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Digital sum moments and substitutions

by

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0. Introduction. In an article published in 1986 Jean Coquet ([C86]) proved that if $s_2(\nu)$ denotes the sum of binary digits of ν, then for each integer k there exist 1-periodic bounded functions $F_{k,0}, F_{k,1}, \ldots, F_{k,k-1}$ such that for any integer x one has, with $l = \log_2 x$,

$$x^{-1} \sum_{\nu<x} (s_2(\nu))^k = (l/2)^k + \sum_{h<k} l^h F_{k,h}(l)\,. \tag{1}$$

The result in the case $k = 1$ was first established by H. Delange ([De75]) and has recently been generalized to the case of "θ-expansion of positive integers" with θ an arbitrary real base > 1 ([GTi91]), with an $o(1)$ correction term; in this case the function $F_{k,0}$ is shown to be continuous; so are $F_{k,0}$ and $F_{k,1}$ in the case $k = 2$, for the binary expansion (see [C86] and [K90]).

Our first aim in this paper is to give a generalization of (1), including the proof of the continuity of the functions $F_{k,h}$, within the framework of the so-called "numeration system associated with a substitution".

More precisely, if σ is a primitive substitution on a finite alphabet $\mathcal{A}$ whose largest eigenvalue satisfies $\theta > 1$, and $s^f(\nu)$ denotes $\sum_{i=1}^n f(m_i)$ (where for ν an integer, $\sum_{i=1}^n |\sigma^{i-1}(m_i)|$ is the unique admissible representation of ν, and f is a map from $\mathcal{A}^*$ to $\mathbb{R}$), then we prove the existence of a real number α and 1-periodic continuous functions $F_{k,h}$ $(h = 0, 1, \ldots, k-1)$ such that for any $x > 0$,

$$x^{-1} \sum_{\nu<x} (s^f(\nu))^k = \alpha^k l^k + \sum_{h<k} l^h F_{k,h}(l) + \varepsilon(x) \tag{2}$$

where $\lim_{x\to\infty} \varepsilon(x) = 0$, and $l = \log_\theta x$.

Moreover, we prove a similar formula for "moments of the sum-of-digits function" in the form

$$x^{-1} \sum_{\nu<x} (s^f(\nu) - \alpha l)^{2k} = (2k-1)(2k-3)\ldots 1\beta^k l^k + \sum_{h<k} l^h G_{k,h}(l) + \eta(x) \tag{3}$$

where the real number β is explicitly determined and $\lim_{x\to+\infty} \eta(x) = 0$, the functions $G_{k,h}$ being 1-periodic and continuous. For odd moments ($2k-1$ in place of $2k$ on the l.h.s. of (3)) the first term of the r.h.s. of (3) disappears.

The functions $F_{k,h}$ of (2) are shown to be nowhere differentiable when $\alpha \neq 0$ and $\alpha \neq f(\omega)$ (ω being the empty word); the functions $G_{k,h}$ of (3) are also nowhere differentiable if $\beta \neq 0$ and $\alpha \neq f(\omega)$.

Note that the constants α and β (in the case $\alpha = 0$) of the formulae (2) and (3) were previously determined (in [D90]) by another method; moreover, the special form of the moments of the sum-of-digits function is clearly connected with the gaussian distribution of this function, and was first conjectured by J. M. Luck ([L]); we hope to give a more precise statement of this idea in a forthcoming paper.

The general framework of "substitutions on a finite alphabet" allows us to study some mathematical models useful in theoretical physics; in some of these cases we give the explicit values of the constants α and β.

To return to the initial formula (1), we also note that when $\mathcal{A} = \{1\}$ and $\sigma(1) = 11\ldots1$ (q times one, q an integer ≥ 2), the substitutive numeration coincides with the ordinary base q numeration, and in this case the remainder terms $\varepsilon(x)$ and $\eta(x)$ of (2) and (3) disappear when x is an integer, according to (1); moreover, we then have $\alpha = (q-1)/2$ and $\beta = (q^2-1)/12$ (if $q = 2$, the result for β agrees with that of P. Kirschenhofer ([K90]).

With the substitutive numeration system, one can also obtain expansions of integers with respect to linear recurrences studied in [GTi91] (see also [Sh88] and [B89]); we make explicit that connection, and give the values of α and β in the last section. For instance, in the Fibonacci case $\alpha = (5-\sqrt{5})/10$ (in accordance with [CV86]) and $\beta = 1/(5\sqrt{5})$.

Finally, we note that in the case of ordinary q-adic expansion (q an integer ≥ 2) some asymptotic formulae involving the sum of digits can also be obtained by other methods, issuing from analytic theory of numbers (cf. [FG] and [MM]).

1. Numeration system and sums associated with a substitution. Let σ be a substitution on a finite alphabet $\mathcal{A} = \{1, 2, \ldots, d\}$, i.e. a map from $\mathcal{A}$ to $\mathcal{A}^* \setminus \omega$, the set of non-empty words of $\mathcal{A}$; let M be the transpose of its matrix ($M_{a,b}$ is the number of occurrences of b in σa); $|m|$ denotes the length of $m \in \mathcal{A}^*$. We assume that M is primitive and that the word $\sigma(1)$ has length at least 2 and begins with the letter 1. By the theorem of Perron–Frobenius, there exists a unique eigenvalue θ of M with maximum modulus. As $|\sigma(1)| \geq 2$, one has $\theta > 1$.

We first recall a representation of integers ([DT89]) which we reset in terms of automata. The state set of the prefix automaton is the alphabet $\mathcal{A}$ of the substitution. The alphabet of the automaton is the set $\mathcal{A}'$ of words m such that $\exists a \in \mathcal{A}$, $m < \sigma a$. There is one arc from a to b with label m iff

$mb \leq \sigma a$. We write $b = a \cdot m$. Every sequence $(m_n, \ldots, m_1)$, $m_i \in \mathcal{A}'$, which is the label of a path with initial state a is called *a-recognizable*. There are exactly $|\sigma^n(a)|$ a-recognizable sequences of length n.

Since $1 < \sigma(1)$, for each integer $\nu \geq 1$ there exists a unique 1-recognizable sequence $(m_n, \ldots, m_1)$ such that $\nu = \sum_{i=1}^{n} |\sigma^{i-1}(m_i)|$ and m_n is not empty; this sequence is called the (admissible) *representation* of ν.

For instance, a special case is the representation in the Fibonacci base ($F_0 = 1$, $F_1 = 2$, $F_{i+1} = F_i + F_{i-1}$). Indeed, if σ is defined by $\sigma(1) = 12$ and $\sigma(2) = 1$, then the m_i are ω (empty word) or 1, and $|\sigma^{i-1}(m_i)|$ is 0 if $m_i = \omega$, and F_{i-1} if $m_i = 1$; the prefix automaton recognizes the sequences $(m_n, \ldots, m_1)$ such that $m_{i+1}m_i \neq 11$ (see Fig. 1).

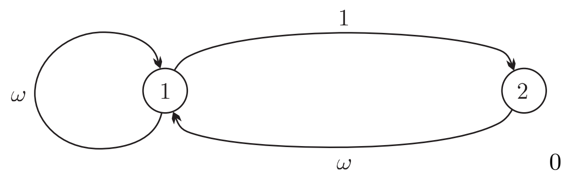

Fig. 1

The q-ary representation (q an integer, $q \geq 2$) is another special case, with the substitution defined on $\mathcal{A} = \{1\}$ by $\sigma(1) = 1^q$ (see Fig. 2).

