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# The Cauchy Problem for Degenerate Parabolic Equations and Newton Polygon

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## 1. Introduction

We are concerned with the homogeneous Cauchy problem for $p$-evolution equations:

$$L(t,x,\partial_t,\partial_x)u(t,x) \equiv \left(\partial_t^m - \sum_{j=1}^{m}\sum_{|\alpha|\le pj} a_{j\alpha}(t,x)\partial_x^\alpha\partial_t^{m-j}\right)u(t,x) = 0, \tag{1}$$

$$\partial_t^{j-1}u(0,x) = \varphi_j(x), \qquad 1 \le j \le m, \tag{2}$$

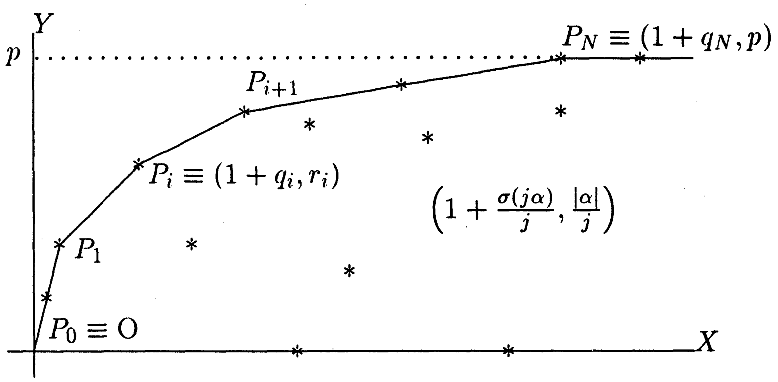
where $t \in [0,T]$, $T > 0$ and $p > 0$. It is known that the Cauchy problem is $H^\infty$-wellposed if the equation (1) is $p$-parabolic (see I. G. Petrowsky [1] and S. Mizohata [2]). There are also several papers on the Cauchy problem for degenerate parabolic equations published in the 1970's (cf. O. A. Oleĭnik [3], M. Miyake [4] and K. Igari [5]). The equations studied there are, however, of first order in $\partial_t$. Recently K. Kitagawa [6], [7] investigated necessary conditions for the Cauchy problem (1)–(2) to be well-posed. Put

$$a_{j\alpha}(t,x) \equiv t^{\sigma(j\alpha)}b_{j\alpha}(t,x), \tag{3}$$

plot the points $\left(1+\dfrac{\sigma(j\alpha)}{j}, \dfrac{|\alpha|}{j}\right)$ on the $XY$-plane and draw the Newton polygon $P_0P_1\cdots P_{N+1}$ with $P_0 \equiv (0,0)$ and $P_{N+1} \equiv (\infty, p)$. He defines a characteristic equation corresponding to each vertex $P_i$ $(1 \le i \le N)$ and one corresponding to each side $P_iP_{i+1}$ $(1 \le i \le N-1)$:

$$p_i^o(x,\lambda,\xi) \equiv \lambda^m - \sum_{(j\alpha)\in\Gamma_i^o} b_{j\alpha}(0,x)(i\xi)^\alpha\lambda^{m-j} = 0, \tag{4}$$

$$p_i(t,x,\lambda,\xi) \equiv \lambda^m - \sum_{(j\alpha)\in\Gamma_i} t^{\sigma(j\alpha)-q_ij}b_{j\alpha}(0,x)(i\xi)^\alpha\lambda^{m-j} = 0, \tag{5}$$




where $P_i \equiv (1+q_i, r_i)$,

$$\Gamma_i^o \equiv \left\{(j\alpha);\ \left(1+\frac{\sigma(j\alpha)}{j}, \frac{|\alpha|}{j}\right) \equiv P_i\right\} \text{ and } \Gamma_i \equiv \left\{(j\alpha);\ \left(1+\frac{\sigma(j\alpha)}{j}, \frac{|\alpha|}{j}\right) \in P_iP_{i+1}\right\}.$$

Note that $p_i(0,x,\lambda,\xi) = p_i^o(x,\lambda,\xi)$ $(1 \le i \le N-1)$. Regarding them as equations of $\lambda$, denote their roots by $\lambda_{ik}^o(x,\xi)$ and by $\lambda_{ik}(t,x,\xi)$ $(1 \le k \le m)$ respectively.

**Theorem 1** (K. Kitagawa [6, Theorem 1], [7, Theorem 1]). *For the Cauchy problem to be $H^\infty$-wellposed, it is necessary that*

$$\text{(6)} \qquad \operatorname{Re} \lambda_{ik}^o(x,\xi) \le 0, \quad (x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n, \ 1 \le i \le N, \ 1 \le k \le m,$$

$$\text{(7)} \quad \operatorname{Re} \lambda_{ik}(t,x,\xi) \le 0, \quad (t,x,\xi) \in (0,\infty) \times \boldsymbol{R}^n \times \boldsymbol{R}^n, \ 1 \le i \le N-1, \ 1 \le k \le m.$$

But, in his papers, nothing is mentioned about sufficient conditions. Note first that the conditions (6) and (7) are not sufficient. In fact, consider the equation:

$$\text{(8)} \qquad (\partial_t - a\partial_x^2 - bt^3\partial_x^3 - ct^5\partial_x^4)u(t,x) = 0,$$

where $a$, $b$ and $c$ are complex constants with $\operatorname{Im} b \neq 0$. In this example, (6) and (7) are equivalent to that $\operatorname{Re} a \ge 0$ and $\operatorname{Re} c \le 0$. But, by applying the theorem of Petrowsky, which will be cited in § 3, we see easily that the Cauchy problem for this equation with datum at $t = 0$ is $H^\infty$-wellposed if and only if $\operatorname{Re} a > 0$ and $\operatorname{Re} c < 0$.

In this paper, we prove

**Theorem 2.** *Suppose the following* A.1–A.4:

A.1 *the coefficients $a_{j\alpha}$ and $b_{j\alpha}$ depend only on $t$ and are continuous in $t \in [0, T_0]$, $T_0 > 0$.*

A.2 *$\sigma(j\alpha)$ are non-negative rational numbers.*
A.3 *there exist $\delta_i > 0$ such that*

$$\text{(9)} \qquad \operatorname{Re} \lambda_{ik}^o(\xi) \le -\delta_i |\xi|^{r_i}, \quad \xi \in \boldsymbol{R}^n, \qquad 1 \le i \le N, \qquad 1 \le k \le m.$$

A.4

$$\text{(10)} \quad \operatorname{Re} \lambda_{ik}(t, \xi) \le 0, \quad (t, \xi) \in (0, \infty) \times \boldsymbol{R}^n, \qquad 1 \le i \le N-1, \qquad 1 \le k \le m.$$

*Then there exists $T > 0$ such that the Cauchy problem* (1)–(2) *is $H^\infty$-wellposed, that is, for any $\varphi_j(x) \in H^\infty(\boldsymbol{R}^n)$, there exists a unique solution $u(t, x) \in C_t^m$ ([0, T]; $H^\infty(\boldsymbol{R}^n)$). Moreover $u(t, x) \in C_t^m((0, T]; H^\infty_{1/p}(\boldsymbol{R}^n))$, where $H^\infty_{1/p}(\boldsymbol{R}^n)$ ($\subset H^\infty(\boldsymbol{R}^n)$) stands for the Gevrey class of exponent $1/p$.*

The outline of the proof is as follows. By the Fourier transform with respect to $x$, we get

$$\text{(11)} \qquad \left( \partial_t^m - \sum_{j=1}^m \sum_{|\alpha| \le pj} a_{j\alpha}(t)(i\xi)^\alpha \partial_t^{m-j} \right) \hat{u}(t, \xi) = 0.$$

The following proposition will be proved in § 3.

**Proposition 3.** *Suppose the conditions* A.1–A.4. *Then there exist positive constants $T$, $C$, $\gamma$ and $\rho$ such that any solution $\hat{u}(t, \xi)$ of* (11) *satisfies*

$$\text{(12)} \qquad \sum_{j=1}^m |\partial_t^{j-1} \hat{u}(t, \xi)| \le C \langle \xi \rangle^\gamma \exp\{-\rho t^{1+q_N} \langle \xi \rangle^p\} \sum_{j=1}^m |\partial_t^{j-1} \hat{u}(0, \xi)|,$$
$$(t, \xi) \in [0, T] \times \boldsymbol{R}^n.$$

Theorem 2 immediately follows from this inequality.

