM_1(1+|t|)^{-n/2}\|\mathbf{x}\|_{\mathscr{H}}\|\mathbf{y}\|_{\mathscr{H}}\,.
\end{aligned}
$$

Next we consider the term $J_2(t)$. Let us put $J_2(t)=(e^{\alpha t}/2\pi)\lim_{|\omega|\to\infty} L_\omega(t)$. Using the identity $(it)^{-1}de^{ist}/ds=e^{ist}$ we have, by integration by parts

$$
\begin{aligned}
L_\omega(t) &= \frac{1}{it}[e^{ist}(1-\rho(s))(\varphi_R((\alpha+is)I-A)^{-1}\mathbf{x},\mathbf{y})_{\mathscr{H}}]_{s=-\omega}^{\omega} \\
&+\frac{-1}{(it)^2}\left[e^{ist}\frac{d}{ds}\{(1-\rho(s))(\varphi_R((\alpha+is)I-A)^{-1}\mathbf{x},\mathbf{y})_{\mathscr{H}}\}\right]_{s=-\omega}^{\omega} \\
&+\cdots+\frac{(-1)^{l-1}}{(it)^l}\left[e^{ist}\frac{d^{l-1}}{ds^{l-1}}\{(1-\rho(s))(\varphi_R((\alpha+is)I-A)^{-1}\mathbf{x},\mathbf{y})_{\mathscr{H}}\}\right]_{s=-\omega}^{\omega} \\
&+\frac{(-1)^l}{(it)^l}\int_{-\omega}^{\omega} e^{ist}\frac{d^l}{ds^l}\{(1-\rho(s))(\varphi_R((\alpha+is)I-A)^{-1}\mathbf{x},\mathbf{y})_{\mathscr{H}}\}ds.
\end{aligned}
$$

Since we have, by (f.1)

$$
\left\|\frac{d^j}{ds^j}((\alpha+is)I-A)^{-1}\right\|\le j!\,M_a^j\|((\alpha+is)I-A)^{-1}\|\qquad \text{for } |s|>a
$$

it follows from (4.3) of Lemma 4.2 that

$$
\left|\left[e^{ist}\frac{d^j}{ds^j}\{(1-\rho(s))(\varphi_R((\alpha+is)I-A)^{-1}\mathbf{x},\mathbf{y})_{\mathscr{H}}\}\right]_{s=-\omega}^{\omega}\right|\to 0\qquad \text{as } |\omega|\to\infty\,.
$$

Let the last term of $L_\omega(t)$ be $(-1)^l L_\omega^{l+1}(t)/(it)^l$. Noting that Lemma 4.2 holds for the adjoint operator, we have

$$
\begin{aligned}
L_\omega^{l+1}(t) &\le l!\int_{d/2\le|s|\le\omega}(1-\rho(s))|((\alpha+is)I-A)^{-1}\mathbf{x},((\alpha-is)I-A^*)^{-1}\varphi_R\mathbf{y})_{\mathscr{H}}|ds \\
&+\sum_{j=0}^{l-1}\binom{l}{j}j!\int_{d/2\le|s|\le\omega}\left|\frac{d^j}{ds^j}\rho(s)\right||(\varphi_R((\alpha+is)I-A)^{-(j+1)}\mathbf{x},\mathbf{y})_{\mathscr{H}}|ds \\
&=K_1+K_2\,.
\end{aligned}
$$

If we take $a<d/2$ we obtain, by (f.1) and (4.4) of Lemma 4.2, that

$$K_1 \leq Cl!M_a^{l-1}\left(\int_{d/2\leq|s|}\|((\alpha+is)I-A)^{-1}\mathbf{x}\|_{\mathscr{H}}^2ds\right)^{1/2} \tag{4.7}$$
$$\times\left(\int_{d/2\leq|s|}\|((\alpha-is)I-A^*)^{-1}\varphi_R\mathbf{y}\|_{\mathscr{H}}^2ds\right)^{1/2}$$
$$\leq C(l, M_a)\|\mathbf{x}\|_{\mathscr{H}}\|\mathbf{y}\|_{\mathscr{H}}.$$

Moreover, by (f.1)

$$K_2 \leq C(l, M_a)\|\mathbf{x}\|_{\mathscr{H}}\|\mathbf{y}\|_{\mathscr{H}} \quad \text{for any } l \geq 1. \tag{4.8}$$

Combining (4.7) and (4.8), we have

$$|J_2(t)| \leq \frac{e^{\alpha t}}{2\pi}|t|^{-l}C(l, M_a)\|\mathbf{x}\|_{\mathscr{H}}\|\mathbf{y}\|_{\mathscr{H}}. \tag{4.9}$$

Letting $\alpha \to 0$ in (4.6) and (4.9), we obtain (4.1) for any $\mathbf{x} \in C_0^\infty(\Omega_{R+1})$. By (f.2), (4.1) holds for any $\mathbf{x} \in \mathscr{H}_R$. □

Theorem 1.2 follows from Proposition 4.1 and Lemma 2.3.

