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On general L-functions

by

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1. Introduction and notation. The aim of the present paper is to develop in a unified way some analytic results for a rather general class of L-functions. This class is defined axiomatically and the axioms are modelled on the basic properties of the zeta and L-functions associated with algebraic number fields and automorphic forms which appear in number theory.

A first attempt in this direction was made in Perelli [14] and Perelli–Puglisi [15]. Here we present a more satisfactory set of axioms, and concentrate our investigations mainly on problems connected with the zero-free regions and real zeros.

Special attention is given to the problem of the uniformity of the results. Among the parameters involved in the definition of our general L-functions, we have chosen to give uniform results only in the main parameter which appears in the functional equation (see Section 2). This is usually the parameter which reflects the more interesting quantities connected with the underlying algebraic structures. At the cost of complications in the details one could obtain results similar to those in Sections 4 and 5, with complete uniformity in all parameters.

A crucial role is played throughout the paper by the concept of irreducibility and its connection with the Rankin–Selberg type convolution for general L-functions. Apparently, in the present abstract setting the problems associated with the distribution of zeros of an L-function are closely related to the analytic properties of the Rankin–Selberg type convolution of the L-function itself.

We finally remark that the present paper can be viewed as a first step toward a satisfactory establishment of the qt-principle asked for in a general setting in Lang's paper [9].

We will use the following basic notation. Further notation will be introduced later on.

K — an algebraic number field, $n = [K : \mathbb{Q}]$,
$\mathfrak{p}$ — a prime ideal of K, $\mathfrak{a}$ — an ideal of K,

$\mathcal{P}_K = \{\text{prime ideals of } K\}$, $\mathcal{I}_K = \{\text{ideals of } K\}$,
$N\mathfrak{a}$ — the norm of $\mathfrak{a}$ over $\mathbb{Q}$,
$\zeta_K(s)$ — the Dedekind zeta function of K,
$\zeta(s)$ — the Riemann zeta function,
$s = \sigma + it$,
$d_k(n)$ — the general divisor function,
$|S|$ — the cardinality of the set S.

We will use the following notation for the Laurent expansion at $s = 1$:

$$\zeta(s) = \sum_{j=-1}^{\infty} r_j (s-1)^j ,$$

$$\zeta_K(s) = \sum_{j=-1}^{\infty} r_j(K)(s-1)^j ,$$

$$L(s,\mathcal{A}) = \sum_{j=-m(\mathcal{A})}^{\infty} r_j(\mathcal{A})(s-1)^j \quad \text{if } m(\mathcal{A}) \geq 1$$

(see Section 2 for the meaning of $L(s,\mathcal{A})$ and $m(\mathcal{A})$).

2. The axioms

Definition 1. Let K be an algebraic number field, called the *base field*. The class $\mathcal{L}_K$ of *general L-functions over K* is defined by the following axioms (A1)–(A4).

(A1) (Euler product with Ramanujan–Petersson condition). There exist a positive integer $M = M(\mathcal{A})$ and a sequence $\mathcal{A} = (A_\mathfrak{p})_{\mathfrak{p}\in\mathcal{P}_K}$ of complex square matrices of order M and with monic characteristic polynomial $P_\mathfrak{p} = P_\mathfrak{p}^{\mathcal{A}} \in \mathbb{C}[X]$ satisfying the following properties: there exists a finite subset $\mathcal{P}_K^0 = \mathcal{P}_K^0(\mathcal{A})$ of $\mathcal{P}_K$ such that

(i) if $\mathfrak{p} \in \mathcal{P}_K \setminus \mathcal{P}_K^0$ then the eigenvalues $\chi_j(\mathfrak{p}) = \chi_j^{\mathcal{A}}(\mathfrak{p})$ satisfy $|\chi_j(\mathfrak{p})| = 1$, $j = 1,\ldots,M$,

(ii) if $\mathfrak{p} \in \mathcal{P}_K^0$ then the eigenvalues $\chi_j(\mathfrak{p}) = \chi_j^{\mathcal{A}}(\mathfrak{p})$ satisfy $0 \leq |\chi_j(\mathfrak{p})| \leq 1$, $j = 1,\ldots,M$.