To get the inequality (12), we need to obtain a series of energy estimates corresponding to the vertexes and the sides of the Newton polygon. One of the most crucial points is to prove the uniform boundedness of the definite integral

$$\text{(13)} \qquad \int_a^b \left| \frac{d}{dt} \lambda_{ik}(\theta, \xi_0) \right| d\theta, \qquad 1 \le i \le N-1, \qquad 1 \le k \le m$$

with respect to $\xi_0 \in S^{n-1}$ (Lemma 5). We essentially use the fact that the coefficients of the characteristic equations (5) are polynomials in $t^{1/\nu}$ with some $\nu \in \boldsymbol{N}$.

This paper is constituted as follows:

In § 2, two algebraic lemmas are prepared, whose proofs are given in § 5. In § 3, Proposition 3 is proved. In § 4, we consider the uniform $H^\infty$-

wellposedness of the Cauchy problem and give sufficient conditions. We should remark that the conditions A.1–A.4 are not sufficient for the uniform $H^\infty$-wellposedness.

**Notation.** We use the following notation in this paper: $x \equiv (x_1, \dots, x_n) \in \boldsymbol{R}^n$, $\xi \equiv (\xi_1, \dots, \xi_n) \in \boldsymbol{R}^n$, $|\xi| \equiv \sqrt{\xi_1^2 + \cdots + \xi_n^2}$, $\langle\xi\rangle \equiv \sqrt{1+|\xi|^2}$, $\partial_t \equiv \dfrac{\partial}{\partial t}$, $\partial_{x_j} \equiv \dfrac{\partial}{\partial x_j}$, $\alpha \equiv (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \boldsymbol{N}^n$, $\boldsymbol{N} \equiv \{0, 1, 2, \dots,\}$, $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n$, $\partial_x^\alpha \equiv \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$, $S^{n-1} \equiv \{\xi \in \boldsymbol{R}^n; |\xi| = 1\}$,

$$H^\infty(\boldsymbol{R}^n) \equiv \bigcap_{s\geq 0} \{f(x) \in L^2(\boldsymbol{R}^n);\ \langle\xi\rangle^s \hat{f}(\xi) \in L^2(\boldsymbol{R}^n)\},$$

$$H^\infty_{1/p}(\boldsymbol{R}^n) \equiv \bigcup_{\rho>0} \{f(x) \in H^\infty(\boldsymbol{R}^n);\ \exp(\rho\langle\xi\rangle^p)\hat{f}(\xi) \in L^2(\boldsymbol{R}^n)\},$$

$C_t^m(I; X)$ denotes the set of $m$ times continuously differentiable functions of $t \in I$ with value in $X$.

## 2. Two algebraic lemmas

The following two lemmas are essential in our proof. They will be proved in § 5.

**Lemma 4.** *Let $A$ be an $m \times m$ matrix of the following form:*

$$A = \begin{pmatrix} a_{11} & 1 & & O \\ a_{21} & & \ddots & \\ \vdots & O & & 1 \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix}$$

*and $\lambda_k (1 \leq k \leq m)$ its eigenvalues. Then there exists a regular matrix $N = (n_{ij})_{1\leq i,j\leq m}$ such that*

(i) $\det N = 1$.

(ii) *$n_{ij}$ are polynomials in $(a_{ij}, \lambda_k)$.*

(iii) $D \equiv N^{-1}AN = \begin{pmatrix} \lambda_1 & & & a_{ij}^* \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_m \end{pmatrix}$, *where $a_{ij}^*$ are polynomials in $(a_{ij}, \lambda_k)$.*

**Lemma 5.** *Let $P(\lambda, t)$ be a polynomial in $(\lambda, t)$, that is,*

$$P(\lambda, t) \equiv \lambda^m + \sum_{j=1}^m a_j(t)\lambda^{m-j}, \qquad a_j(t) \equiv \sum_{k=1}^n a_{jk}t^k \qquad \text{and} \qquad a_{jk} \in \boldsymbol{C}$$

*and denote continuous roots of $P(\lambda, t) = 0$ by $\lambda_k(t)$ $(1 \leq k \leq m)$. Let $M > 0$ and* *suppose* $|a_{jk}| \le M$ *for all j, k. Then for any* $-\infty < a < b < \infty$, *there exists a positive constant* $C = C(m, n, a, b, M)$ *such that*

$$\int_a^b \left| \frac{d}{dt} \lambda_k(t) \right| dt \le C(m, n, a, b, M), \qquad 1 \le k \le m. \tag{14}$$

## 3. Energy estimate

In order to prove Proposition 3, we prepare two propositions. The following proposition enables us to get energy estimates corresponding to the vertexes of the Newton polygon. Let $\sigma_i^+$ and $\sigma_i^-$ $(1 \le i \le N)$ stand for the slopes of the sides $P_i P_{i+1}$ and those of the sides $P_{i-1} P_i$ respectively.

**Proposition 6.** *Assume* (9), *then there exist positive constants* $\beta_i > 0$ $(1 \le i \le N)$ *such that estimates of the following form hold for any solution* $\hat{u}(t, \xi)$ *of* (11): *for every fixed constants* $\Theta_i$ *satisfying* $0 < \Theta_i \le \beta_i$, *there exist positive constants* $C^o = C^o(\Theta_i)$ *such that*

$$\begin{aligned} \sum_{j=1}^m |\partial_t^{j-1} \hat{u}(t, \xi)| &\le C^o(\Theta_i) |\xi|^{\mu_i^o} \exp[-3K_i |\xi|^{s_i} \\ &\quad \times \{(t|\xi|^{\sigma_i^+})^{1+q_i} - (\Theta_i^{-1} |\xi|^{-(\sigma_i^- - \sigma_i^+)})^{1+q_i}\}] \\ &\quad \times \sum_{j=1}^m |\partial_t^{j-1} \hat{u}(\Theta_i^{-1} |\xi|^{-\sigma_i^-}, \xi)|, \end{aligned} \tag{15}$$

$$(t, \xi) \in [\Theta_i^{-1} |\xi|^{-\sigma_i^-}, \Theta_i |\xi|^{-\sigma_i^+}] \times \{\xi \in \boldsymbol{R}^n; |\xi|^{-(\sigma_i^- - \sigma_i^+)/2} 2^{1/2(1+q_i)} \le \Theta_i\},$$

*where* $\mu_i^o \equiv (m-1)(r_i - \sigma_i^- q_i)$, $K_i \equiv \dfrac{\delta_i}{4(1+q_i)}$ *and* $s_i \equiv r_i - \sigma_i^+(1+q_i)$, $1 \le i \le N$.

*Proof.* To avoid confusion, omit a suffix $\boldsymbol{i}$. Put

$$\hat{u}_j^0(t, \xi) \equiv t^{(m-j)q} |\xi|^{(m-j)r} \partial_t^{j-1} \hat{u}(t, \xi), \qquad 1 \le j \le m,$$

$$\hat{U}^o(t, \xi) \equiv {}^t(\hat{u}_1^o, \dots, \hat{u}_m^o), \qquad \xi = \xi_0 |\xi| \ (\xi_0 \in S^{n-1}),$$

then

$$\partial_t \hat{u}_j^o = t^q |\xi|^r \hat{u}_{j+1}^o + \frac{(m-j)q}{t} \hat{u}_j^o, \qquad 1 \le j \le m-1,$$

$$\begin{aligned} \partial_t \hat{u}_m^o = \sum_{j=1}^m \Bigg[ &\sum_{(j\alpha) \in \Gamma^o} b_{j\alpha}(0)(i\xi_0)^\alpha + \sum_{(j\alpha) \in \Gamma^o} \{b_{j\alpha}(t) - b_{j\alpha}(0)\}(i\xi_0)^\alpha \\ &+ \sum_{(j\alpha) \notin \Gamma^o} t^{\sigma(j\alpha) - qj} b_{j\alpha}(t)(i\xi_0)^\alpha |\xi|^{|\alpha| - rj} \Bigg] \hat{u}_{m-j+1}^o t^q |\xi|^r. \end{aligned}$$