§5. Behavior of $R(\lambda)$ near $\lambda = 0$

Our purpose in this section is to show (f.3) in §4. Namely, we shall investigate the behavior of the resolvent in a neighborhood of $\lambda = 0$. When $n \geq 3$, (f.3) was proved by Shibata [14, Part 1], so that we shall discuss the case that $n = 2$ only.

5.1 Reduction to a simple case. Let us consider the following exterior Dirichlet problem

$$(P_\lambda) \qquad (\lambda - \Delta)u = f \quad \text{in } \Omega \subset \boldsymbol{R}^2 \quad \text{and} \quad u = 0 \quad \text{on } \Gamma,$$

where $\lambda \in S_{r,\varepsilon} = \{\lambda \in \boldsymbol{C}\backslash\{0\} \,|\, |\lambda| < r, |\arg\lambda| < \pi - \varepsilon\}$, $0 < r < 1$ and $0 < \varepsilon < \pi/2$. The main step in proving (f.3) is the following theorem.

Theorem 5.1. *There exist an r and an $A(\lambda) \in Anal(S_{r,\varepsilon}; \mathscr{L}(L_{comp}, H^2(\Omega)))$ such that*

$$(\lambda - \Delta)A(\lambda)f = f \quad \text{in } \Omega \quad \text{and} \quad A(\lambda)f = 0 \quad \text{on } \Gamma,$$

for $f \in L_{comp}$ and $\lambda \in S_{r,\varepsilon}$, where

$$L_{comp} = \bigcup_{R \geq r_0} L_R^2(\Omega).$$

Moreover, for any $R \geq r_0$ and $\varphi_R \in C_0^\infty(\boldsymbol{R}^2)$ such that $\varphi_R = 1$ for $|x| \leq R$ and

$= 0$ *for* $|x| \geq R + 1$ *there exists a* $C = C(\varphi_R, R)$ *such that*

$$\|\varphi_R A(\lambda)f\|_1 \leq C\|f\|, \tag{5.1.1}$$

$$\left\| \varphi_R \frac{d}{d\lambda} A(\lambda)f \right\|_1 \leq C|\mathrm{Im}\,\lambda|^{-1}\|f\|, \tag{5.1.2}$$

$$\left\| \varphi_R \frac{d^2}{d\lambda^2} A(\lambda)f \right\|_1 \leq C|\mathrm{Im}\,\lambda|^{-2}\|f\| \tag{5.1.3}$$

for $f \in L^2_R(\Omega)$ *and* $\lambda \in S_{r,\varepsilon}$.

Postponing the proof of Theorem 5.1, we shall show (f.3). The following lemma immediately follows from Lemma 3.4 of [14].

Lemma 5.2. *Let* $\mathscr{B}$ *be a Banach space with norm* $|\cdot|$. *Let* $f(\tau) \in C^2(\boldsymbol{R}^1\backslash\{0\}; \mathscr{B})$. *If* $\left|\left(\frac{d}{d\tau}\right)^j f(\tau)\right| \leq C(f)|\tau|^{-j}$, $\forall\tau \in \boldsymbol{R}^1\backslash\{0\}$, $j = 0, 1, 2$, *then,*

$$\int_{-\infty}^{\infty} |f(\tau + 2h) - 2f(\tau + h) + f(\tau)|d\tau \leq C(f)|h|.$$

Combining Theorem 5.1 and Lemma 5.2, we have the following lemma.

Lemma 5.3. *For any* $f \in L^2_R(\Omega)$, $g \in H_D(\Omega)$, α *such that* $0 < \alpha < 2r/3$ *and* $\tilde{\rho}(s) \in C_0^\infty(\boldsymbol{R})$ *such that* $\operatorname{supp}\tilde{\rho}(s) \subset \{|s| \leq 2r/3\}$, *we have*

$$\langle\!\langle \tilde{\rho}(\cdot)(\varphi_R A(\alpha + i\cdot)f, g)_D \rangle\!\rangle_{1,\boldsymbol{R}} \leq M_2\|f\|\,\|\nabla g\|, \tag{5.1.4}$$

where M_2 *is a constant depending essentially on* $\tilde{\rho}$, R *and* φ_R *only.*

Proof of (f.3). In terms of $A(\lambda)$, we shall represent $(\lambda I - A)^{-1}$. If we put

$$(\lambda I - A)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

for $\begin{bmatrix} u \\ v \end{bmatrix} \in D(A)$, then we have

$$v = \lambda u - f \quad \text{and} \quad \{\lambda(\lambda + 1) - \Delta\}u = (\lambda + 1)f + g \qquad \text{in } \Omega.$$