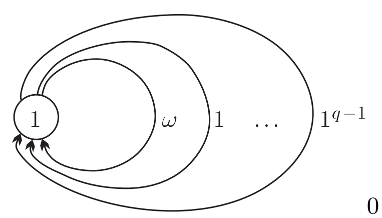

Fig. 2

We define the *sum-of-digits function* relative to a map $f : \mathcal{A}^* \to \mathbb{R}$ by

$$s^f(\nu) = \sum_{i=1}^{n} f(m_i) \qquad ((m_n, \ldots, m_1) \text{ being the representation of } \nu).$$

A *moment of order* k is

$$S^f_{k,\lambda}(x) = \sum_{0<\nu<x} (s^f(\nu) - \lambda \log_\theta x)^k \qquad \text{for } x \in \mathbb{R}^*_+ \text{ and } k \in \mathbb{N},\ \lambda \in \mathbb{R}.$$

We will also use, for the computation of the asymptotic expansion of $S^f_{k,\lambda}(x)$, the vector V^k_n defined by

$$(V^k_n)_a = \sum_{m_n, \ldots, m_1} (g(m_n) + \ldots + g(m_1))^k \qquad \text{for any } a \in \mathcal{A},\ n \text{ an integer } \geq 1$$

(g defined by $g(m) = f(m) - \lambda$; summation over all a-recognizable sequences of length n),

$$(V_0^k)_a = \begin{cases} 0 & \text{if } k \neq 0\,, \\ 1 & \text{if } k = 0\,; \end{cases}$$

and the matrix $A^{(k)}$ defined by

$$A_{a,b}^{(k)} = \sum_m g(m)^k \qquad (\text{sum over all } m \in \mathcal{A}^* \text{ such that } mb \leq \sigma a)\,.$$

When f is a morphism and $f \circ \sigma = f$ (this means $f(mm') = f(m) + f(m')$ and $f(\sigma(m)) = f(m)$ for all m, m' in $\mathcal{A}^*$), one has

$$S_{k,\lambda}^f(x) = \sum_{0<\nu<x} (f(u_1 \ldots u_\nu) - \lambda \log_\theta x)^k\,,$$

$(u_n)_{n\geq 1}$ being the fixed point of σ extended by concatenation to $\mathcal{A}^{\mathbb{N}}$, with $u_1 = 1$. Indeed (see [DT89]), if $u_1 \ldots u_\nu = \sigma^{n-1}(m_n) \ldots \sigma^0(m_1)$, then $f(u_1 \ldots u_\nu) = s^f(\nu)$. We note that a formula for $\sum f(u_1 \ldots u_\nu)$, with weaker assumptions on f, was given in [DT91].

2. Computation of V_n^k

Lemma 1.

$$V_{n+1}^k = MV_n^k + \sum_{h<k} \binom{k}{h} A^{(k-h)} V_n^h\,.$$

Proof. From the definition of V_n^k we deduce

$$(V_{n+1}^k)_a = \sum_{m_{n+1},b} \sum_{m_n,\ldots,m_1} (g(m_{n+1}) + \ldots + g(m_1))^k$$

(sum over (m_{n+1}, b) such that $m_{n+1}b \leq \sigma a$, and over $m_n, \ldots, m_1$ b-recognizable). By the binomial formula,

$$(g(m_{n+1}) + \ldots + g(m_1))^k = \sum_{h\leq k} \binom{k}{h} (g(m_{n+1}))^{k-h} (g(m_n) + \ldots + g(m_1))^h\,.$$

Thus

$$V_{n+1}^k = \sum_{h\leq k} \binom{k}{h} A^{(k-h)} V_n^h\,.$$

As $A^{(0)} = M$, we obtain the assertion.

Remark 1. If $k = 0$, this relation becomes $V_{n+1}^0 = MV_n^0$; hence

$$V_n^0 = M^n \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = (|\sigma^n(a)|)_{a\in\mathcal{A}}\,.$$

Let

$$P(X)=\prod_{j=1}^{d'}(X-\theta_j)^{\alpha_j}$$

be the minimal polynomial of the matrix M, with $\theta=\theta_1>|\theta_2|\geq|\theta_3|\geq\ldots\geq|\theta_{d'}|$ and $\theta_j\neq\theta_{j'}$ for $j\neq j'$. The following lemma states that the sequence $V^k=(V_n^k)_{n\in\mathbb{N}}$ satisfies a recurrence equation related to the polynomial $P(X)^{k+1}$.

Let $S:(\mathbb{C}^d)^{\mathbb{N}}\to(\mathbb{C}^d)^{\mathbb{N}}$ be the shift (defined by $S((x_n)_{n\in\mathbb{N}})=(x_{n+1})_{n\in\mathbb{N}}$).

LEMMA 2. *The sequence* $V^k=(V_n^k)_{n\in\mathbb{N}}$ *satisfies* $(P(S))^{k+1}(V^k)=0$.

Proof. This is true for $k=0$: by Remark 1, V^0 is the sequence $n\to M^n\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$, hence $P(S)(V^0)$ is the sequence $n\to(P(M))M^n\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$; now $P(M)=0$. Suppose the lemma to be true for any $h\leq k-1$. The formula of Lemma 1 can be written as

$$S(V^k)=MV^k+\sum_{h<k}\binom{k}{h}A^{(k-h)}V^h\,.$$

Applying $(P(S))^k$ and using the fact that $(P(S))^k$ commutes with any matrix, we obtain

$$((P(S))^k\circ S)(V^k)=(P(S))^k(MV^k)\,.$$

But $(P(S))^k\circ S=S\circ(P(S))^k$ and $(P(S))^k(MV^k)=M((P(S))^k(V^k))$, so we have

$$S(W^k)=MW^k,\qquad \text{where } W^k=(P(S))^k(V^k)\,.$$

We deduce by induction $S^i(W^k)=M^iW^k$ for all $i\in\mathbb{N}$. Hence

$$(P(S))(W^k)=(P(M))W^k,\ \text{ i.e. } (P(S))^{k+1}(V^k)=0W^k=0\,.$$

LEMMA 3. *There exist polynomials* $p_{jka}(X)\in\mathbb{C}[X]$ *such that*

(i) $(V_n^k)_a=\sum_{j=1}^{d'}p_{jka}(n)\theta_j^n$ *for any* $a\in\mathcal{A}$,

(ii) $(V_n^k)_a=p_{1ka}(n)\theta^n+O(n^{k'}|\theta_2|^n)$ *with* $k'=(k+1)\alpha_2-1$,

(iii) $p_{1ka}(X)\in\mathbb{R}[X]$ *and* $d^\circ(p_{1ka})\leq k$.

Proof. (i) Let $V^{k,a}$ be the sequence $n\to(V_n^k)_a$. Lemma 2 implies $(P(S'))^{k+1}(V^{k,a})=0$, where S' is the shift on $\mathbb{C}^{\mathbb{N}}$. It is known that the kernel of $(P(S'))^{k+1}$ is generated by the sequences $n\to n^l\theta_j^n$, with $1\leq j\leq d'$ and l less than the order of θ_j in $(P(X))^{k+1}$, i.e. $l<(k+1)\alpha_j$. In other words, there exist polynomials $p_{jka}(X)$ of degree at most $(k+1)\alpha_j-1$, satisfying (i).