Set

$$\partial_t \hat{U}^o \equiv \left[\begin{pmatrix} 0 & 1 & & O \\ & & \ddots & \\ & O & & 1 \\ a_m^o & \cdots & \cdots & a_1^o \end{pmatrix} + \begin{pmatrix} & & & \\ & & O & \\ & & & \\ b_m^o & \cdots & \cdots & b_1^o \end{pmatrix}\right.$$

$$\left. + \begin{pmatrix} d_1^o & & & O \\ & \ddots & & \\ O & & d_{m-1}^o & \\ c_m^o & \cdots & \cdots & c_1^o \end{pmatrix}\right] t^q |\xi|^r \hat{U}^o$$

$$\equiv [A^o(\xi_0) + B^o(t, \xi_0) + C^o(t, \xi)] t^q |\xi|^r \hat{U}^o,$$

where

$$a_j^o = a_j^o(\xi_0) \equiv \sum_{(j\alpha) \in \Gamma^o} b_{j\alpha}(0)(i\xi_0)^\alpha,$$

$$b_j^o = b_j^o(t, \xi_0) \equiv \sum_{(j\alpha) \in \Gamma^o} \{b_{j\alpha}(t) - b_{j\alpha}(0)\}(i\xi_0)^\alpha,$$

$$c_j^o = c_j^o(t, \xi) \equiv \sum_{(j\alpha) \notin \Gamma^o} t^{\sigma(j\alpha) - qj} b_{j\alpha}(t)(i\xi_0)^\alpha |\xi|^{|\alpha| - rj},$$

$$d_j^o = d_j^o(t, \xi) \equiv (m-j) q t^{-(1+q)} |\xi|^{-r}, \qquad 1 \le j \le m.$$

Note that for the roots $\lambda_k^o(\xi)$ $(1 \le k \le m)$ of the characteristic equation $p^o(\lambda, \xi) = 0$, $|\xi|^{-r} \lambda_k^o(\xi) = \lambda_k^o(\xi_0)$ and they are eigenvalues of the matrix $A^o(\xi_0)$. By the assumption A.3,

$$\operatorname{Re} \lambda_k^o(\xi_0) \le -\delta, \qquad \xi_0 \in S^{n-1}.$$

By Lemma 4, there exists a regular matrix $N^o(\xi_0) \equiv (n_{jk}^o(\xi_0))_{1 \le j,k \le m}$ satisfying the following conditions:

(i) $\det N^o \equiv 1$, $\xi_0 \in S^{n-1}$.

(ii) $n_{jk}^0$ are polynomials in $(a_j^o, \lambda_k^o)$.

(iii) $D^o \equiv N^{o-1} A^o N^o = \begin{pmatrix} \lambda_1^o & & & a_{jk}^* \\ & \lambda_2^o & & \\ & & \ddots & \\ O & & & \lambda_m^o \end{pmatrix}$, where $a_{jk}^*$ are polynomials in $(a_j^o, \lambda_k^o)$.

Put

$$I_\varepsilon \equiv \begin{pmatrix} 1 & & & O \\ & \varepsilon & & \\ & & \ddots & \\ O & & & \varepsilon^{m-1} \end{pmatrix}$$

and $N_\varepsilon^o \equiv N^o I_\varepsilon$, then

$$D_\varepsilon^o \equiv N_\varepsilon^{o-1} A^o N_\varepsilon^o = \begin{pmatrix} \lambda_1^o & & & \varepsilon^{k-j} a_{jk}^* \\ & \lambda_2^o & & \\ & & \ddots & \\ O & & & \lambda_m^o \end{pmatrix}.$$

We shall determine the size of $\varepsilon$ later. Put $W_\varepsilon^o \equiv N_\varepsilon^{o-1} \hat{U}^o \equiv {}^t(w_1^o, \dots, w_m^o)$ and $S_\varepsilon^o \equiv |W_\varepsilon^o|^2 \equiv \sum_{j=1}^m |w_j^o|^2$, then

$$\partial_t W_\varepsilon^o = (D_\varepsilon^o + N_\varepsilon^{o-1} B^o N_\varepsilon^o + N_\varepsilon^{o-1} C^o N_\varepsilon^o) t^q |\xi|^r W_\varepsilon^o.$$

Since

$$\begin{aligned} \operatorname{Re}(D_\varepsilon^o W_\varepsilon^o, W_\varepsilon^o) &= \operatorname{Re}\left\{\sum_{j=1}^m \left(\lambda_j^o w_j^o + \sum_{k=j+1}^m \varepsilon^{k-j} a_{jk}^* w_k^o, w_j^o\right)\right\} \\ &\le \sum_{j=1}^m (\operatorname{Re} \lambda_j^o) |w_j^o|^2 + \sum_{j=1}^m \sum_{k=j+1}^m \varepsilon^{k-j} |a_{jk}^*| |w_k^o| |w_j^o| \\ &\le (1 - \delta + const.\varepsilon) S_\varepsilon^o, \end{aligned}$$

where *const.* is a positive constant independent of $\varepsilon$, we have

$$\begin{aligned} \frac{d}{dt} S_\varepsilon^o(t,\xi) &= 2 \operatorname{Re}\{(D_\varepsilon^o W_\varepsilon^o, W_\varepsilon^o) + (N_\varepsilon^{o-1} B^o N_\varepsilon^o W_\varepsilon^o, W_\varepsilon^o) + (N_\varepsilon^{o-1} C^o N_\varepsilon^o W_\varepsilon^o, W_\varepsilon^o)\} t^q |\xi|^r \\ &\le \{-2\delta + const.\varepsilon + 2(|B^o| + |C^o|) |N_\varepsilon^{o-1}| |N_\varepsilon^o|\} t^q |\xi|^r S_\varepsilon^o(t,\xi). \end{aligned}$$

Here estimate $|B^o|$, $|C^o|$, $|N_\varepsilon^o|$ and $|N_\varepsilon^{o-1}|$ in turn. Assume $t \in [\Theta^{-1}|\xi|^{-\sigma^-}, \Theta|\xi|^{-\sigma^+}]$. First, since

$$|b_j^o(t,\xi_0)| \le \max_{0 \le t \le \Theta} |b_{j\alpha}(t) - b_{j\alpha}(0)| \sum_{(j\alpha)} 1 \qquad \text{and} \qquad b_{j\alpha}(t) \in C^0([0,T_0]; \boldsymbol{C}),$$

for any $\varepsilon > 0$, there exists a positive constant $\beta_1 = \beta_1(\varepsilon)$ such that

$$|b_j^o(t,\xi_0)| \le \frac{\varepsilon^m}{\sqrt{m}}, \qquad 0 < \Theta \le \beta_1,$$

that is,

$$|B^o(t, \xi_0)| = \sqrt{\sum_{j=1}^{m} |b_j^o(t, \xi_0)|^2} \leq \varepsilon^m, \qquad 0 \leq \Theta \leq \beta_1.$$

Second, since

$$|c_j^o(t, \xi)| \leq \max_{0 \leq t \leq \Theta} |b_{j\alpha}(t)| \sum_{(j\alpha) \notin \Gamma^o} t^{\sigma(j\alpha) - qj} |\xi|^{|\alpha| - rj}$$

and

$$|\alpha| - rj - \sigma^{\pm}(\sigma(j\alpha) - qj) \leq 0, \qquad (j\alpha) \notin \Gamma^o,$$

we decompose and estimate the summation as follows:

$$|c_j^o(t, \xi)| \leq const. \left\{ \sum_{(j\alpha) \notin \Gamma^o, \sigma(j\alpha) > qj} \Theta^{\sigma(j\alpha) - qj} |\xi|^{|\alpha| - rj - \sigma^+(\sigma(j\alpha) - qj)} + \sum_{(j\alpha) \notin \Gamma^o, \sigma(j\alpha) = qj} |\xi|^{|\alpha| - rj} \right.$$

$$\left. + \sum_{(j\alpha) \notin \Gamma^o, \sigma(j\alpha) < qj} \Theta^{-(\sigma(j\alpha) - qj)} |\xi|^{|\alpha| - rj - \sigma^-(\sigma(j\alpha) - qj)} \right\}$$

$$\leq const. \left\{ \sum_{(j\alpha) \notin \Gamma^o, \sigma(j\alpha) \neq qj} \Theta^{|\sigma(j\alpha) - qj|} + \sum_{(j\alpha) \notin \Gamma^o, \sigma(j\alpha) = qj} |\xi|^{|\alpha| - rj} \right\}.$$