We take $r' < r$ so small that there exists an $\varepsilon' < \pi/2$ such that $\lambda(\lambda + 1) \in S_{r,\varepsilon}$ if $\lambda \in S_{r',\varepsilon'}$. We expect to get

$$u = A(\lambda(\lambda + 1))\{(\lambda + 1)f + g\}$$

and

$$v = \lambda A(\lambda(\lambda + 1))\{(\lambda + 1)f + g\} - f$$

for $\mathbf{x} \equiv \begin{bmatrix} f \\ g \end{bmatrix} \in \mathscr{H}_{comp}$. From this consideration, if we put

$$R(\lambda) = \begin{bmatrix} (\lambda + 1)A(\lambda(\lambda + 1)) & A(\lambda(\lambda + 1)) \\ \lambda(\lambda + 1)A(\lambda(\lambda + 1)) - 1 & \lambda A(\lambda(\lambda + 1)) \end{bmatrix},$$

we have

$$R(\lambda)\mathbf{x} = (\lambda I - A)^{-1}\mathbf{x} \qquad \text{for } \mathbf{x} \in \mathscr{H}_{comp} \quad \text{and} \quad \lambda \in S_{r',\varepsilon'},$$

because $R(\lambda)\mathbf{x} \in D(A)$ as it follows from the fact that $A(\lambda(\lambda + 1)) \in \mathscr{L}(L_{comp}, H^2(\Omega) \cap H_D(\Omega))$. By Theorem 5.1 and Lemma 5.3, we also know that $R(\lambda)$ satisfies all the properties mentioned in (f.3) with $d = 2r'/3$. □

5.2 Potential theory and integral equations. In order to express a solution to (P_λ), we shall deal with potential theory. Our strategy follows Borchers and Varnhorn [2] mainly.

A fundamental solution E_λ satisfying the distributional identity $(\lambda - \Delta)E_\lambda = \delta$ can be written as follows:

$$E_\lambda(x) = F^{-1}\left[\frac{1}{\lambda + i0 + |\xi|^2}\right] = \frac{1}{2\pi} K_0(|x|\sqrt{\lambda}).$$

Here and hereafter, F^{-1} denotes the Fourier inverse transform, $\sqrt{\lambda} \in \boldsymbol{C}$ denotes the particular square root of $\lambda \in S_{r,\varepsilon}$ with $\operatorname{Re}\sqrt{\lambda} \geq 0$, and K_n $(n \in \boldsymbol{N} \cup \{0\})$ the modified Bessel function of order n. Especially in the case that $\lambda = 0$,

$$E_0(x) = \frac{1}{2\pi} \log \frac{1}{|x|}.$$

Let us introduce the boundary layer potentials with source densities $\Psi \in C^0(\Gamma)$. We define the single layer potential by the formula:

$$E_\lambda \Psi(x) = \int_\Gamma E_\lambda(x - y)\Psi(y) do_y .$$

Now $E_\lambda(x - y)$ has the following form:

$$(5.2.1) \qquad E_\lambda(x - y) = E_0(x - y) + \frac{1}{2\pi}\left\{\log \frac{1}{\sqrt{\lambda}} + \log 2 - \gamma + E_\lambda^0(x - y)\right\},$$

where $\gamma = -\dfrac{\Gamma'(1)}{\Gamma(1)}$ (Γ being the Gamma function),

$$(5.2.2) \qquad E_\lambda^0(x - y) = \tilde{O}(|\log \lambda||\lambda|), \quad \frac{d}{d\lambda} E_\lambda^0(x - y) = \tilde{O}(|\log \lambda|)$$

$$\text{and} \quad \frac{d^2}{d\lambda^2} E_\lambda^0(x - y) = \tilde{O}\left(\frac{1}{|\lambda|}\right)$$

(cf. [1]). Here and hereafter, $\tilde{O}(f(\lambda))$ represents the terms satisfying the following estimate:

$$|\tilde{O}(f(\lambda))| \le Cf(\lambda) \quad \forall \lambda \in S_{r,\varepsilon}, \qquad |x| \text{ and } |y| \text{ being bounded}.$$

We define the double layer potential by the formula:

$$D_\lambda \Psi(x) = \int_\Gamma D_\lambda(x, y)\Psi(y)do_y ,$$

where

$$D_\lambda(x, y) = \nabla_x E_\lambda(x - y)\cdot N(y) = -\frac{1}{2\pi}K_1(|x - y|\sqrt{\lambda})\frac{\sqrt{\lambda}}{|x - y|}(x - y)\cdot N(y);$$

$$D_0(x, y) = \frac{1}{2\pi}\nabla_x \log\frac{1}{|x - y|}\cdot N(y) = -\frac{(x - y)\cdot N(y)}{2\pi|x - y|^2}.$$

Here $N(y)$ denotes the interior unit normal of Γ at $y \in \Gamma$. $D_\lambda(x, y)$ has the following form:

(5.2.3) $$D_\lambda(x, y) = D_0(x, y) + D_\lambda^0(x, y),$$

where

(5.2.4) $$D_\lambda^0(x, y) = \tilde{O}(|\log\lambda||\lambda|), \qquad \frac{d}{d\lambda}D_\lambda^0(x, y) = \tilde{O}(|\log\lambda|)$$

$$\text{and} \quad \frac{d^2}{d\lambda^2}D_\lambda^0(x, y) = \tilde{O}\left(\frac{1}{|\lambda|}\right)$$