(iii) We have

$$(V_n^k)_a = \overline{(V_n^k)_a} = \sum_{j=1}^{d'} \overline{p_{jka}(n)}\,\overline{\theta_j^n} \quad \text{and } \overline{\theta_1} = \theta_1 .$$

Using the unicity of the decomposition of $V^{k,a}$, we obtain $\overline{p_{1ka}(n)} = p_{1ka}(n)$ for all $n \in \mathbb{N}$, so $p_{1ka}(X) \in \mathbb{R}[X]$. We have $\alpha_1 = 1$ by Perron–Frobenius, hence $d^\circ(p_{1ka}) \le (k+1) - 1$.

Remark 2. For $k = 0$, the polynomial $p_{10a}(X)$ has degree 0, i.e. it is a constant ε_a. The relation (ii) becomes

$$(V_n^0)_a = \varepsilon_a \theta^n + O(n^{\alpha_2 - 1}|\theta_2|^n) .$$

Using Remark 1 we deduce

$$\varepsilon_a = \lim_{n\to+\infty} \theta^{-n}|\sigma^n(a)| .$$

For every word $m = a_1 \ldots a_n$, we will denote by $\varepsilon(m)$ the sum $\sum_{i=1}^n \varepsilon_{a_i}$.

3. Asymptotic expansion for $S_{k,\lambda}^f(x)$

PROPOSITION 1. *There exist bounded functions $F_{k,h} : \mathbb{R} \to \mathbb{R}$ with period* 1 *such that*

$$S_{k,\lambda}^f(x) = x \sum_{h=0}^{k} l^h F_{k,h}(l) + O(\varphi_k(l)) \qquad (\textit{as } x \to +\infty,\ \textit{with } l = \log_\theta x) ,$$

$$\varphi_k(l) = \begin{cases} l^{k'}|\theta_2|^l & \textit{if } |\theta_2| > 1 ,\ \textit{with } k' = (k+1)\alpha_2 - 1 , \\ l^{k'+1} & \textit{if } |\theta_2| = 1 , \\ l^k & \textit{if } |\theta_2| < 1 . \end{cases}$$

Note that in all cases $\varphi_k(l)$ is $o(x)$. The functions $F_{k,k}$ are in fact constants (cf. §5).

We first need a lemma about the representation, relative to a substitution σ, of real positive numbers.

LEMMA 4. *For any $x \in \mathbb{R}_+^*$, there exists an integer $n = n(x)$ and a unique infinite* 1*-recognizable sequence $(m_n, m_{n-1}, \ldots)$ such that*

$$x = \sum_{i=-\infty}^{n} \varepsilon(m_i)\theta^{i-1} ,$$

m_n is not empty and $m_i a_i \ne \sigma(a_{i+1})$ for infinitely many i (with $a_i = 1 \cdot m_n \cdot \ldots \cdot m_i$). We also have $n(x) = \log_\theta x + O(1)$.

Proof. In [DT89], we define the representation of real numbers belonging to $[0, \varepsilon(1)[$. Now, if $x \in \mathbb{R}_+^*$, define $n(x)$ as the unique integer n such that $\varepsilon(1)\theta^{n-1} \le x < \varepsilon(1)\theta^n$. So $x\theta^{-n} \in [0, \varepsilon(1)[$ and has the representation

$x\theta^{-n} = \sum_{i=1}^{+\infty} \varepsilon(\mu_i)\theta^{-i}$. μ_1 is not empty, else x would be less than $\varepsilon(1)\theta^{n-1}$. Lemma 4 is thus proved with $m_i = \mu_{n-i+1}$ $\forall i \le n$.

Proof of Proposition 1. We will define $F_{k,h}$ as a sum of two functions $F^1_{k,h}$ and $F^2_{k,h}$.

In order to define these, we denote by $b_p(k,a)$ the coefficients of the polynomial $p_{1ka}(X)$ of Lemma 3, i.e.

$$p_{1ka}(X) = \sum_{p=0}^{k} b_p(k,a)X^p\,.$$

Given a real number l, $x = \theta^l$ has, by Lemma 4, the representation

$$x = \sum_{i=-\infty}^{n} \varepsilon(m_i)\theta^{i-1} \tag{1}$$

and we set $x_i = f(m_n) + \ldots + f(m_{i+2}) - \lambda(l-i)$ for $-\infty < i \le n-2$. We define

$$F^1_{k,h}(l) = \theta^{-l} \sum_{i,p,q} \binom{k}{q} S(x,i,p,q) \binom{p}{h} (i-l)^{p-h}\theta^i$$

(sum over $-\infty < i \le n-2$ and $h \le p \le q \le k$) with

$$S(x,i,p,q) = \sum_{m,a} (x_i + f(m))^{k-q} b_p(q,a)$$

(sum over $(m,a) \in \mathcal{A}^* \times \mathcal{A}$ such that $ma \le m_{i+1}$).

We define $F^2_{k,h}$ just as $F^1_{k,h}$, with the condition $-\infty < i \le n-2$ replaced by $-\infty < i \le n-1$, and with $S(x,i,p,q)$ replaced by

$$S'(x,i,p,q) = \sum_{m,a} (f(m) - \lambda(l-i))^{k-q} b_p(q,a)$$

(sum over $1 \le m < ma \le \sigma(1)$ if $i \le n-2$; over $1 \le m < ma \le m_n$ if $i = n-1$). In these definitions, we assume that $0^0 = 1$.

Then we estimate the sum

$$x \sum_{h=0}^{k} l^h F^1_{k,h}(l)\,. \tag{2}$$

We replace $F^1_{k,h}(l)$ by its value and we sum first over h, $0 \le h \le p$, then over p, $0 \le p \le q$. We find that (2) is the sum of

$$\varphi(i,q,m,a) = \binom{k}{q} (x_i + f(m))^{k-q} p_{1qa}(i)\theta^i$$

over $-\infty < i \le n-2$, $0 \le q \le k$ and $ma \le m_{i+1}$.

When $i \geq 0$ and (for example) $l \geq 1$, we use Lemma 3(ii) and the estimate $n = l + O(1)$ of Lemma 4 to obtain

$$\left|\varphi(i,q,m,a) - \binom{k}{q}(x_i + f(m))^{k-q}(V_i^q)_a\right| \leq Cl^{k-q}(i+1)^{q'}|\theta_2|^i$$

with C a constant independent of i and l, and $q' = \alpha_2(q+1) - 1$. When $i \leq -1$ and $l \geq 1$, we have

$$|x_i + f(m)| \leq C'|i|l \quad \text{and} \quad |\varphi(i,q,m,a)| \leq C''l^{k-q}|i|^k\theta^i$$

with C' and C'' constants. But $\sum_{i=0}^{n-2} l^{k-q}(i+1)^{q'}|\theta_2|^i$ and $\sum_{i=-\infty}^{-1} l^{k-q}|i|^k\theta^i$ are $O(\varphi_k(l))$ as l tends to $+\infty$. Finally, (2) is equal to

$$\sum_{i,q,m,a} \binom{k}{q}(x_i + f(m))^{k-q}(V_i^q)_a + O(\varphi_k(l))$$

(sum over $0 \leq i \leq n-2$, $0 \leq q \leq k$ and $ma \leq m_{i+1}$).

Using the definition of $(V_i^q)_a$ and the binomial formula, it is also equal to

$$\sum_{i,m,m'_i,\ldots,m'_1} (x_i + f(m) + g(m'_i) + \ldots + g(m'_1))^k + O(\varphi_k(l)) \tag{3}$$

(sum over $0 \leq i \leq n-2$ and $m_n \ldots m_{i+2}mm'_i \ldots m'_1$ 1-recognizable, with $m < m_{i+1}$).