We assume $\mathcal{P}_K^0$ to be minimal, i.e. $|\chi_j(\mathfrak{p})| < 1$ for some j for every $\mathfrak{p} \in \mathcal{P}_K^0$. The prime ideals of $\mathcal{P}_K^0$ are called *exceptional primes* of $\mathcal{A}$.

We write $P_\mathfrak{p}(X) = X^M + e_{M-1}X^{M-1} + \ldots + e_0$, $e_j = e_j^{\mathcal{A}}(\mathfrak{p})$.

The *general L-function* $L(s,\mathcal{A})$ is defined, in $\sigma > 1$, by

$$L(s,\mathcal{A}) = \prod_{\mathfrak{p}} \prod_{j=1}^{M} (1 - \chi_j(\mathfrak{p}) N\mathfrak{p}^{-s})^{-1} . \tag{2.1}$$

(A2) (Meromorphic continuation). The function $L(s,\mathcal{A})$ has an analytic continuation over $\mathbb{C}$ except possibly at $s=1$, where it has a pole of order $m(\mathcal{A})\ge 0$.

(A3) (Growth condition). There exists an absolute constant $0<\delta<1/2$ such that for every $\varepsilon>0$

$$L(s,\mathcal{A}) \ll_{\mathcal{A},\varepsilon} \exp(\exp(\varepsilon|t|)), \qquad |t|\to\infty\,,$$

uniformly in $-\delta\le\sigma\le 1+\delta$.

(A4) (Functional equation). There exists a sequence $\mathcal{A}^*=(A^*_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{P}_K}$ of complex square matrices, satisfying the same kind of properties as $\mathcal{A}$ in (A1), such that $L(s,\mathcal{A}^*)$ satisfies conditions of the same kind as $L(s,\mathcal{A})$ in (A2) and (A3). Moreover, there exist $N=N(\mathcal{A})\in\mathbb{N}$, $Q_{\mathcal{A}}>0$, $\alpha_i=\alpha_i(\mathcal{A})\in\mathbb{R}^+$, $i=1,\ldots,N$, $\beta_i=\beta_i(\mathcal{A})\in\mathbb{C}$, $i=1,\ldots,N$, $w_{\mathcal{A}}\in\mathbb{C}$ such that

$$\Phi(s,\mathcal{A})=w_{\mathcal{A}}\Phi(1-s,\mathcal{A}^*)\,,$$

where

$$\Phi(s,\mathcal{A})=Q^s_{\mathcal{A}}\prod_{i=1}^{N}\Gamma(\alpha_i s+\beta_i)L(s,\mathcal{A})\,,$$

$$\Phi(s,\mathcal{A}^*)=Q^s_{\mathcal{A}}\prod_{i=1}^{N}\Gamma(\alpha_i s+\beta_i)L(s,\mathcal{A}^*)\,.$$

We will denote the quantities introduced in (A1)–(A3) adding $*$ when referred to $L(s,\mathcal{A}^*)$. We write $\Delta(s)=\prod_{i=1}^N\Gamma(\alpha_i s+\beta_i)$ and $A=A(\mathcal{A})=\sum_{i=1}^N\alpha_i$.

Definition 2. Let $L(s,\mathcal{A})\in\mathcal{L}_K$. The degree n of the base field K and the quantities $M, M^*, N, w_{\mathcal{A}}, \alpha_i, \beta_i, i=1,\ldots,N$, are called the *parameters* of $\mathcal{A}$ (or of $L(s,\mathcal{A})$). The quantity $Q_{\mathcal{A}}$ is called the *main parameter* of $\mathcal{A}$ (or of $L(s,\mathcal{A})$). We will denote by Q_K the main parameter of $\zeta_K(s)$. Note that the parameters of $\zeta_K(s)$ are essentially reduced to n.