(cf. [1]). To represent the normal derivative of $E_\lambda \Phi$ at Γ, let us define $H_\lambda \Psi(x)$ ($\lambda \in S_{r,\varepsilon} \cup \{0\}$) in a neighborhood U of Γ by the formula:

$$H_\lambda \Psi(x) = \int_\Gamma H_\lambda(x, y)\Psi(y)do_y , \qquad H_\lambda(x, y) = -\nabla_x E_\lambda(x - y)\cdot N(\tilde{x}),$$

where $x \in U$, and $\tilde{x} \in \Gamma$ denotes the unique projection of x on Γ. From the definition we have

(5.2.5a) $$(H_\lambda \Psi)^\pm(x) = -(\nabla_x E_\lambda \Psi)^\pm(x)\cdot N(x),$$

(5.2.5b) $$\langle D_\lambda \Phi, \Psi\rangle = \langle \Phi, H_{\bar{\lambda}}\Psi\rangle,$$

where

$$w^\pm(x) = \lim_{t\to 0+} w(x \pm tN(x)) \quad \text{for } x \in \Gamma, \qquad \langle \Phi, \Psi\rangle = \int_\Gamma \Phi\bar{\Psi}do.$$

Since we know

(5.2.6) $|K_n(\sqrt{\lambda}|x|)| \leq C \exp(-c\sqrt{|\lambda|}|x|)$ as $\sqrt{|\lambda|}|x| \geq r > 0$ and $\lambda \in S_{r,\varepsilon}$

with some constants C and c depending on ε and r (cf. [1]), we have

(5.2.7) $$|\partial_x^\alpha E_\lambda(x-y)|, |\partial_x^\alpha D_\lambda(x, y)| \leq C(\alpha, r, \varepsilon) \exp(-c\sqrt{|\lambda|}|x-y|)$$

when $\sqrt{|\lambda|}|x-y| \geq r$ and $\lambda \in S_{r,\varepsilon}$. We shall use the following well-known results of classical potential theorem (cf. [4, Chapter 3]).

Proposition 5.4. *The double layer potential $D_0\beta$ with constant source density $\beta \in \boldsymbol{C}$ satisfies the following relations*

(5.2.8) $$(D_0\beta)(x) = \begin{cases} \beta & x \in \Omega^c, \\ \beta/2 & x \in \Gamma, \\ 0 & x \in \Omega. \end{cases}$$

Proposition 5.5. *Let $\Psi \in C^0(\Gamma)$ be given. Then we have*

(5.2.9) $$(E_\lambda \Psi)^- = E_\lambda \Psi = (E_\lambda \Psi)^+$$

(5.2.10) $$(D_\lambda \Psi)^- - D_\lambda \Psi = \frac{1}{2}\Psi = D_\lambda \Psi - (D_\lambda \Psi)^+$$

(5.2.11) $$(H_\lambda \Psi)^- - H_\lambda \Psi = -\frac{1}{2}\Psi = H_\lambda \Psi - (H_\lambda \Psi)^+ .$$

Let us consider the exterior Dirichlet problem (Q_λ) of the form

(Q_λ) $$(\lambda - \Delta)u = 0 \quad \text{in } \Omega \quad \text{and} \quad u|_\Gamma = b \quad \text{on } \Gamma,$$

where $b \in C^0(\Gamma)$ is given. Concerning the uniqueness of classical solutions of (Q_0), we have the following lemma:

Lemma 5.6. *The solution of (Q_0) is unique provided that*

(5.2.12) $$u(x) = O(1), \quad \nabla u(x) = O(|x|^{-1}) \quad \text{as } |x| \to \infty .$$

Next lemma describes a decay property of the potential $E_0\Phi$.

Lemma 5.7. *Let $\Phi \in C^0(\Gamma)$ with $\int_\Gamma \Phi do = 0$. Then the single layer potential $E_0\Phi$ satisfies the following decay property*:

$$E_0\Phi = O(|x|^{-1}) \quad \text{as } |x| \to \infty .$$

Since Lemmas 5.6 and 5.7 can be proved by similar arguments as in [2], we omit the proofs.

In order to prove the existence of a solution u of (Q_λ) $(\lambda \in S_{r,\varepsilon})$ and (Q_0), let us introduce a boundary integral operator B_λ by the formula:

$$B_\lambda \Phi(x) = D_\lambda \Phi(x) + \frac{2\pi}{|\Gamma| \log \sqrt{\lambda}} E_\lambda \Phi(x) \qquad \text{for } \lambda \in S_{r,\varepsilon}, \tag{5.2.13a}$$

$$B_0 \Phi(x) = D_0 \Phi(x) - \Phi_M \tag{5.2.13b}$$

for $\Phi \in C^0(\Gamma)$ where

$$\Phi_M = \frac{1}{|\Gamma|} \int_\Gamma \Phi do \quad \text{and} \quad |\Gamma| = \int_\Gamma do .$$