These 1-recognizable sequences may be interpreted as the representations of all the integers ν belonging to the interval $\{N_1, N_1+1, \ldots, N-1\}$ (see Section 1), with

$$N_1 = |\sigma^{n-1}(m_n)| \quad \text{and} \quad N = \sum_{i=1}^{n} |\sigma^{i-1}(m_i)|\,;$$

and (3) as the sum

$$\sum_{N_1 \leq \nu < N} (s^f(\nu) - \lambda l)^k + O(\varphi_k(l))\,.$$

Now the representations of the integers ν belonging to $\{1, 2, \ldots, N_1 - 1\}$ are the 1-recognizable sequences $mm'_i \ldots m'_1$ such that $0 \leq i \leq n-2$ and $1 \leq m < \sigma(1)$, or $i = n-1$ and $1 \leq m < m_n$. We obtain

$$x \sum_{h \leq k} l^h F^2_{k,h}(l) = \sum_{0 < \nu < N_1} (s^f(\nu) - \lambda l)^k + O(\varphi_k(l))\,.$$

There remains to estimate $\sum_{N \leq \nu < x}(s^f(\nu) - \lambda l)^k$ (or $\sum_{x \leq \nu < N}(s^f(\nu) - \lambda l)^k$ if $x < N$). This sum is $O(l^k|N - x|)$. We have

$$N - x = \sum_{i=1}^{n}(|\sigma^{i-1}(m_i)| - \varepsilon(m_i)\theta^{i-1}) + O(1)\,,$$

$$|\sigma^{i-1}(m_i)| - \varepsilon(m_i)\theta^{i-1} = O(i^{\alpha_2 - 1}|\theta_2|^i) \quad \text{(see Remarks 1 and 2)}.$$

Hence $l^k|N - x| = O(\varphi_k(l))$.

$F^1_{k,h}$ is bounded because $|\theta^{-l}(x_i + f(m))^{k-q}(i-l)^{p-h}\theta^i| = O((n-i)^k\theta^{i-n})$ and $\sum_{i=-\infty}^{n-2}(n-i)^k\theta^{i-n}$ does not depend on n. In the same way, $F^2_{k,h}$ is bounded.

Periodicity. Let $l' = l + 1$ and $x' = \theta^{l'}$. The representation of x' is connected with that of x by $n(x') = n(x) + 1$ and $m'_i = m_{i-1}$ for $i \le n + 1$; so $S(x', i, p, q) = S(x, i - 1, p, q)$ and $F^1_{k,h}(l + 1) = F^1_{k,h}(l)$, and similarly for $F^2_{k,h}$.

Remark 3. In the case where s^f is the sum-of-digits in the q-ary expansion (see Section 1), one has $\theta = q$ and $\#\mathcal{A} = 1$; so in Lemma 3, $d' = 1$ and $(V^k_n)_a = p_{1ka}(n)q^n$.

The representation of real numbers in Lemma 4 ($x = \sum_{i=-\infty}^{n}\varepsilon(m_i)\theta^{i-1}$) coincides with their q-ary expansion. If x is an integer, m_i is empty for $i \le 0$; then in the proof of Proposition 1, we can replace the condition $-\infty < i$ by $0 \le i$. We obtain

$$S^f_{k,\lambda}(x) = x\sum_{h=0}^{k} l^h F_{k,h}(l) \quad \text{for any integer } x \ge 1 \text{ and } l = \log_\theta x.$$

4. Continuity

PROPOSITION 2. *The functions $F_{k,h}$ are continuous on $\mathbb{R}$.*

Proof. Let $\widetilde{S} : [1, +\infty[\to \mathbb{R}$ be the continuous piecewise affine function such that

$$\widetilde{S}(n) = S^f_{k,\lambda}(n) \quad \text{for any } n \in \mathbb{N}.$$

As $\widetilde{S}(x) = S^f_{k,\lambda}(x) + O(\varphi_k(l))$ we can replace, in Proposition 1, $S^f_{k,\lambda}(x)$ by $\widetilde{S}(x)$. Next we define, for $0 \le h < k$, the functions

$$(1) \qquad \widetilde{S}_{k,h}(x) = \widetilde{S}(x) - x\sum_{h'=h+1}^{k} l^{h'}F_{k,h'}(l) \quad (x \in [1, +\infty[,\ l = \log_\theta x).$$

If $h = k$, then $\widetilde{S}_{k,k}(x)$ is equal to $\widetilde{S}(x)$, which is continuous.

We want to establish a relation between $\widetilde{S}_{k,h}$ and $F_{k,h}$. We deduce from Proposition 1 that

$$\widetilde{S}_{k,h}(x) = xl^hF_{k,h}(l) + O(x|l|^{h-1}),$$

or, equivalently,

$$F_{k,h}(l) = \theta^{-l}l^{-h}\widetilde{S}_{k,h}(\theta^l) + O(|l|^{-1});$$

hence

$$F_{k,h}(l+n) = \theta^{-l-n}(l+n)^{-h}\widetilde{S}_{k,h}(\theta^{l+n}) + O(|l+n|^{-1}) \quad \text{for } n \in \mathbb{N}\,.$$

But $F_{k,h}(l+n) = F_{k,h}(l)$. Moreover, given a compact set K and $\varepsilon > 0$ we have, for n large enough, $|l+n|^{-1} < \varepsilon$ for any $l \in K$. In other words,

$$F_{k,h}(l) = \lim_{n\to+\infty}(\theta^{-l-n}(l+n)^{-h}\widetilde{S}_{k,h}(\theta^{l+n})) \quad \text{(uniformly on compact sets)}\,.$$

By this relation and by (1), we obtain successively the continuity of $F_{k,k}$, $\widetilde{S}_{k,k-1}, F_{k,k-1}, \ldots, \widetilde{S}_{k,0}, F_{k,0}$.

5. The main result. We can specify the asymptotic expansion given in Proposition 1, by computing $F_{k,h}(l)$ for the maximal h such that $F_{k,h} \neq 0$. We will use the eigenvectors of M. M is primitive, hence has a unique row-eigenvector ξ defined by

$$\xi M = \theta\xi \quad \text{and} \quad \xi\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = 1\,.$$

The vector $\varepsilon = (\varepsilon_a)_{a\in\mathcal{A}}$ is a column-eigenvector since, using Remarks 1 and 2, we have

$$\varepsilon = \lim_{n\to+\infty}\theta^{-n}M^n\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix},$$

hence $M\varepsilon = \theta\varepsilon$ and $\xi\varepsilon = 1$. We define the constant

$$\alpha = \theta^{-1}\sum_{a,m,b}\xi_a f(m)\varepsilon_b \quad \text{(sum over } (a,m,b) \in \mathcal{A}\times\mathcal{A}^*\times\mathcal{A} \text{ with } mb \le \sigma a).$$

In the case $\lambda = \alpha$, we will also use

$$\beta = \theta^{-1}\xi(A^{(2)}\varepsilon + 2A^{(1)}v)$$

where the vector v is defined, modulo $\mathbb{R}\varepsilon$, by

$$(\theta I - M)v = A^{(1)}\varepsilon\,.$$

β does not depend on the choice of v, because

$$\theta^{-1}\xi A^{(1)}\varepsilon = \alpha - \lambda\,, \tag{1}$$

which is zero in the case $\lambda = \alpha$. Such a v exists: consider the hyperplane $(\theta I - M)(\mathbb{R}^d)$ and the hyperplane orthogonal to ξ, which contains the vector $A^{(1)}\varepsilon$ in the case $\lambda = \alpha$.