We will need the following fifth axiom:

(A5) (Tensor product). Let $L(s,\mathcal{A})\in\mathcal{L}_K$ and define, in $\sigma>1$,

$$L(s,\mathcal{A}\otimes\mathcal{A})=\prod_{\mathfrak{p}}\prod_{i,j=1}^{m(\mathcal{A})}(1-\chi_i(\mathfrak{p})\chi_j(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}\,,$$

$$L(s,\mathcal{A}\otimes\overline{\mathcal{A}})=\prod_{\mathfrak{p}}\prod_{i,j=1}^{m(\mathcal{A})}(1-\chi_i(\mathfrak{p})\overline{\chi_j(\mathfrak{p})}N\mathfrak{p}^{-s})^{-1}\,.$$

Then

(i) $L(s,\mathcal{A}\otimes\mathcal{A})=P(s,\mathcal{A}\otimes\mathcal{A})L(s,\widetilde{\mathcal{A}\otimes\mathcal{A}})$,

(ii) $L(s,\mathcal{A}\otimes\overline{\mathcal{A}})=P(s,\mathcal{A}\otimes\overline{\mathcal{A}})L(s,\widetilde{\mathcal{A}\otimes\overline{\mathcal{A}}})$,

where $L(s,\widetilde{\mathcal{A}\otimes\mathcal{A}})$ and $L(s,\widetilde{\mathcal{A}\otimes\overline{\mathcal{A}}})$ belong to $\mathcal{L}_K$ and

$$P(s,\mathcal{A}\otimes\mathcal{A})=\prod_{\mathfrak{p}\in\mathcal{P}}\prod_{j=1}^{M(\mathcal{A})^2}(1-\lambda_j(\mathfrak{p})N\mathfrak{p}^{-s}),$$

and similarly for $P(s,\mathcal{A}\otimes\overline{\mathcal{A}})$, where $\mathcal{P}=\mathcal{P}_K(\mathcal{A}\otimes\mathcal{A})$ (resp. $\overline{\mathcal{P}}=\mathcal{P}_K(\mathcal{A}\otimes\overline{\mathcal{A}})$) is a suitable finite, possibly empty, subset of $\mathcal{P}_K$ and $\lambda_j(\mathfrak{p})=\lambda_j(\mathfrak{p},\mathcal{A}\otimes\mathcal{A})$ (resp. $\lambda_j(\mathfrak{p},\mathcal{A}\otimes\overline{\mathcal{A}})$) are suitable complex numbers satisfying $0\le|\lambda_j(\mathfrak{p})|\le 1$. Whenever we refer to the parameters and main parameter of $\mathcal{A}\otimes\mathcal{A}$ or $L(s,\mathcal{A}\otimes\mathcal{A})$ (resp. $\mathcal{A}\otimes\overline{\mathcal{A}}$ or $L(s,\mathcal{A}\otimes\overline{\mathcal{A}})$) we will always mean the parameters and main parameter of the general L-function $L(s,\widetilde{\mathcal{A}\otimes\mathcal{A}})$ (resp. $L(s,\widetilde{\mathcal{A}\otimes\overline{\mathcal{A}}})$), which will be denoted, by abuse of notation, by $\mathcal{A}\otimes\mathcal{A}$ (resp. $\mathcal{A}\otimes\overline{\mathcal{A}}$) instead of $\widetilde{\mathcal{A}\otimes\mathcal{A}}$ (resp. $\widetilde{\mathcal{A}\otimes\overline{\mathcal{A}}}$). Moreover, we assume that

$$\sum_{\mathfrak{p}\in\mathcal{P}}\log N\mathfrak{p}\ll\log(Q_{\mathcal{A}\otimes\mathcal{A}}+2)\,,\tag{2.2}$$

$$\sum_{\mathfrak{p}\in\overline{\mathcal{P}}}\log N\mathfrak{p}\ll\log(Q_{\mathcal{A}\otimes\bar{\mathcal{A}}}+2)\,,\tag{2.3}$$

where the $\ll$-symbol may depend on the parameters of $\mathcal{A}\otimes\mathcal{A}$ (resp. $\mathcal{A}\otimes\overline{\mathcal{A}}$) but not on its main parameter. We will use the notation

$$Q=\max(Q_{\mathcal{A}},Q_{\mathcal{A}\otimes\mathcal{A}},Q_{\mathcal{A}\otimes\bar{\mathcal{A}}},Q_K)$$

unless explicitly stated.