Obviously $(\lambda - \Delta) B_\lambda \Phi = 0$ in Ω for any $\Phi \in C^0(\Gamma)$, so that the problem is how to compensate the boundary value. To consider this problem let us introduce the operator K_λ: $C^0(\Gamma) \to C^0(\Gamma)$ by the formula:

$$K_\lambda \Phi = \left(-\frac{1}{2} + D_\lambda + \frac{2\pi}{|\Gamma| \log \sqrt{\lambda}} E_\lambda \right) \Phi \qquad \text{for } \lambda \in S_{r,\varepsilon}, \tag{5.2.14a}$$

$$K_0 \Phi = \left(-\frac{1}{2} + D_0 \right) \Phi - \Phi_M . \tag{5.2.14b}$$

In view of Proposition 5.5 we have $(B_\lambda \Phi)^+(x) = K_\lambda \Phi(x)$ for $x \in \Gamma$ and $\lambda \in S_{r,\varepsilon} \cup \{0\}$. If we shall show the existence of the inverse operator K_λ^{-1} of K_λ, then (Q_λ) is solved by the formula: $u = B_\lambda K_\lambda^{-1} b$, so that the following lemma as well as the next one is a key of our discussion.

Lemma 5.8. *Given $b \in C^0(\Gamma)$, there exists a unique solution $\Phi \in C^0(\Gamma)$ of the equation*: $K_0 \Phi = b$ *on* Γ.

Proof. We employ essentially the same argument as in the proof of Theorem 3.4 of [2]. Since K_0 is a Fredholm operator on $C^0(\Gamma)$ we study the following homogeneous equation for the adjoint operator:

$$K_0^* \Phi = \left(-\frac{1}{2} + H_0 \right) \Phi - \Phi_M = 0 \qquad \text{on } \Gamma, \tag{5.2.15}$$

where we have used (5.2.5b). Let $\Phi \in C^0(\Gamma)$ be a solution of (5.2.15). Then from (5.2.9) and (5.2.11) we obtain

$$(H_0 \Phi)^- = -\frac{1}{2} \Phi + H_0 \Phi = \Phi_M \qquad \text{on } \Gamma.$$

Now we get $\Phi_M = 0$. In fact, for any $\beta \in \boldsymbol{C}$ we have $K_0 \beta = -\beta$ because $D_0 \beta = \beta/2$ on Γ from (5.2.8), hence $-\langle \beta, \Phi \rangle = \langle K_0 \beta, \Phi \rangle = \langle \beta, K_0^* \Phi \rangle = 0$, which implies $\Phi_M = 0$. Therefore,

$$(H_0\Phi)^- = 0 \quad \text{on } \Gamma. \tag{5.2.16}$$

By (5.2.11) we have

$$\Phi = (H_0\Phi)^+ - (H_0\Phi)^- = (H_0\Phi)^+ \quad \text{on } \Gamma. \tag{5.2.17}$$

Putting $u = E_0\Phi$ we obtain, from Green's first identity, (5.2.5a) and (5.2.16),

$$\int_{\Omega^c} |\nabla u|^2 dy = 0.$$

Therefore by (5.2.9) we have $u = C$ (= const.) on $\overline{\Omega^c}$. If we put $\bar{u} = u - C$, $\bar{u}$ is a solution of (Q_0) in Ω with $b = 0$ on Γ. Since $\Phi_M = 0$, it follows from Lemma 5.7 that $\bar{u}$ satisfies (5.2.12). By Lemma 5.6 $u = C$ in Ω. Therefore by (5.2.5a), $(H_0\Phi)^+ = 0$ on Γ, which together with (5.2.17) yields $\Phi = 0$. Applying the Fredholm alternative theorem, we have the lemma. □

Lemma 5.9. *Let $\lambda \in \boldsymbol{C}$ and let K_λ and K_0: $C^0(\Gamma) \to C^0(\Gamma)$ be the boundary integral operators defined by* (5.2.14). *Then there exists an $r \in (0, 1)$ such that for $\lambda \in S_{r,\varepsilon}$ the inverse K_λ^{-1} of K_λ exists. Moreover, we have the following estimates*:

$$\|K_\lambda^{-1}\| \le 2\|K_0^{-1}\|, \tag{5.2.18a}$$

$$\left\| \frac{d}{d\lambda} K_\lambda^{-1}\Phi \right\|_{L^\infty(\Gamma)} \le \frac{C}{|\lambda||\log \lambda|^2} \|\Phi\|_{L^\infty(\Gamma)}, \tag{5.2.18b}$$

$$\left\| \frac{d^2}{d\lambda^2} K_\lambda^{-1}\Phi \right\|_{L^\infty(\Gamma)} \le \frac{C}{|\lambda|^2|\log \lambda|^2} \|\Phi\|_{L^\infty(\Gamma)}. \tag{5.2.18c}$$

$$\|(K_0 - K_\lambda)\Phi\|_{L^\infty(\Gamma)} \le C\,|\log \lambda|^{-1} \|\Phi\|_{L^\infty(\Gamma)}, \tag{5.2.19a}$$

$$\left\| \frac{d}{d\lambda}(K_0 - K_\lambda)\Phi \right\|_{L^\infty(\Gamma)} \le \frac{C}{|\lambda||\log \lambda|^2} \|\Phi\|_{L^\infty(\Gamma)}, \tag{5.2.19b}$$

$$\left\| \frac{d^2}{d\lambda^2}(K_0 - K_\lambda)\Phi \right\|_{L^\infty(\Gamma)} \le \frac{C}{|\lambda|^2|\log \lambda|^2} \|\Phi\|_{L^\infty(\Gamma)} \tag{5.2.19c}$$

for any $\Phi \in C^0(\Gamma)$.