THEOREM. *There exist continuous functions $F_{k,h}$ and $G_{k,h}$ with period* 1 *such that* (*as* $x \to +\infty$)

$$S^f_{k,0}(x) = (\alpha l)^k x + x \sum_{h<k} l^h F_{k,h}(l) + O(\varphi_k(l)),$$

$$S^f_{k,\alpha}(x) = \begin{cases} \dfrac{k!}{(k/2)!}\left(\dfrac{\beta l}{2}\right)^{k/2} x + x \displaystyle\sum_{h<k/2} l^h G_{k,h}(l) + O(\varphi_k(l)) & \textit{if } k \textit{ is even},\\ x \displaystyle\sum_{h<k/2} l^h G_{k,h}(l) + O(\varphi_k(l)) & \textit{if } k \textit{ is odd}, \end{cases}$$

with $l = \log_\theta x$ *and* $\varphi_k(l)$ *defined in Proposition* 1.

The next lemmas concern the polynomial of Lemma 3(ii). By (ii) and (iii) of that lemma, there exists a polynomial $P_k(X) = \sum_{p=0}^{k} b_p(k) X^p$ with $\mathbb{R}^d$ coefficients $b_p(k)$ such that

$$V_n^k = \theta^n P_k(n) + o(\theta^n).$$

Writing now the formula of Lemma 1 modulo $o(\theta^n)$, and using the fact that two polynomials asymptotically equal are identical, we obtain

Lemma 5.

$$\theta P_k(n+1) = M P_k(n) + \sum_{h<k} \binom{k}{h} A^{(k-h)} P_h(n).$$

Next we compute the term of degree k in $P_k(X)$.

Lemma 6.

$$b_k(k) = (\alpha - \lambda)^k \varepsilon.$$

Proof. This is true for $k = 0$ (see Remark 2). Suppose

(2) $$b_{k-1}(k-1) = (\alpha - \lambda)^{k-1} \varepsilon.$$

Identifying the terms of degree k in the formula of Lemma 5, we obtain

$$\theta b_k(k) = M b_k(k).$$

Thus $b_k(k)$ is an eigenvector and there exists $t \in \mathbb{R}$ such that $b_k(k) = t\varepsilon$ (θ being a simple eigenvalue by Perron–Frobenius).

Identifying the terms of degree $k-1$ in the formula of Lemma 5, we obtain

$$\theta k b_k(k) + \theta b_{k-1}(k) = M b_{k-1}(k) + k A^{(1)} b_{k-1}(k-1)$$

and, multiplying on the left by ξ and using $\xi\varepsilon = 1$ and $\xi M = \theta\xi$,

$$\theta k t + \theta \xi b_{k-1}(k) = \theta \xi b_{k-1}(k) + k \xi A^{(1)} b_{k-1}(k-1).$$

Then we can compute t, and from (1) and (2) we obtain the assertion.

LEMMA 7. *If $\lambda = \alpha$, we have $d^\circ(P_k) \le [k/2]$ and*

$$b_{[k/2]}(k) = \beta_k \varepsilon \qquad \text{if } k \text{ is even},$$
$$(\theta I - M) b_{[k/2]}(k) = \beta_k A^{(1)} \varepsilon \quad \text{if } k \text{ is odd, with } \beta_k = \frac{k!}{[k/2]!} \left(\frac{\beta}{2}\right)^{[k/2]}.$$

Proof. For $k = 0$, this formula is the same as in Lemma 6. Suppose $k \ge 1$ and the formula is true for $0, 1, \ldots, k-1$. Then, by Lemma 5, the degree of the polynomial $\theta P_k(n+1) - MP_k(n)$ is at most $k' = [(k-1)/2]$.

Let $p = d^\circ(P_k)$. If $p \ge k' + 2$ we obtain, identifying the terms of degree p and $p-1$ in Lemma 5,

$$\theta b_p(k) = M b_p(k) \quad \text{and} \quad \theta p b_p(k) + \theta b_{p-1}(k) = M b_{p-1}(k).$$

By the same computation as in Lemma 6, we obtain $b_p(k) = 0$, contrary to $p = d^\circ(P_k)$.

Hence $p \le k' + 1$. Identifying the terms of degree $k' + 1$ and k' in Lemma 5, we obtain

$$\theta b_{k'+1}(k) = M b_{k'+1}(k),$$

$$\theta(k'+1) b_{k'+1}(k) + \theta b_{k'}(k) - M b_{k'}(k)$$

$$= \begin{cases} kA^{(1)} b_{k'}(k-1) & \text{if } k \text{ is odd}, \quad (3)\\ kA^{(1)} b_{k'}(k-1) + \dfrac{k(k-1)}{2} A^{(2)} b_{k'}(k-2) & \text{if } k \text{ is even}. \quad (4)\end{cases}$$

By the same computation as in Lemma 6, we obtain $b_{k'+1}(k) = 0$ if k is odd (using (1), (3) and $b_{k'}(k-1) \in \mathbb{R}\varepsilon$ by the induction hypothesis). Then by (3) we obtain the value of $(\theta I - M) b_{k'}(k)$ and prove the assertion of the lemma for k.

If k is even we again have $b_{k'+1}(k) = t\varepsilon$ with $t \in \mathbb{R}$ and, multiplying (4) by ξ on the left,

$$\theta(k'+1)t = k\xi A^{(1)} b_{k'}(k-1) + \frac{k(k-1)}{2} \xi A^{(2)} b_{k'}(k-2).$$

By the induction hypothesis, the vector $v = (1/\beta_{k-1}) b_{k'}(k-1)$ satisfies $(\theta I - M)v = A^{(1)}\varepsilon$, and the vector $b_{k'}(k-2)$ is equal to $b_{(k-2)/2}(k-2) = \beta_{k-2}\varepsilon$. Using the definition of β we obtain the assertion.

Proof of the Theorem. We compute $F_{k,k}(l)$, replacing h by k in the proof of Proposition 1. We obtain

$$F^1_{k,k}(l) = \theta^{-l} \sum_{i,m,a} b_k(k,a)\theta^i \quad (\text{sum over } -\infty < i \le n-2 \text{ and } ma \le m_{i+1})$$

and, by Lemma 6,

$$F_{k,k}^1(l) = \theta^{-l}(\alpha - \lambda)^k \sum_{i=-\infty}^{n-2} \varepsilon(m_{i+1})\theta^i .$$

In the same way,

$$F_{k,k}^2(l) = \theta^{-l}(\alpha - \lambda)^k \Big((\varepsilon(m_n) - \varepsilon(1))\theta^{n-1} + \sum_{i=-\infty}^{n-2} (\theta - 1)\varepsilon(1)\theta^i \Big) .$$

Hence $F_{k,k}(l) = (\alpha - \lambda)^k$ and we obtain the conclusion of the Theorem in the case $\lambda = 0$.

In the case $\lambda = \alpha$, we first compute $F_{k,h}(l)$ for $h > [k/2]$. This condition, together with $h \le p \le q \le k$, implies $p > [q/2]$; then $b_p(q, a)$ is zero by Lemma 7, and $F_{k,h}(l) = 0$. Next we compute $F_{k,k/2}(l)$ for k even. The condition $k/2 \le p \le q \le k$ implies $p = q/2 = k/2$, else $p > q/2$ and $b_p(q, a) = 0$. The computation of $F_{k,k/2}(l)$ is the same as that of $F_{k,k}(l)$ and leads to $F_{k,k/2}(l) = \beta_k$.

Now we will check that the sum

$$Z_{k,\alpha}^f(x) = \sum_{0<\nu<x} (s^f(\nu) - \alpha \log_\theta \nu)^k$$

has the same equivalent as $S_{k,\alpha}^f(x)$ if k is even. This sum is studied in [D90] and [GoL87].