Then for any $\varepsilon > 0$, there exists a positive constant $\beta_2 = \beta_2(\varepsilon)$ such that

$$|c_j^o(t, \xi)| \leq \frac{\varepsilon^m}{\sqrt{2m-1}}, \qquad |\xi|^{-(\sigma^- - \sigma^+)/2} 2^{1/2(1+q)} \leq \Theta \leq \beta_2.$$

Next, since

$$|d_j^o(t, \xi)| \leq (m-1) q \Theta^{1+q} |\xi|^{\sigma^-(1+q) - r} \leq (m-1) q \Theta^{1+q},$$

for any $\varepsilon > 0$, there exists a positive constant $\beta_3 = \beta_3(\varepsilon)$ such that

$$|d_j^o(t, \xi)| \leq \frac{\varepsilon^m}{\sqrt{2m-1}}, \qquad 0 < \Theta \leq \beta_3.$$

The estimates above imply that

$$|C^o(t, \xi)| = \sqrt{\sum_{j=1}^{m} \{|c_j^o(t, \xi)|^2 + |d_j^o(t, \xi)|^2\}} \leq \varepsilon^m,$$

$$|\xi|^{-(\sigma^- - \sigma^+)/2} 2^{1/2(1+q)} \leq \Theta \leq \min(\beta_2, \beta_3).$$

Next, since $n_{jk}^o(\xi_0)$ are bounded,

$$|N_\varepsilon^o(\xi_0)| \leq |N^o(\xi_0)| |I_\varepsilon| \leq const.$$

Moreover, since cofactors $\tilde{n}^o_{jk}(\xi_0)$ of the matrix $N^o(\xi_0)$ are bounded,

$$|N^{o-1}_\varepsilon(\xi_0)| \le |I^{-1}_\varepsilon||N^{o-1}(\xi_0)| \le const.\varepsilon^{1-m}.$$

Put $\beta \equiv \min(\beta_1, \beta_2, \beta_3)$. Then, for any positive constant $\Theta$ satisfying $0 < \Theta \le \beta$, the following inequalities hold:

$$|B^o| \le \varepsilon^m, \quad |C^o| \le \varepsilon^m, \quad |N^o_\varepsilon| \le const. \qquad \text{and}$$
$$|N^{o-1}_\varepsilon| \le const.\varepsilon^{1-m}, \quad |\xi| \ge (\Theta 2^{-1/2(1+q)})^{-2/(\sigma^- - \sigma^+)}.$$

Then

$$\frac{d}{dt} S^o_\varepsilon(t,\xi) \le (-2\delta + const.\varepsilon)t^q|\xi|^r S^o_\varepsilon(t,\xi)$$

Since *const.* is independent of $\varepsilon$, we can choose the size of $\varepsilon$ satisfying

$$\frac{d}{dt} S^o_\varepsilon(t,\xi) \le -\frac{3\delta}{2} t^q|\xi|^r S^o_\varepsilon(t,\xi),$$

so we obtain

$$S^o_\varepsilon(t,\xi) \le \exp\left[\int_{\Theta^{-1}|\xi|^{-\sigma^-}}^t \left(-\frac{3\delta}{2} x^q|\xi|^r\right)dx\right] S^o_\varepsilon(\Theta^{-1}|\xi|^{-\sigma^-},\xi).$$

Then

$$S^o_\varepsilon(t,\xi) \le \exp[-6K|\xi|^s\{(t|\xi|^{\sigma^+})^{1+q} - (\Theta^{-1}|\xi|^{-(\sigma^- - \sigma^+)})^{1+q}\}]S^o_\varepsilon(\Theta^{-1}|\xi|^{-\sigma^-},\xi),$$
$$(t,\xi) \in [\Theta^{-1}|\xi|^{-\sigma^-}, \Theta|\xi|^{-\sigma^+}] \times \{\xi \in \boldsymbol{R}^n; |\xi| \ge (\Theta 2^{-1/2(1+q)})^{-2/(\sigma^- - \sigma^+)}\}.$$

Using the inequalities $S^{o1/2}_\varepsilon = |W^o_\varepsilon| \le |N^{o-1}_\varepsilon||\hat{U}^o|$ and $|\hat{U}^o| \le |N^o_\varepsilon||W^o_\varepsilon| = |N^o_\varepsilon|S^{o1/2}_\varepsilon$ and replacing by $\hat{u}$, we obtain (15).

*Remark.* In (15), consider in the case $t = \Theta|\xi|^{-\sigma^+}$. Noting the inequality $|\xi|^{-(\sigma^- - \sigma^+)/2}2^{1/2(1+q)} \le \Theta$, we have

$$-3K|\xi|^s\{\Theta^{1+q} - (\Theta^{-1}|\xi|^{-(\sigma^- - \sigma^+)})^{1+q}\} \le -\frac{3K}{2}\Theta^{1+q}|\xi|^s,$$

thus we obtain

$$\sum_{j=1}^m |\partial^{j-1}_t \hat{u}(\Theta|\xi|^{-\sigma^+},\xi)| \le C^o(\Theta)|\xi|^{\mu^o} \exp\left(-\frac{3K}{2}\Theta^{1+q}|\xi|^s\right)\sum_{j=1}^m |\partial^{j-1}_t \hat{u}(\Theta^{-1}|\xi|^{-\sigma^-},\xi)|,$$
$$|\xi| \ge (\Theta 2^{-1/2(1+q)})^{-2/(\sigma^- - \sigma^+)}.$$

The following proposition enables us to get energy estimates corresponding to the sides of the Newton polygon.

**Proposition 7.** *Assume* (10), *then for the constants* $\Theta_i$ $(1 \le i \le N-1)$ *given by Proposition* 6, *there exist positive constants* $C = C(\Theta_i)$ *and* $R = R(\Theta_i)$ *such that for any solution* $\hat{u}(t,\xi)$ *of* (11),

$$(16)\quad \sum_{j=1}^{m} |\partial_t^{j-1}\hat{u}(t,\xi)| \le C(\Theta_i)|\xi|^{\mu_i} \exp\left(\frac{K_i}{2}\Theta_i^{1+q_i}|\xi|^{s_i}\right) \sum_{j=1}^{m} |\partial_t^{j-1}\hat{u}(\Theta_i|\xi|^{-\sigma_i^+},\xi)|,$$

$$(t,\xi) \in [\Theta_i|\xi|^{-\sigma_i^+}, \Theta_i^{-1}|\xi|^{-\sigma_i^+}] \times \{\xi \in \boldsymbol{R}^n; |\xi| \ge R(\Theta_i)\},$$

*where* $\mu_i \equiv (m-1)(r_i - \sigma_i^+ q_i)$, $1 \le i \le N-1$.