Proof. We use (5.2.2) and (5.2.4). The proof is the same as in [2, Proposition 3.8] and omitted. □

5.3 Proof of Theorem 5.1. Put

$$A(\lambda)f = (\lambda - \Delta)^{-1}Ef - B_\lambda K_\lambda^{-1} f_\lambda \qquad \text{for } f \in L_{comp}, \tag{5.3.1}$$

where $(\lambda - \Delta)^{-1}Ef = \int E_\lambda(x-y)Ef(y)dy$, E is an extension operator by zero from $f \in L_{comp}$ to $Ef \in L^2(\boldsymbol{R}^2)$ and $f_\lambda = (\lambda - \Delta)^{-1}Ef|_\Gamma$ is the restriction of the whole space solution to the boundary Γ. Since $(\lambda - \Delta)^{-1}Ef \in H^2(\boldsymbol{R}^2)$ for $\lambda \in S_{r,\varepsilon}$ as follows from Parseval's formula, the Sobolev's imbedding theorem implies $f_\lambda \in C^0(\Gamma)$ for $\lambda \in S_{r,\varepsilon}$. Therefore $A(\lambda) \in Anal(S_{r,\varepsilon}; \mathscr{L}(L_{comp}, H^2(\Omega))$. Moreover $u = A(\lambda)f$ satisfies the equations (P_λ). Our main task is to show (5.1.1), (5.1.2) and (5.1.3). Let us start with the following proposition.

Proposition 5.10. *Let $0 < \varepsilon < \pi$ and $\lambda \in S_{r,\varepsilon}$. Then $(\lambda - \Delta)^{-1}Ef$ is decomposed as follows:*

$$(5.3.2) \qquad (\lambda - \Delta)^{-1}Ef = -\log \lambda R^0 f + R^0_\lambda f, \qquad \text{for } f \in L^2_R(\Omega),$$

where

$$R^0 f = \frac{1}{4\pi}\int_{\boldsymbol{R}^2} Ef(y)dy,$$

$$R^0_\lambda f = \frac{1}{2\pi}\int_{\boldsymbol{R}^2}\left(\log\frac{1}{|x-y|} + \log 2 - \gamma + E^0_\lambda(x-y)\right)Ef(y)dy.$$

Moreover the following estimates hold for $\lambda \in S_{r,\varepsilon}$:

$$(5.3.3a) \quad \|\varphi_R R^0 f\|_{0,\boldsymbol{R}^2} \le C(R)\|f\|_{0,\Omega}, \qquad \|\varphi_R R^0_\lambda f\|_{0,\boldsymbol{R}^2} \le C(R)\|f\|_{0,\Omega},$$

$$(5.3.3b) \qquad \left\|\varphi_R \frac{d}{d\lambda}R^0_\lambda f\right\|_{0,\boldsymbol{R}^2} \le C(R)|\log\lambda|\,\|f\|_{0,\Omega},$$

$$(5.3.3c) \qquad \left\|\varphi_R \frac{d^2}{d\lambda^2}R^0_\lambda f\right\|_{0,\boldsymbol{R}^2} \le C(R)\frac{1}{|\lambda|}\|f\|_{0,\Omega}.$$

Proof. From (5.2.1) and (5.2.2) we have the decomposition (5.3.2) and the estimate (5.3.3a) by Schwarz's inequality. By (5.2.1) and (5.2.2) we have also (5.3.3b) and (5.3.3c). □

Hereafter we assume that f is a function in $L^2_R(\Omega)$ and Φ is a function in $C^0(\Gamma)$. According to (5.3.2) we set

$$f_\lambda = -\log\lambda R^0 f + R^1_\lambda f,$$

where

$$R^1_\lambda f = R^0_\lambda f|_\Gamma.$$

Then by (5.2.2) we have

$$(5.3.4a) \qquad \|R^0 f\|_{L^\infty(\Gamma)} \le C(R)\|f\|, \qquad \|R^1_\lambda f\|_{L^\infty(\Gamma)} \le C(R)\|f\|,$$

(5.3.4b) $$\left\|\frac{d}{d\lambda}R_\lambda^1 f\right\|_{L^\infty(\Gamma)} \le C(R)|\log\lambda|\,\|f\|\,, \qquad \left\|\frac{d^2}{d\lambda^2}R_\lambda^1 f\right\|_{L^\infty(\Gamma)} \le C(R)\frac{1}{|\lambda|}\|f\|\,.$$

According to (5.2.1) and (5.2.3), E_λ and D_λ are decomposed as follows:

(5.3.5) $$E_\lambda\Phi = -\frac{|\Gamma|}{4\pi}\log\lambda\Phi_M + \frac{1}{2\pi}E_\lambda^0\Phi\,,$$