COROLLARY. $Z_{k,\alpha}^f(x) = O(l^{[k/2]}x)$ *(with* $l = \log_\theta x$*), and if* k *is even, then*

$$Z_{k,\alpha}^f(x) = \frac{k!}{(k/2)!}\left(\frac{\beta l}{2}\right)^{k/2} x + O(l^{k/2-1}x) .$$

Proof. It is sufficient to prove this for integer x. We have

$$\begin{aligned} Z_{k,\alpha}^f(x) &= \sum_{0<\nu<x} (\lambda_\nu + \mu_\nu)^k \qquad \text{with } \lambda_\nu = s^f(\nu) - \alpha l \text{ and } \mu_\nu = \alpha \log_\theta(x/\nu) \\ &= S_{k,\alpha}^f(x) + \sum_{0<\nu<x} \sum_{i=0}^{k-1} \binom{k}{i} \lambda_\nu^i \mu_\nu^{k-i} . \end{aligned}$$

Thus, it is sufficient to prove that for $i \le k-1$ and $j \le k$, $\sum_{0<\nu<x} \lambda_\nu^i \mu_\nu^j = O(l^{[(k-1)/2]}x)$. We remark that

$$\sum_{0<\nu<x} \lambda_\nu^i \mu_\nu^j = \sum_{0<\nu<x} (\mu_\nu^j - \mu_{\nu+1}^j) \sum_{\nu'=1}^{\nu} \lambda_{\nu'}^i .$$

We deduce from the theorem that $\sum_{\nu'=1}^{\nu} \lambda_{\nu'}^i = O(\nu l^{[i/2]} \mu_\nu^i)$, and from the mean value theorem

$$\mu_\nu^j - \mu_{\nu+1}^j = O(\mu_\nu^{j-1}/\nu)\,.$$

Hence

$$\sum_{0<\nu<x} \lambda_\nu^i \mu_\nu^j = O\Big(l^{[i/2]} \sum_{0<\nu<x} \mu_\nu^{i+j-1}\Big)\,. \tag{1}$$

In the case $i=0$ this estimate becomes

$$\sum_{0<\nu<x} \mu_\nu^j = O\Big(\sum_{0<\nu<x} \mu_\nu^{j-1}\Big)$$

and by induction $\sum_{0<\nu<x} \mu_\nu^j = O(x)$. Then we deduce from (1) that

$$\sum_{0<\nu<x} \lambda_\nu^i \mu_\nu^j = O(l^{[(k-1)/2]}x)\,.$$

Remark. A more precise computation should give the existence of functions $H_{k,h}$ such that

$$Z_{k,\alpha}^f(x) = \frac{k!}{(k/2)!}\left(\frac{\beta l}{2}\right)^{k/2} x + x \sum_{h<k/2} l^h H_{k,h}(l) + O(\varphi_k(l))$$

for k even, and the same without the first term of the r.h.s. if k is odd.

The functions $H_{k,h}$ are related to the functions $G_{k,h}$ of the Theorem (and $G_{k,k/2} = \frac{k!}{(k/2)!}\big(\frac{\beta}{2}\big)^{k/2}$ if k even):

$$H_{k,h}(l) = G_{k,h}(l) - \sum \binom{k}{i}\binom{k-i}{k-i'}\binom{j}{h} \frac{i(-1)^i \alpha^{i'}}{\theta^l \operatorname{Log}\theta} \times \int_0^{\theta^l} (\log_\theta \nu - l)^{i'+j-h-1} G_{k-i',j}(\log_\theta \nu)\, d\nu$$

(sum over $h \le j \le [(k-1)/2]$ and $1 \le i \le i' \le k-2j$).

Of course $H_{k,h}$ is periodic and continuous, and differentiable iff $G_{k,h}$ is.

6. Nondifferentiability of $F_{k,h}$ and $G_{k,h}$

PROPOSITION 3. *If $\alpha \ne 0$ and $\alpha \ne f(\omega)$, then the functions $F_{k,h}$ of the Theorem are nowhere differentiable for $h<k$. If $\beta \ne 0$ and $\alpha \ne f(\omega)$, then $G_{k,h}$ are nowhere differentiable for $h<k/2$.*

Proof. For fixed k and $h<k$, we define a mapping $(x,l) \to \phi_x(l)$ from $\mathbb{R}_+^* \times \mathbb{R}$ to $\mathbb{R}$ such that, in the case $x=\theta^l$, $\phi_x(l)$ should be equal to $xF_{k,h}(l)$ (with $F_{k,h}(l)$ defined in the proof of Proposition 1).

We set

$$\phi_x(l) = \sum_{i=-\infty}^{n-1} \phi_{x,i}(l), \qquad \phi_{x,i}(l) = \phi_{x,i}^1(l) + \phi_{x,i}^2(l)\,,$$

$$\phi^1_{x,i}(l) = \begin{cases} \displaystyle\sum_{h\le p\le q\le k} \sum_{ma\le m_{i+1}} \binom{k}{q} (x_{i,m} - \lambda(l-i))^{k-q} \binom{p}{h} b_p(q,a)(i-l)^{p-h}\theta^i \\ \qquad \text{if } i \le n-2\,, \\ 0 \quad \text{if } i = n-1 \end{cases}$$

(where $x_{i,m} = f(m_n) + \ldots + f(m_{i+2}) + f(m)$ depends on the representation $(m_i)_{-\infty<i\le n}$ of x, but not on l).

$\phi^2_{x,i}(l)$ is defined in the same way as $\phi^1_{x,i}(l)$, with $x_{i,m}$ replaced by $f(m)$, and with the condition $ma \le m_{i+1}$ replaced by $1 \le m < ma \le \sigma(1)$, or $1 \le m < ma \le m_n$ if $i = n-1$.

$\phi^1_{x,i}(l)$ is a polynomial in l whose coefficients have absolute value less than $C(|i|+1)^k\theta^i$, with C a constant independent of i and l. As the series $\sum_{i=-\infty}^{n-1}(|i|+1)^k\theta^i$ converges, we deduce that $\phi_x(l)$ is also a polynomial in l.

Suppose $F_{k,h}$ is differentiable at the point l; let $x = \theta^l$. Then x has a representation $m_n m_{n-1}\ldots = (m_i)_{-\infty<i\le n}$ by Lemma 4.

For any $j \le n$, we define two real numbers u_j and v_j : u_j has representation $m_n m_{n-1}\ldots m_j\omega^{\mathbb{N}}$, and v_j has representation $m_n m_{n-1}\ldots m_j \omega^{\nu-1} a\omega^{\mathbb{N}}$.

Fix now ν such that $|\sigma^\nu(b)| \ge 2$ for any $b \in \mathcal{A}$, and set $a = 1\cdot m_n\cdot\ldots\ldots\cdot m_j\cdot\omega^\nu$; then $m_n\ldots m_j\omega^{\nu-1}a\omega^{\mathbb{N}}$ is 1-recognizable. We have

$$v_j - u_j = \varepsilon(a)\theta^{j-\nu-1}\,.$$

Let

$$\Delta_j = \frac{v_j F_{k,h}(l'_j) - u_j F_{k,h}(l_j)}{v_j - u_j} \qquad (l'_j = \log_\theta v_j \text{ and } l_j = \log_\theta u_j)\,.$$

This is the rate of variation of the function $t \to tF_{k,h}(\log_\theta t)$ between the points u_j and v_j. As j tends to $-\infty$, $u_j - x = O(|v_j - u_j|)$; thus Δ_j tends to the derivative of this function at the point x.