*Proof.* We omit a suffix $\boldsymbol{i}$ also here. For $\hat{u}_j^o(t,\xi)$ $(1 \le j \le m)$ given by Proposition 6, put $\theta \equiv t|\xi|^{\sigma^+}$, $\theta \in [\Theta, \Theta^{-1}]$, $\hat{u}_j(\theta,\xi) \equiv \hat{u}_j^o(\theta|\xi|^{-\sigma^+},\xi)$ and $\hat{U}(\theta,\xi) \equiv {}^t(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_m)$, then set

$$\begin{aligned}\partial_\theta \hat{u}_j &= \{\hat{u}_{j+1} + (m-j)q\theta^{-(1+q)}|\xi|^{-s}\hat{u}_j\}\theta^q|\xi|^s \\ &\equiv \{\hat{u}_{j+1} + d_j(t,\xi)\hat{u}_j\}\theta^q|\xi|^s, \qquad 1 \le j \le m-1,\end{aligned}$$

$$\begin{aligned}\partial_\theta \hat{u}_m = \sum_{j=1}^{m} \Bigg[ &\sum_{(j\alpha)\in\Gamma} \theta^{\sigma(j\alpha)-qj} b_{j\alpha}(0)(i\xi_0)^\alpha + \sum_{(j\alpha)\in\Gamma} \theta^{\sigma(j\alpha)-qj}\{b_{j\alpha}(\theta|\xi|^{-\sigma^+}) - b_{j\alpha}(0)\}(i\xi_0)^\alpha \\ &+ \sum_{(j\alpha)\notin\Gamma} \theta^{\sigma(j\alpha)-qj} b_{j\alpha}(\theta|\xi|^{-\sigma^+})(i\xi_0)^\alpha |\xi|^{|\alpha|-rj-\sigma^+(\sigma(j\alpha)-qj)} \Bigg] \hat{u}_{m-j+1}\theta^q|\xi|^s \\ \equiv \sum_{j=1}^{m} &\{a_j(\theta,\xi_0) + b_j(\theta,\xi) + c_j(\theta,\xi)\}\hat{u}_{m-j+1}\theta^q|\xi|^s.\end{aligned}$$

Set

$$\begin{aligned}\partial_\theta \hat{U} = &\left[\begin{pmatrix} 0 & 1 & & O \\ & & \ddots & \\ & O & & 1 \\ a_m & \cdots & \cdots & a_1 \end{pmatrix} + \begin{pmatrix} & & & \\ & & O & \\ & & & \\ b_m & \cdots & \cdots & b_1 \end{pmatrix}\right. \\ &\left. + \begin{pmatrix} d_1 & & & O \\ & \ddots & & \\ O & & d_{m-1} & \\ c_m & \cdots & \cdots & c_1 \end{pmatrix}\right] \theta^q|\xi|^s\hat{U} \\ \equiv\ & [A(\theta,\xi_0) + B(\theta,\xi) + C(\theta,\xi)]\theta^q|\xi|^s\hat{U}.\end{aligned}$$

Now, note that $|\xi|^{-r}\lambda_k(t,\xi) = \lambda_k(\theta,\xi_0)$ $(1 \le k \le m)$ and they are eigenvalues of the matrix $A(\theta,\xi_0)$ for the roots $\lambda_k(t,\xi)$ of the characteristic equation

$p(t, \lambda, \xi) = 0$. By the assumption A.4,

$$\mathrm{Re}\ \lambda_k(\theta, \xi_0) \leq 0, \qquad (\theta, \xi_0) \in [\Theta, \Theta^{-1}] \times S^{n-1}.$$

By virtue of Lemma 4, there exists a regular matrix $N(\theta, \xi_0) \equiv (n_{jk}(\theta, \xi_0))_{1 \leq j,k \leq m}$ satisfying the following conditions:

(i) $\det N \equiv 1$, $(\theta, \xi_0) \in [\Theta, \Theta^{-1}] \times S^{n-1}$.

(ii) $n_{jk}$ are polynomials in $(a_j, \lambda_k)$.

(iii) $D \equiv N^{-1}AN = \begin{pmatrix} \lambda_1 & & & a_{jk}^* \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_m \end{pmatrix}$, where $a_{jk}^*$ are polynomials in $(a_j, \lambda_k)$.

In the same way as Proposition 6, put $N_\varepsilon \equiv NI_\varepsilon$, $W_\varepsilon \equiv N_\varepsilon^{-1}\hat{U} \equiv {}^t(w_1, w_2, \dots, w_m)$, $S_\varepsilon \equiv |W_\varepsilon|^2 \equiv \sum_{j=1}^m |w_j|^2$ and $D_\varepsilon \equiv N_\varepsilon^{-1}AN_\varepsilon$. We shall determine the size of $\varepsilon$ later. Then

$$\partial_\theta W_\varepsilon = \left\{(D_\varepsilon + N_\varepsilon^{-1}BN_\varepsilon + N_\varepsilon^{-1}CN_\varepsilon)\theta^q|\xi|^s + \left(\frac{d}{d\theta} N_\varepsilon^{-1}\right)N_\varepsilon\right\}W_\varepsilon,$$

$$\begin{aligned}\frac{d}{d\theta} S_\varepsilon(\theta, \xi) &= 2\ \mathrm{Re}\left[\left\{(D_\varepsilon W_\varepsilon, W_\varepsilon) + (N_\varepsilon^{-1}BN_\varepsilon W_\varepsilon, W_\varepsilon) + (N_\varepsilon^{-1}CN_\varepsilon W_\varepsilon, W_\varepsilon)\right\}\theta^q|\xi|^s \right.\\ &\qquad \left. + \left(\left(\frac{d}{d\theta} N_\varepsilon^{-1}\right)N_\varepsilon W_\varepsilon, W_\varepsilon\right)\right] \\ &\leq \left[\{const.\varepsilon + 2(|B| + |C|)|N_\varepsilon^{-1}||N_\varepsilon|\}\theta^q|\xi|^s + 2\left|\frac{d}{d\theta} N_\varepsilon^{-1}\right||N_\varepsilon|\right]S_\varepsilon(\theta, \xi).\end{aligned}$$

Here and hereafter *const.* denotes a positive constant depending on $\Theta$. In the same way as Proposition 6, for any $\varepsilon > 0$, there exists a positive constant $R = R(\Theta, \varepsilon)$ such that the following inequalities hold:

$$|B| \leq \varepsilon^m, \quad |C| \leq \varepsilon^m, \quad |N_\varepsilon| \leq const. \ \text{ and } \ |N_\varepsilon^{-1}| \leq const.\varepsilon^{1-m}, \quad |\xi| \geq R(\Theta, \varepsilon).$$

Here note that we can obtain the estimates above for any constant $\Theta$ given by Proposition 6 because $\sigma^+ > 0$ ($\neq 0$) and $|\alpha| - rj - \sigma^+(\sigma(j\alpha) - qj) < 0$ ($\neq 0$) when $(j\alpha) \notin \Gamma$. Since

$$\left|\frac{d}{d\theta} N_\varepsilon(\theta, \xi_0)^{-1}\right| \leq \varepsilon^{1-m}\left|\frac{d}{d\theta} N(\theta, \xi_0)^{-1}\right| \leq const.\varepsilon^{1-m}\left(1 + \sum_{j=1}^m \left|\frac{d}{d\theta} \lambda_j(\theta, \xi_0)\right|\right),$$

$$\frac{d}{d\theta} S_\varepsilon(\theta, \xi) \leq \left\{const.\varepsilon|\xi|^s + const.\varepsilon^{1-m}\left(1 + \sum_{j=1}^m \left|\frac{d}{d\theta} \lambda_j(\theta, \xi_0)\right|\right)\right\}S_\varepsilon(\theta, \xi),$$

then

$$S_\varepsilon(\theta,\xi)\le \exp\left[\int_\Theta^{\Theta^{-1}}\left\{const.\varepsilon|\xi|^s+const.\varepsilon^{1-m}\left(1+\sum_{j=1}^m\left|\frac{d}{d\theta}\lambda_j(\theta,\xi_0)\right|\right)\right\}d\theta\right]S_\varepsilon(\Theta,\xi).$$

By Lemma 5,

$$S_\varepsilon(\theta,\xi)\le \exp\{const.(\varepsilon|\xi|^s+\varepsilon^{1-m})\}S_\varepsilon(\Theta,\xi).$$

Thus, for the constant $\Theta$, we can take a small constant $\varepsilon$ such that the following energy estimate holds:

$$S_\varepsilon(\theta,\xi)\le \exp\,(K\Theta^{1+q}|\xi|^s+const.)S_\varepsilon(\Theta,\xi),$$

$$(\theta,\xi)\in[\Theta,\Theta^{-1}]\times\{\xi\in\boldsymbol{R}^n;\ |\xi|\ge R(\Theta,\varepsilon)\}.$$

Replacing the estimate above by $\hat{u}$, we obtain (16).