(5.3.6) $$D_\lambda\Phi = D_0\Phi + D_\lambda^0\Phi\,,$$

where

$$E_\lambda^0\Phi = \int_\Gamma(-\log|x-y| + \log 2 - \gamma + E_\lambda^0(x-y))\Phi(y)do_y\,,$$

$$D_\lambda^0\Phi = \int_\Gamma D_\lambda^0(x, y)\Phi(y)do_y\,.$$

By (5.2.2) and (5.2.4), we have that

(5.3.7a) $$\|\varphi_R\Phi_M\| \le C\|\Phi\|_{L^\infty(\Gamma)}\,, \qquad \|\varphi_R E_\lambda^0\Phi\| \le C\|\Phi\|_{L^\infty(\Gamma)}\,,$$
$$\|\varphi_R E_\lambda\Phi\| \le C\,|\log\lambda|\,\|\Phi\|_{L^\infty(\Gamma)}\,,$$

(5.3.7b) $$\left\|\varphi_R\frac{d}{d\lambda}E_\lambda^0\Phi\right\| \le C\,|\log\lambda|\,\|\Phi\|_{L^\infty(\Gamma)}\,, \qquad \left\|\varphi_R\frac{d}{d\lambda}E_\lambda\Phi\right\| \le C\frac{1}{|\lambda|}\|\Phi\|_{L^\infty(\Gamma)}\,,$$

(5.3.7c) $$\left\|\varphi_R\frac{d^2}{d\lambda^2}E_\lambda^0\Phi\right\| \le C\frac{1}{|\lambda|}\|\Phi\|_{L^\infty(\Gamma)}\,, \qquad \left\|\varphi_R\frac{d^2}{d\lambda^2}E_\lambda\Phi\right\| \le C\frac{1}{|\lambda|^2}\|\Phi\|_{L^\infty(\Gamma)}\,;$$

(5.3.8a) $$\|\varphi_R D_0\Phi\| \le C\|\Phi\|_{L^\infty(\Gamma)}\,, \qquad \|\varphi_R D_\lambda^0\Phi\| \le C|\lambda|\,|\log\lambda|\,\|\Phi\|_{L^\infty(\Gamma)}\,,$$
$$\|\varphi_R D_\lambda\Phi\| \le C\|\Phi\|_{L^\infty(\Gamma)}\,,$$

(5.3.8b) $$\left\|\varphi_R\frac{d}{d\lambda}D_\lambda\Phi\right\| = \left\|\varphi_R\frac{d}{d\lambda}D_\lambda^0\Phi\right\| \le C\,|\log\lambda|\,\|\Phi\|_{L^\infty(\Gamma)}\,,$$

(5.3.8c) $$\left\|\varphi_R\frac{d^2}{d\lambda^2}D_\lambda\Phi\right\| = \left\|\varphi_R\frac{d^2}{d\lambda^2}D_\lambda^0\Phi\right\| \le C\frac{1}{|\lambda|}\|\Phi\|_{L^\infty(\Gamma)}\,,$$

where $C = C(R)$. Let us calculate $B_\lambda K_\lambda^{-1}f_\lambda$. To get the formula

(5.3.9) $$D_\lambda K_\lambda^{-1}f_\lambda = -\log\lambda D_\lambda^0 K_0^{-1}R^0 f + D_\lambda K_\lambda^{-1}R_\lambda^1 f$$
$$-\log\lambda D_\lambda K_0^{-1}(K_0 - K_\lambda)K_\lambda^{-1}R^0 f\,,$$

we use the fact that $D_0K_0^{-1}R^0f = 0$, which follows from $K_0^{-1}R^0f = R^0fK_0^{-1}1 = -R^0f$ and $D_0 1 = 0$ on Ω (cf. (5.2.8)). The fact that $K_0^{-1}1 = -1$ follows from the following observation. Since by Proposition 5.4 $D_0(-1) + 1/2 = 0$ on Γ, we have $K_0(-1)=1$, which implies that $K_0^{-1}1=-1$. To get the formula:

$$\frac{2\pi}{|\Gamma|\log\sqrt{\lambda}}E_\lambda K_\lambda^{-1}f_\lambda = -\log\lambda R^0 f - \frac{2}{|\Gamma|}E_\lambda^0 K_0^{-1}R^0 f + \frac{2\pi}{|\Gamma|\log\sqrt{\lambda}}E_\lambda K_\lambda^{-1}R_\lambda^1 f - \frac{4\pi}{|\Gamma|}E_\lambda K_0^{-1}(K_0-K_\lambda)K_\lambda^{-1}R^0 f, \tag{5.3.10}$$

we use the fact that $|\Gamma|(K_0^{-1}R^0 f)_M=\int_\Gamma - R^0 f do = -R^0 f|\Gamma|$. In view of (5.3.2) and (5.3.10), the worst term of $A(\lambda):\log\lambda R^0 f$ is cancelled, so that we have

$$\begin{aligned} A(\lambda)f &= R_\lambda^0 f + \log\lambda D_\lambda^0 K_0^{-1}R^0 f - D_\lambda K_\lambda^{-1}R_\lambda^1 f \\ &\quad + \log\lambda D_\lambda K_0^{-1}(K_0-K_\lambda)K_\lambda^{-1}R^0 f + \frac{2}{|\Gamma|}E_\lambda^0 K_0^{-1}R^0 f \\ &\quad - \frac{2\pi}{|\Gamma|\log\sqrt{\lambda}}E_\lambda K_\lambda^{-1}R_\lambda^1 f + \frac{4\pi}{|\Gamma|}E_\lambda K_0^{-1}(K_0-K_\lambda)K_\lambda^{-1}R^0 f. \end{aligned} \tag{5.3.11}$$