We will deduce that the rate of variation

$$\Delta'_j = \frac{\phi_{v_j}(l_j) - \phi_{u_j}(l_j)}{v_j - u_j}$$

also has a limit. By the mean value theorem, there exists a real number l''_j between l_j and l'_j such that

$$\Delta_j - \Delta'_j = \frac{(l'_j - l_j)\phi'_{v_j}(l''_j)}{v_j - u_j}$$

(where ϕ'_{v_j} is the derivative of the function $t \to \phi_{v_j}(t)$). We have

$$\phi'_{v_j}(l''_j) = \phi'_x(l''_j) + O(|l''_j - j|^k\theta^j)$$

(since the representations of v_j and x coincide between the indices j and n). As j tends to $-\infty$, $\phi'_x(l''_j)$ tends to $\phi'_x(l)$, since ϕ'_x is a polynomial. Thus $\phi'_{v_j}(l''_j) \to \phi'_x(l)$, $\Delta_j - \Delta'_j \to (x\ln\theta)^{-1}\phi'_x(l)$ and

$$\Delta'_j \to L = F_{k,h}(l) + (\ln\theta)^{-1}F'_{k,h}(l) - (x\ln\theta)^{-1}\phi'_x(l)\,.$$

Now

$$\Delta'_j = \varepsilon(a)^{-1}\sum_{h\le p\le q\le k}\binom{k}{q}(x_{j,\omega} + (\nu-1)f(\omega) - \lambda(l_j - j + \nu + 1))^{k-q} \times \binom{p}{h} b_p(q,a)(j-\nu-1-l_j)^{p-h}\,.$$

Fixing j, we consider Δ'_j as a polynomial in ν; its degree is at most $k-h$. We compute the coefficient c_{k-h} of the term of degree $k-h$; it is obtained for $p=q$; using Lemma 6 and the binomial formula we obtain

$$c_{k-h} = \binom{k}{h}(f(\omega)-\alpha)^{k-h}(\alpha-\lambda)^h \qquad (\text{independent of } j)\,.$$

But this coefficient is also equal to $(1/(k-h)!)\Delta^{k-h}(\Delta'_j)$, where Δ is the operator which associates with every polynomial $P(X)$ the polynomial $P(X+1)-P(X)$. As $\lim_{j\to-\infty}\Delta^{k-h}(\Delta'_j) = \Delta^{k-h}(L) = 0$, we obtain $c_{k-h}=0$.

So if the function $F_{k,h}$ of Proposition 1 is differentiable, we have necessarily $\alpha = f(\omega)$ or $\lambda = \alpha$.

In the case $\alpha = f(\omega)$ a counterexample is given in [DT91].

In the case $\lambda = \alpha$, we must prove that $F_{k,h}$ is not differentiable for $h < k/2$ except in the cases $\alpha = f(\omega)$ or $\beta = 0$. We have seen (Section 5) that $b_p(q,a)$ is zero if $p > q/2$; hence Δ'_j is a polynomial in ν of degree at most $k-2h$, because $k-q+p-h \le k-2p+p-h \le k-2h$; the term of degree $k-2h$ is obtained for $p=h$ and $q=2h$. As $h<k/2$, $k-2h$ is positive hence $\Delta^{k-2h}(L)=0$.

We deduce that

$$0 = \lim_{j\to-\infty}\Delta^{k-2h}(\Delta'_j) = (k-2h)!\varepsilon(a)^{-1}\binom{k}{2h}(f(\omega)-\alpha)^{k-2h}b_h(2h,a)\,,$$

$$b_h(2h,a) = \frac{(2h)!}{h!}\left(\frac{\beta}{2}\right)^h\varepsilon(a)\,,$$

hence $\alpha = f(\omega)$ or $\beta = 0$.

7. Application to the sequence $(n\omega)_{n\ge1}$ for some quadratic ω. Consider the sequence $\varepsilon = (\varepsilon_n)_{n\ge1}$ defined by

$$\varepsilon_n = \begin{cases} 1 & \text{if } \operatorname{frac}(n\omega/2) < 1/2\,,\\ -1 & \text{otherwise,}\end{cases}$$

where ω is a quadratic number such that

$$\omega = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \omega}} \qquad \text{with } a_1 \in \mathbb{N}^*,\ a_2 = 2\nu,\ \nu \in \mathbb{N}^* .$$

Godrèche, Luck and Vallet ([GoL87]) asked about the asymptotic expansion of

$$\frac{1}{N} \sum_{n=1}^{N} \left(s(n) - \frac{a_1}{2} \log_\theta n \right)^2$$

where $s(n) = \sum_{i=1}^{n} \varepsilon_i$ and $\theta = a_1(a_2 + \omega) + 1$.

We obtain (Section 5)

$$\frac{1}{N} \sum_{n=1}^{N} \left(s(n) - \frac{a_1}{2} \log_\theta n \right)^2 = \beta \log_\theta N + H_{2,0}(\log_\theta N) + o(1)$$

with

$$\beta = \frac{(\theta - a_1\nu - 1)(a_1^2\nu + 3a_1 + 2\nu)}{12a_1\nu(a_1\nu + 2)} .$$

Indeed, the sequence ε may be obtained from the substitution σ on the alphabet $\mathcal{A} = \{1, 2, 3\}$ defined by

$$\sigma(1) = m^\nu 1 \quad (\text{where } m = 1^{a_1} 2\, 3^{a_1 - 1}) ,$$
$$\sigma(2) = m^{\nu+1} 3, \qquad \sigma(3) = m^\nu 3$$

and the output function

$$f(1) = 1, \qquad f(2) = f(3) = -1$$

(i.e. $\varepsilon_i = f(u_i)$ where $u_1 = 1$ and $(u_i)_{i \geq 1}$ is the fixed point of σ).

This substitution is the same as the one of [GoL87], Section 4 (a_2 even), upon replacing their letters a and c by the letter 1, b by 2 and d by 3.

The matrix of σ is

$$M = \begin{pmatrix} a_1\nu + 1 & \nu & (a_1 - 1)\nu \\ a_1(\nu + 1) & \nu + 1 & a_1(\nu + 1) - \nu \\ a_1\nu & \nu & (a_1 - 1)\nu + 1 \end{pmatrix} ,$$

the eigenvectors defined in Section 5 are here

$$\xi = \left(\frac{1}{2}, \frac{\theta - 1}{2a_1} - \nu, \frac{1}{2} + \nu - \frac{\theta - 1}{2a_1} \right) \qquad \text{and} \qquad \varepsilon = \frac{\theta}{\theta + 1} \begin{pmatrix} 1 \\ 1 + \frac{1}{\nu} - \frac{1}{\theta\nu} \\ 1 \end{pmatrix} .$$

The condition $f(\sigma(m)) = f(m)$ (see Section 1) is satisfied. An easy calculation gives, for α and β defined in Section 5, $\alpha = a_1/2$ and β as indicated above.

8. Sum-of-digits function in the case of finite Parry expansion. Now we compute α and β in the case of "normal numeration" associated with linear finite recurrence expansion with canonical initial values (cf. [B89] and [GTi91]; see also [Fro] and [Sh88]).