*Proof of Proposition* 3. For the constants $\beta_i$ $(1\le i\le N)$ given by Proposition 6, put $T\equiv\min_{1\le i\le N}\beta_i$. Then we can choose a common constant $T$ for the energy estimates corresponding to the vertexes $P_i$ $(1\le i\le N)$ and the sides $P_iP_{i+1}$ $(1\le i\le N-1)$ of the Newton polygon. First, note that, by simple computation, there exists a positive constant $C_0$ such that for any solution $\hat{u}(t,\xi)$ of (11),

$$(17)\qquad \sum_{j=1}^m|\partial_t^{j-1}\hat{u}(t,\xi)|\le C_0\sum_{j=1}^m|\partial_t^{j-1}\hat{u}(0,\xi)|,\qquad (t,\xi)\in[0,T^{-1}|\xi|^{-\sigma_0^+}]\times\boldsymbol{R}^n.$$

Combining Proposition 6, Proposition 7 and (17), we get the following estimates: there exist positive constants $C$, $\gamma$ and $M$ such that for any solution $\hat{u}(t,\xi)$ of (11),

$$(18)\quad \sum_{j=1}^m|\partial_t^{j-1}\hat{u}(t,\xi)|\le C|\xi|^\gamma\exp\left[-\sum_{v=0}^{i-1}K_vT^{1+q_v}|\xi|^{s_v}-3K_i|\xi|^{r_i}\right.$$
$$\left.\times\{t^{1+q_i}-(T^{-1}|\xi|^{-\sigma_i^-})^{1+q_i}\}\right]\sum_{j=1}^m|\partial_t^{j-1}\hat{u}(0,\xi)|,$$
$$(t,\xi)\in[T^{-1}|\xi|^{-\sigma_i^-},T|\xi|^{-\sigma_i^+}]\times\{\xi\in\boldsymbol{R}^n;|\xi|\ge M\},\qquad 1\le i\le N,$$

$$(19)\quad \sum_{j=1}^m|\partial_t^{j-1}\hat{u}(t,\xi)|\le C|\xi|^\gamma\exp\left(-\sum_{v=0}^{i}K_vT^{1+q_v}|\xi|^{s_v}\right)\sum_{j=1}^m|\partial_t^{j-1}\hat{u}(0,\xi)|,$$
$$(t,\xi)\in[T|\xi|^{-\sigma_i^+},T^{-1}|\xi|^{-\sigma_i^+}]\times\{\xi\in\boldsymbol{R}^n;|\xi|\ge M\},\quad 1\le i\le N-1,$$

where $q_0=-1$, $s_0=0$ and $K_0$ is some real constant.

If $t \in [T^{-1}|\xi|^{-\sigma_i^-}, T|\xi|^{-\sigma_i^+}]$ $(1 \le i \le N)$, then

$$t^{1+q_i}|\xi|^{r_i} = (t^{1+q_N}|\xi|^p)(t^{q_i-q_N}|\xi|^{r_i-p}),$$

$$t^{q_i-q_N}|\xi|^{r_i-p} \ge (T|\xi|^{-\sigma_i^+})^{q_i-q_N}|\xi|^{r_i-p} \ge T^{q_i-q_N}.$$

If $t \in [T|\xi|^{-\sigma_i^+}, T^{-1}|\xi|^{-\sigma_i^+}]$ $(1 \le i \le N-1)$, then

$$T^{1+q_i}|\xi|^{s_i} = (t^{1+q_N}|\xi|^p)(t^{-1-q_N}T^{1+q_i}|\xi|^{s_i-p}),$$

$$t^{-1-q_N}T^{1+q_i}|\xi|^{s_i-p} \ge (T^{-1}|\xi|^{-\sigma_i^+})^{-1-q_N}T^{1+q_i}|\xi|^{s_i-p} \ge T^{2+q_i+q_N}.$$

Hence, by the inequalities (17), (18) and (19), there exists a positive constant $\rho$ such that

$$\sum_{j=1}^{m} |\partial_t^{j-1}\hat{u}(t,\xi)| \le C|\xi|^\gamma \exp\{-\rho t^{1+q_N}|\xi|^p\} \sum_{j=1}^{m} |\partial_t^{j-1}\hat{u}(0,\xi)|, \tag{20}$$

$$(t,\xi) \in [0,T] \times \{\xi \in \boldsymbol{R}^n; |\xi| \ge M\},$$

that is, we get the inequality (12).

Theorem 2 immediately follows from Proposition 3 and the following well-known theorem due to I. G. Petrowsky [1].

**Petrowsky's theorem A.** *Assume* A.1. *Then the Cauchy problem* (1)–(2) *is $H^\infty$-wellposed if and only if there exist two positive constants $C$ and $l$ such that for any solution $\hat{u}(t,\xi)$ of* (11),

$$\sum_{j=1}^{m} |\partial_t^{j-1}\hat{u}(t,\xi)| \le C\langle\xi\rangle^l \sum_{j=1}^{m} |\partial_t^{j-1}\hat{u}(0,\xi)|, \qquad (t,\xi) \in [0,T] \times \boldsymbol{R}^n. \tag{21}$$

## 4. Uniform $H^\infty$-wellposedness

Consider the non-homogeneous Cauchy problem instead of (1)–(2):

$$L(t,x,\partial_t,\partial_x)u(t,x) = f(t,x), \qquad (t,x) \in [t_0,T] \times \boldsymbol{R}^n, \tag{22}$$

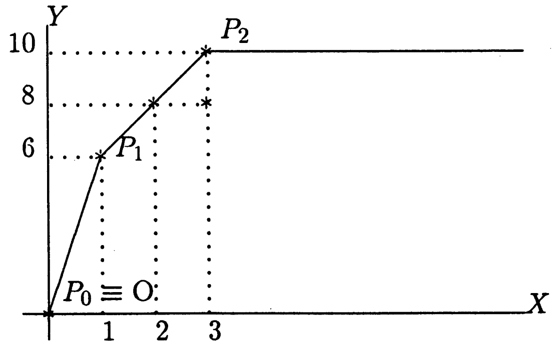
$$\partial_t^{j-1}u(t_0,x) = \varphi_j(x), \qquad 1 \le j \le m. \tag{23}$$

Following I. G. Petrowsky [1], we introduce

**Definition 8** (uniform $H^\infty$-wellposedness). We say that (22)–(23) is uniformly $H^\infty$-wellposed, if for any

$$\varphi_j(x) \in H^\infty(\boldsymbol{R}^n), \qquad 1 \le j \le m,\ t_0 \in [0,T), \qquad f(t,x) \in C_t^0([0,T]; H^\infty(\boldsymbol{R}^n)),$$

there exists a unique solution $u(t,x) \in C_t^m([t_0,T]; H^\infty(\boldsymbol{R}^n))$ of (22)–(23).




Concerning the uniform $H^\infty$-wellposedness, we know

**Petrowsky's theorem B.** *Assume* A.1. *Then the Cauchy problem* (22)–(23) *is uniformly $H^\infty$-wellposed if and only if there exist two positive constants $C$ and $l$ such that for any solution $\hat{u}(t,\xi)$ of* (11),

$$\sum_{j=1}^{m} |\partial_t^{j-1}\hat{u}(t,\xi)| \le C\langle\xi\rangle^l \sum_{j=1}^{m} |\partial_t^{j-1}\hat{u}(t_0,\xi)|, \qquad (t,\xi) \in [t_0, T] \times \boldsymbol{R}^n, \tag{24}$$

*where $C$ and $l$ are independent of $t_0$.*

The assumption A.1–A.4 are not sufficient for the uniform $H^\infty$-wellposedness. The following example is due to K. Kitagawa [8].