Applying (5.2.18), (5.2.19), (5.3.3), (5.3.4), (5.3.7) and (5.3.8) to (5.3.11) we have

$$\|\varphi_R A(\lambda)f\| \le C(R)\|f\|, \tag{5.3.12a}$$

$$\left\|\varphi_R\frac{d}{d\lambda}A(\lambda)f\right\| \le \frac{C(R)}{|\lambda||\log\lambda|}\|f\|, \tag{5.3.12b}$$

$$\left\|\varphi_R\frac{d^2}{d\lambda^2}A(\lambda)f\right\| \le \frac{C(R)}{|\lambda|^2|\log\lambda|}\|f\|. \tag{5.3.12c}$$

To get (5.1.1), (5.1.2) and (5.1.3) from (5.3.12) we use the facts that $u=A(\lambda)f$ satisfies (P_λ) and

$$\|\nabla(\varphi_R u)\|^2 = (\{-\varphi_R\Delta\varphi_R + \nabla\cdot(\varphi_R\nabla\varphi_R)\}u, u) - \mathrm{Re}(\varphi_R u, \varphi_R\Delta u);$$

$$\Delta\frac{d}{d\lambda}A(\lambda)f = A(\lambda)f + \lambda\frac{d}{d\lambda}A(\lambda)f \quad \text{on } \Omega,$$

$$\Delta\frac{d^2}{d\lambda^2}A(\lambda)f = 2\frac{d}{d\lambda}A(\lambda)f + \lambda\frac{d^2}{d\lambda^2}A(\lambda)f \quad \text{on } \Omega,$$

which complete the proof of Theorem 5.1.

References

[1] Abramowitz, M. and Stegun, I. A., *Handbook of Mathematical Functions,* New York: Dover, 1970.

[2] Borchers, W. and Varnhorn, W., On the boundedness of the Stokes semigroup in two-dimensional exterior domains, Math. Z. **213** (1993), 275–299.

[3] Chang, I-Dee and Finn, R., On the solutions of an equations occurring in continuum mechanics with application to the Stokes paradox, Arch. Rational Mech. Anal. **7** (1961), 389–401.

[4] Folland, G. B., *Introduction to Partial Differential Equations*/Preliminary informal notes of university courses and seminars in mathematics, Princeton University Press and University of Tokyo Press, Princeton, New Jersey, 1976.

[5] Huang, F. L., Characteristic condition for exponential stability of linear dynamical systems in Hibert spaces, Ann. of Diff. Eqns. **1** (1985), 43–56.

[6] Leis, R., *Initial Boundary Value Problems in Mathematical Physics*, B. G. Teubner—Verlag Stuttgart; John Wiley & Sons, Chichester et al., 1986.

[7] Liu, Z. and Zheng, S., Uniform exponential stability and approximation in control of thermoelastic system, preprint (1991).

[8] Matsumura, A., On the asymptotic behaviour of solutions of semi-linear wave equations, Publ. RIMS, Kyoto Univ. **13** (1976), 363–386.

[9] Melrose, R. B., Singularities and energy decay in acoustic scattering, Duke Math. J. **46** (1979), 43–59.

[10] Morawetz, C. S., Decay of solutions of the exterior problem for the wave equation, Comm. Pure Appl. Math. **28** (1975), 229–264.

[11] Morawetz, C. S., Ralston, J. V. and Strauss, W. A., Decay of solution outside non-trapping obstacles, *ibid* **30** (1977), 447–508.

[12] Mizohata, S., *The Theory of Partial Differential Equations*, Cambridge University Press, London, 1973.

[13] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[14] Shibata, Y., On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain, Tsukuba J. Math. **7** (1983), 1–68.

[15] Shibata, Y. and Soga, H., Scattering theory for the elastic wave equation, Publ. RIMS, Kyoto Univ. **25** (1989), 861–887.

[16] Shibata, Y. and Tsutsumi, Y., On a global existence theorem of small amplitude solutions for nonlinear wave equations in an exterior domain, Math. Z. **191** (1986), 165–199.

[17] Varnhorn, W., An explicit potential theory for the Stokes resolvent boundary value problem in three dimensions, Manus. Math. **70** (1991), 339–361.

[18] Weck, N. and Witsch, K. J., Exterior Dirichlet problem for the reduced wave equation: asymptotic analysis of low frequencies, Comm. Partial Differential Equations **16** (1991), 173–195.

nuna adreso:

Institute of Mathematics
University of Tsukuba
Tsukuba-shi, Ibaraki 305
Japan

(Ricevita la 14-an de januaro, 1994)