Let $(u_i)_{i\geq 1}$ be a strictly increasing sequence of positive integers with $u_1 = 1$; the *normal representation* of the integer $N \geq 1$ with respect to $(u_i)_{i\geq 1}$ is the finite sequence of n integers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ uniquely determined by the "greedy algorithm":

$$u_n \leq N < u_{n+1}, \quad \varepsilon_n = [N/u_n], \quad r_n = N - \varepsilon_n u_n\,,$$

and, for $2 \leq i \leq n$,

$$\varepsilon_{i-1} = [r_i/u_{i-1}], \quad r_{i-1} = r_i - \varepsilon_{i-1}u_{i-1}\,.$$

If $(\varepsilon'_i)_{i=1,\ldots,n}$ are integers such that $N = \sum_{i=1}^n \varepsilon'_i u_i$, then the sequence $(\varepsilon'_i)_{i=1,\ldots,n}$ is the normal representation of N if and only if

$$\forall j = 1, \ldots, n \quad \sum_{i=1}^{j} \varepsilon'_i u_i < u_{j+1} \quad \text{(cf. [Fr])}\,.$$

Now let $d \geq 1$ be an integer and $a_1, a_2, \ldots, a_d$ non-negative integers with $a_d \geq 1$ satisfying the "Parry condition": if $d \geq 2$, then

$$\forall j = 2, \ldots, d \quad a_j \ldots a_d 0^{j-1} <_l a_1 a_2 \ldots a_d$$

($<_l$ being the lexicographic order), and if $d = 1$, then $a_1 \geq 2$.

Let $(u_i)_{i\geq 1}$ be such that $u_1 = 1$ and

$$u_i = \begin{cases} a_1 u_{i-1} + a_2 u_{i-2} + \ldots + a_{i-1}u_1 + 1 & (2 \leq i \leq d)\,, \\ a_1 u_{i-1} + a_2 u_{i-2} + \ldots + a_d u_{i-d} & (i > d)\,. \end{cases}$$

Then $(u_i)_{i\geq 1}$ is strictly increasing (because $a_1, a_d \geq 1$).

Lemma. *Define $\mathcal{A} = \{1, 2, \ldots, d\}$ and let σ be the substitution over $\mathcal{A}$ given by*

$$\sigma(j) = \begin{cases} 1^{a_j}(j+1) & \text{if } 1 \leq j \leq d-1\,, \\ 1^{a_d} & \text{if } j = d\,. \end{cases}$$

(i) *For any $i \geq 1$, $u_i = |\sigma^{i-1}(1)|$.*

(ii) *If $N = \sum_{i=1}^n |\sigma^{i-1}(m_i)|$ is the admissible representation of N relative to the substitution σ, then the sequence $\varepsilon_i = |m_i|$ $(i = 1, \ldots, n)$ is the normal representation of N with respect to $(u_i)_{i\geq 1}$.*

Proof. (i) Immediate using the definition of σ.

(ii) Using [DT89], Lemma 1.1, we have for $0 \leq j \leq n$

$$\sum_{i=1}^{j} |\sigma^{i-1}(m_i)| < |\sigma^j(k_j)| \quad \text{for some } k_j \in \mathcal{A}\,.$$

Thus, it remains to prove that for any $k \in \mathcal{A}$ and $j \in \mathbb{N}$,

$$|\sigma^j(k)| \le |\sigma^j(1)|\,.$$

We have

$$\sigma^j(k) = \begin{cases} (\sigma^{j-1}(1))^{a_k}(\sigma^{j-2}(1))^{a_{k+1}}\dots 1^{a_{k+j-1}}(k+j), & 0 \le j \le d-k\,,\\ (\sigma^{j-1}(1))^{a_k}(\sigma^{j-2}(1))^{a_{k+1}}\dots(\sigma^{j-d+k-1}(1))^{a_d}, & j > d-k\,. \end{cases}$$

Now we will use the fact that for the admissible representation, a lexicographic inequality between two representations implies an ordinary inequality between the represented numbers.

If $k \ge 2$, then by the Parry condition in the first case

$$|\sigma^{j-1}(1^{a_k})| + |\sigma^{j-2}(1^{a_{k+1}})| + \dots + |1^{a_{k+j-1}}|$$

is the admissible representation of $|\sigma^j(k)| - 1$, and in the second case

$$|\sigma^{j-1}(1^{a_k})| + |\sigma^{j-2}(1^{a_{k+1}})| + \dots + |\sigma^{j-d+k-1}(1^{a_d})|$$

is the admissible representation of $|\sigma^j(k)|$. Moreover, these representations are $<_l 1\omega^j$ and the proof is complete.

The matrix of σ and the eigenvectors satisfying the conditions of Section 5 are

$$M = \begin{pmatrix} a_1 & 1 & 0 & \dots & 0\\ a_2 & 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ a_d & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad \xi = \frac{\theta-1}{\theta^d-1}(\theta^{d-1},\dots,\theta,1)$$

and

$$\varepsilon = \frac{\theta^d-1}{(\theta-1)\theta P'(\theta)}\begin{pmatrix} \theta\\ \theta^2 - a_1\theta\\ \vdots\\ \theta^d - a_1\theta^{d-1} - \dots - a_{d-1}\theta \end{pmatrix}$$

where θ is the root of $P(X) = X^d - a_1X^{d-1} - \dots - a_d$ with maximum modulus.

We compute the constants α and β of Section 5; the output function is here $f(m) = |m|$ and we check the vector v with first coordinate 0; we obtain

$$\alpha = \frac{Q(\theta)}{\theta P'(\theta)} \quad \text{with } Q(X) = a_1'X^{d-1} + \dots + a_d' \text{ and}$$

$$a_i' = a_i\Big(\tfrac{1}{2}(a_i-1) + \sum_{j<i} a_j\Big)\,,$$

$$\beta = \frac{R(\theta)}{\theta P'(\theta)} - \alpha^2 \quad \text{with } R(X) = a_1''X^{d-1} + \dots + a_d''$$

and

$$a_i'' = a_i\Big(\tfrac{1}{3}(a_i-1)(a_i-\tfrac{1}{2}) - (a_i-1-2\alpha)A_i + \sum_{j<i}(a_j^2 - 2(a_j-\alpha)A_j)\Big)\,,$$

$$A_i = \sum_{k=i}^{d}(a_k-\alpha)\,.$$

For instance, in the case of the ordinary numeration system $d=1$, $\sigma(1)=1^q$ and $\alpha=(q-1)/2$, $\beta=(q^2-1)/12$.

In the case of the Fibonacci expansion $d=2$, $\sigma(1)=12$ and $\sigma(2)=1$, $a_1=a_2=1$, $\alpha=(5-\sqrt{5})/10$ and $\beta=1/(5\sqrt{5})$.

Remark ([Fa92]). If $(a_i)_{i\ge1}$ is an eventually periodic sequence $a_1\ldots a_m(a_{m+1}\ldots a_{m+l})^\infty$ $(m\ge0, l\ge1)$ satisfying the Parry condition

$$\forall j\ge2 \quad a_ja_{j+1}\ldots <_l a_1a_2\ldots$$

and $(u_i)_{i\ge1}$ the sequence $u_1=1$, $u_i=a_1u_{i-1}+a_2u_{i-2}+\ldots+a_{i-1}u_1+1$ $(i\ge2)$, then the numeration associated with the substitution σ on the finite alphabet $\mathcal{A}=\{1,2,\ldots,m+l\}$ given by

$$\sigma(j) = \begin{cases} 1^{a_j}(j+1) & (j=1,2,\ldots,m+l-1)\,,\\ 1^{a_{m+l}}(m+1) & (j=m+l)\end{cases}$$

is the same as the normal representation of integers with respect to $(u_i)_{i\ge1}$ (same proof). We leave the computation of α and β in this case to the reader.

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