**Example.** Consider the operator

$$L(t,\partial_t,\partial_x) \equiv \partial_t - \partial_x^6(t^2\partial_x^4 + 2t\partial_x^2 + 1) - t^2\partial_x^8.$$

Drawing the Newton polygon, we see easily that the conditions (9) and (10) are satisfied. Then, for the operator, the Cauchy problem (1)–(2) is $H^\infty$-wellposed for sufficient small $T > 0$, however, is not uniformly $H^\infty$-wellposed. In fact, following to K. Kitagawa [8], putting $t = 1/\xi(\xi-1)$ and $t_0 = 1/\xi(\xi+1)$ $(\xi > 1)$ for the solution of the equation

$$(\partial_t + \xi^6(t^2\xi^4 - 2t\xi^2 + 1) - t^2\xi^8)\hat{u}(t,\xi) = 0$$

with datum at $t = t_0$, we have

$$\begin{aligned}\hat{u}\left(\frac{1}{\xi(\xi-1)},\xi\right) &= \exp\left[\int_{1/\xi(\xi+1)}^{1/\xi(\xi-1)} \{-\xi^6(t\xi^2-1)^2 + t^2\xi^8\}dt\right]\hat{u}\left(\frac{1}{\xi(\xi+1)},\xi\right) \\ &= \exp\left\{\frac{4\xi^5}{3(\xi^2-1)^2}\right\}\hat{u}\left(\frac{1}{\xi(\xi+1)},\xi\right),\end{aligned}$$

implying that the Cauchy problem is not uniformly $H^\infty$-wellposed.

Now, replace the assumption A.4 by the following one:
A.4′

(25) $\quad \operatorname{Re} \lambda_{ik}(t, \xi) \le -\delta_i |\xi|^{r_i}, \quad (t, \xi) \in (0, \infty) \times \boldsymbol{R}^n, \quad 1 \le i \le N-1, \quad 1 \le k \le m.$

Then, it is easy to obtain the estimate (24). In fact, setting $\theta = t|\xi|^{\sigma_i^+}$, we have

(26) $\quad \operatorname{Re} \lambda_{ik}(\theta, \xi_0) \le -\delta_i, \quad (\theta, \xi_0) \in (0, \infty) \times S^{n-1}, \quad 1 \le i \le N-1, \quad 1 \le k \le m$

and, in the same way as the proof of Proposition 6, we can get the following energy estimates instead of (16):

$$\sum_{j=1}^{m} |\partial_t^{j-1} \hat{u}(t, \xi)| \le C(\Theta_i)|\xi|^{\mu_i} \exp\left[-3K_i|\xi|^{s_i}\{(t|\xi|^{\sigma_i^+})^{1+q_i} - \Theta_i^{1+q_i}\}\right]$$
$$\times \sum_{j=1}^{m} |\partial_t^{j-1} \hat{u}(\Theta_i|\xi|^{-\sigma_i^+}, \xi)|,$$
$$(t, \xi) \in [\Theta_i|\xi|^{-\sigma_i^+}, \Theta_i^{-1}|\xi|^{-\sigma_i^+}] \times \{\xi \in \boldsymbol{R}^n;\ |\xi| \ge R(\Theta_i)\}, \quad 1 \le i \le N-1.$$

Thus we obtain

**Theorem 9.** *Suppose* A.1–A.3 *and* A.4′. *Then there exists* $T > 0$ *such that the Cauchy problem* (22)–(23) *is uniformly* $H^\infty$*-wellposed.*

## 5. Proof of lemmas

*Proof of Lemma* 4. Put $A_m \equiv A \equiv (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m)$. Let $\lambda$ be one of $\lambda_k$ and $\mathbf{x} \equiv {}^t(x_1, x_2, \dots, x_m)$ an eigenvector corresponding to it. Then $A_m \mathbf{x} = \lambda \mathbf{x}$, that is,

$$\begin{cases} a_{11}x_1 + x_2 = \lambda x_1 \\ a_{21}x_1 + x_3 = \lambda x_2 \\ \quad \vdots \qquad\qquad\qquad \vdots \\ a_{m-1,1}x_1 + x_m = \lambda x_{m-1} \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \cdots + a_{mm}x_m = \lambda x_m \end{cases}$$

If we assume $x_1 = 0$, then $\mathbf{x} = \mathbf{o}$, thus we can choose an eigenvector $\mathbf{x} = {}^t(1, x_2, \dots, x_m)$, where $x_{j+1} = \lambda x_j - a_{j1}$ $(1 \le j \le m-1)$. Now put $\mathbf{e_j} \equiv {}^t(0, \dots, \overset{j}{1}, \dots, 0)$ and $N_m \equiv (\mathbf{x}, \mathbf{x} + \mathbf{e_2}, \dots, \mathbf{x} + \mathbf{e}_m) \equiv (\mathbf{n_1}, \dots, \mathbf{n_m})$, then

$$\det N_m = \det(\mathbf{x}, \mathbf{e_2}, \ldots, \mathbf{e_m}) = 1,$$

$$\begin{aligned} A_m N_m &= (A_m\mathbf{x}, A_m\mathbf{x} + A_m\mathbf{e_2}, \ldots, A_m\mathbf{x} + A_m\mathbf{e_m}) \\ &= \lambda(\mathbf{x}, \ldots, \mathbf{x}) + (\mathbf{o}, \mathbf{a_2}, \ldots, \mathbf{a_m}). \end{aligned}$$

Each column vector $\mathbf{a_j}$ $(2 \leq j \leq m)$ is a linear combination of $\mathbf{n_1}, \ldots, \mathbf{n_m}$ as follows:

$$\begin{aligned} \mathbf{a_2} &= \mathbf{e_1} + a_{m2}\mathbf{e_m} \\ &= \left(\mathbf{n_1} - \sum_{j=2}^{m} x_j\mathbf{e_j}\right) + a_{m2}(\mathbf{n_m} - \mathbf{n_1}) \\ &= \mathbf{n_1} - \sum_{j=2}^{m} x_j(\mathbf{n_j} - \mathbf{n_1}) + a_{m2}(\mathbf{n_m} - \mathbf{n_1}) \\ &= \left(\sum_{j=1}^{m} x_j - a_{m2}\right)\mathbf{n_1} - \sum_{j=2}^{m-1} x_j\mathbf{n_j} + (-x_m + a_{m2})\mathbf{n_m}, \qquad (x_1 = 1) \\ \mathbf{a_j} &= \mathbf{e_{j-1}} + a_{mj}\mathbf{e_m} \\ &= -(1 + a_{mj})\mathbf{n_1} + \mathbf{n_{j-1}} + a_{mj}\mathbf{n_m} \qquad (3 \leq j \leq m). \end{aligned}$$

Thus

$$A_m N_m = N_m \begin{pmatrix} \lambda & \lambda + \sum_{j=1}^{m} x_j - a_{m2} & \lambda - 1 - a_{m3} & \cdots & \lambda - 1 - a_{mm} \\ 0 & -x_2 & 1 & & O \\ \vdots & \vdots & & \ddots & \\ 0 & -x_{m-1} & O & & 1 \\ 0 & -x_m + a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix}.$$

Then, set $D_m \equiv N_m^{-1} A_m N_m \equiv \begin{pmatrix} \lambda & * \\ 0 & A_{m-1} \end{pmatrix}$, where $*$ denote polynomials in $(a_{ij}, \lambda_k)$ and the $(m-1) \times (m-1)$ matrix $A_{m-1}$ has the same form with $A_m$. Now we use induction on $m$. The claim is trivial for $m = 1$; assume it is true for $m - 1$ $(m \geq 2)$. Now put $\lambda = \lambda_1$. Since $N_m^{-1} A_m N_m = \begin{pmatrix} \lambda_1 & * \\ 0 & A_{m-1} \end{pmatrix}$, there exists an $(m-1) \times (m-1)$ matrix $N_{m-1}$ such that $N_{m-1}^{-1} A_{m-1} N_{m-1} = \begin{pmatrix} \lambda_2 & & * \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$. Put $T \equiv \begin{pmatrix} 1 & 0 \\ 0 & N_{m-1} \end{pmatrix}$ and $N_m^* \equiv N_m T$, then $N_m^{*-1} A_m N_m^* =$

$$\begin{pmatrix} 1 & 0 \\ 0 & N_{m-1}^{-1} \end{pmatrix} \begin{pmatrix} \lambda_1 & * \\ 0 & A_{m-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & N_{m-1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_m \end{pmatrix}.$$

Thus $N_m$ satisfies the conditions (i)–(iii).

*Proof of Lemma* 5. Let $\lambda(t)$ be one of the continuous roots of the equation $P(\lambda, t) = 0$. Write $\lambda(t) \equiv x(t) + iy(t)$ and $P(x + iy, t) \equiv P_1(x, y, t) + iP_2(x, y, t)$, where $P_1(x, y, t)$ and $P_2(x, y, t)$ are polynomials in $(x, y, t)$ with real coefficients. Hereafter consider the polynomial ring $R[x, y, t]$ with coefficient field $\boldsymbol{R}$.

First, we show that the resultant $Q(y, t) \equiv R(P_1(\cdot, y, t), P_2(\cdot, y, t)) \not\equiv 0$, which is, by the Euclidean algorithm and Gauss' lemma, equivalent to that $P_1(x, y, t)$ and $P_2(x, y, t)$ have no common divisor as a polynomial including $x$. Consider in the case $m = even.$ We can also show in the case $m = odd$ in the same way. Put

$$P_1(x, y, t) = P_1(x) \equiv x^m + p_1 x^{m-1} + p_2 x^{m-2} + \cdots + p_m,$$

$$P_2(x, y, t) = P_2(x) \equiv q_1 x^{m-1} + q_2 x^{m-2} + \cdots + q_m,$$

then

$$\operatorname{order}_y p_j(y, t) \begin{cases} \leq j - 1 & (j = \text{odd}) \\ = j & (j = \text{even}) \end{cases}, \qquad \operatorname{order}_y q_j(y, t) \begin{cases} = j & (j = \text{odd}) \\ \leq j - 1 & (j = \text{even}) \end{cases}$$

and

$$Q(y, t) = \begin{vmatrix} 1 & p_1 & p_2 & \cdots & p_{m-1} & p_m & & & \\ & 1 & p_1 & p_2 & \cdots & p_{m-1} & p_m & & \\ & & \ddots & \ddots & \ddots & & \ddots & \ddots & \\ & & & 1 & p_1 & p_2 & \cdots & p_{m-1} & p_m \\ q_1 & q_2 & \cdots & q_{m-1} & q_m & & & & \\ & q_1 & q_2 & \cdots & q_{m-1} & q_m & & & \\ & & \ddots & \ddots & & \ddots & \ddots & & \\ & & & \ddots & \ddots & & \ddots & \ddots & \\ & & & & q_1 & q_2 & \cdots & q_{m-1} & q_m \end{vmatrix}.$$

Denoting the highest order terms of $p_j(\cdot, t)$ and those of $q_j(\cdot, t)$ by $p_j^o(\cdot, t)$ and by $q_j^o(\cdot, t)$ respectively, by simple computation, we have

$$
\begin{matrix}\text{the highest order} \\ \text{term of } Q(\cdot, t)\end{matrix} = \left|\begin{array}{ccccccccccc}
1 & 0 & p_2^o & \cdots & 0 & p_m^o & & & & & \\
 & 1 & 0 & p_2^o & \cdots & 0 & p_m^o & & & & \\
 & & \ddots & \ddots & \ddots & & \ddots & \ddots & & & \\
 & & & & 1 & 0 & p_2^o & \cdots & 0 & p_m^o & \\
q_1^o & 0 & \cdots & q_{m-1}^o & 0 & & & & & & \\
 & q_1^o & 0 & \cdots & q_{m-1}^o & 0 & & & & & \\
 & & \ddots & \ddots & & \ddots & \ddots & & & & \\
 & & & \ddots & \ddots & & \ddots & \ddots & & & \\
 & & & & q_1^o & 0 & \cdots & q_{m-1}^o & 0 & &
\end{array}\right| .
$$

Now put $(x+iy)^m \equiv P_1^o(x, y) + iP_2^o(x, y)$ and $Q^o(y) \equiv R(P_1^o(\cdot, y), P_2^o(\cdot, y))$, then the determinant above is just equal to $Q^o(y)$. Thus, "the highest order term of $Q(\cdot, t)$" $= Q^o(\cdot)$. The fact that $P_1^o(x, y)$ and $P_2^o(x, y)$ have some common divisor as a polynomial including $x$ is equivalent to that $Q^o(y) \equiv 0$. By the theorem on unique factorization in prime elements in polynomial ring, it is obvious that $P_1^o(x, y)$ and $P_2^o(x, y)$ have no common divisor including $x$. Thus $Q^o(y) \not\equiv 0$, that is, $Q(y, t) \not\equiv 0$.

Second, we take out values of $t$ such that $\frac{d}{dt} y(t)$ does not exist in $t \in (a, b)$. By the theorem on implicit function, if $\frac{\partial}{\partial y} Q(y, t) \neq 0$,

$$
\frac{d}{dt} y(t) = -\left(\frac{\partial Q}{\partial t} \Big/ \frac{\partial Q}{\partial y}\right).
$$

Then we take out values of $t$ satisfying $\frac{\partial}{\partial y} Q(y, t) = 0$. We can assume that $Q(y, t)$ is irreducible without loss of generality. Consider the simultaneous equations:

$$
\begin{cases} Q(y, t) = 0 \\ \dfrac{\partial}{\partial y} Q(y, t) = 0 \end{cases}
$$

Now put $T^o(t) \equiv R\left(Q(\cdot, t), \frac{\partial}{\partial y} Q(\cdot, t)\right)$. $T^o(t)$ is a polynomial in $t$ with real coefficients, whose degree is depending only on $m$ and $n$. If $T^o(t) \equiv 0$, then $Q(y, t)$ is reducible, thus $T^o(t) \not\equiv 0$. Then denote the zeros of $T^o(t)$ in $t \in (a, b)$ by $t_1^o, t_2^o, \ldots, t_{l^o}^o$, where the integer $l^o$ has an upper bound depending only on $m$ and $n$. Putting

$$
\Delta^o : t_0^o \equiv a < t_1^o < \cdots < t_{l^o}^o < t_{l^o+1}^o \equiv b,
$$

then we have

$$\frac{d}{dt}\,y(t) = -\left(\frac{\partial Q}{\partial t}\Big/\frac{\partial Q}{\partial y}\right) \in C^0((t_k^o, t_{k+1}^o); \boldsymbol{R}), \qquad 1 \le k \le l^o.$$

Next, we take out values of $t$ such that $\frac{d}{dt}\,y(t)$ changes the sign in $t \in (a,b)$. By the theorem on implicit function, if $\frac{d}{dt}\,y(t) = 0$, then $\frac{\partial}{\partial t}\,Q(y,t) = 0$. Then consider the following simultaneous equations:

$$\begin{cases} Q(y,t) = 0 \\ \dfrac{\partial}{\partial t}\,Q(y,t) = 0 \end{cases}$$

Putting $T^*(t) \equiv R\left(Q(\cdot,t), \frac{\partial}{\partial t}\,Q(\cdot,t)\right)$, in the same way as $T^o(t)$, we denote the zeros of $T^*(t)$ in $t \in (a,b)$ by $t_1^*, t_2^*, \dots, t_{l^*}^*$, where the integer $l^*$ has an upper bound depending only on $m$ and $n$. Put

$$\varDelta^* : t_0^* \equiv a < t_1^* < \cdots < t_{l^*}^* < t_{l^*+1}^* \equiv b$$

and

$$\varDelta \equiv \varDelta^o \cup \varDelta^* : t_0 \equiv a < t_1 < \cdots < t_l < t_{l+1} \equiv b \qquad (l \le l^o + l^*),$$

then $\frac{d}{dt}\,y(t)$ is continuous and of definite sign in $t \in (t_k, t_{k+1})$ $(0 \le k \le l)$. Thus

$$\int_a^b \left|\frac{d}{dt}\,y(t)\right| dt = \sum_{k=0}^{l} \left|\int_{t_k}^{t_{k+1}} \frac{d}{dt}\,y(t)dt\right| = \sum_{k=0}^{l} |y(t_{k+1}) - y(t_k)| \le 2 \max_{t \in [a,b]} |y(t)| \cdot (l+1).$$

Then there exists a positive constant $C_1 = C_1(m,n,a,b,M)$ such that

$$\int_a^b \left|\frac{d}{dt}\,y(t)\right| dt \le C_1(m,n,a,b,M).$$

In the same way, there exists a positive constant $C_2 = C_2(m,n,a,b,M)$ such that

$$\int_a^b \left|\frac{d}{dt}\,x(t)\right| dt \le C_2(m,n,a,b,M).$$

Hence the inequality (14) holds.

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