We need some more definitions.

Definition 3. Let $L(s,\mathcal{A})\in\mathcal{L}_K$ satisfy (A5). We say that $\mathcal{A}$ (or $L(s,\mathcal{A})$) is *irreducible over* K if $L(s,\mathcal{A}\otimes\overline{\mathcal{A}})$ has a simple pole at $s=1$.

Definition 4. Let $L(s,\mathcal{A})\in\mathcal{L}_K$ be irreducible over K. We say that

(i) $\mathcal{A}$ (or $L(s,\mathcal{A})$) is *complex* if $L(s,\mathcal{A}\otimes\mathcal{A})$ is holomorphic at $s=1$,
(ii) $\mathcal{A}$ (or $L(s,\mathcal{A})$) is *real* if $L(s,\mathcal{A}\otimes\mathcal{A})$ has a simple pole at $s=1$,
(iii) $\mathcal{A}$ (or $L(s,\mathcal{A})$) is *totally real* if $P_{\mathfrak{p}}\in\mathbb{R}[X]$ for every $\mathfrak{p}\in P_K$.

By (2.1) we can write $L(s,\mathcal{A})$ and $-\frac{L'}{L}(s,\mathcal{A})$ as Dirichlet series over K, i.e. in the form

$$\sum_{\mathfrak{a}\in\mathcal{I}_K}c(\mathfrak{a})N\mathfrak{a}^{-s}\,.$$

We will use the following notation:

$$L(s,\mathcal{A})=\sum_{\mathfrak{a}\in\mathcal{I}_K}c(\mathfrak{a},\mathcal{A})N\mathfrak{a}^{-s}\qquad(\sigma>1)\,,$$

$$-\frac{L'}{L}(s,\mathcal{A}) = \sum_{\mathfrak{a}\in\mathcal{I}_K} \Lambda(\mathfrak{a},\mathcal{A})N\mathfrak{a}^{-s} \qquad (\sigma > 1)$$

and similarly for the other functions described above. We can also write $L(s,\mathcal{A})$ and $-\frac{L'}{L}(s,\mathcal{A})$ over $\mathbb{Q}$, i.e. as ordinary Dirichlet series (see also Remark 1(ii)), in which case we use the notation

$$L(s,\mathcal{A}) = \sum_{m=1}^{\infty} c(m,\mathcal{A})m^{-s} \qquad (\sigma > 1)$$

and similarly for the other functions. The above notations cause no confusion if $L(s,\mathcal{A}) \in \mathcal{L}_{\mathbb{Q}}$.

Definition 5. Let $L(s,\mathcal{A}) \in \mathcal{L}_K$. We say that $\mathcal{A}$ (or $L(s,\mathcal{A})$) is *positive* if $\Lambda(\mathfrak{a},\mathcal{A}) \geq 0$ for every $\mathfrak{a} \in \mathcal{I}_K$.

Remark 1 (Comments on the axioms and definitions).

(i) Given $L(s,\mathcal{A}) \in \mathcal{L}_K$, one may consider any sequence of complex square matrices $\mathcal{A}' = (A'_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{P}_K}$ where $A'_{\mathfrak{p}}$ has the same eigenvalues as $A_{\mathfrak{p}}$. Clearly $\mathcal{A}'$ gives rise to the same general L-function. Hence $\mathcal{L}_K$ may be regarded as the quotient of the "good sequences of complex matrices" modulo the relation "the $\mathfrak{p}$th matrices have the same eigenvalues". We have chosen the matrix approach since it is in some sense operational, as (A5) shows. Moreover, given $L(s,\mathcal{A}), L(s,\mathcal{B}) \in \mathcal{L}_K$, we write

$$L(s,\mathcal{A}+\mathcal{B}) = L(s,\mathcal{A})L(s,\mathcal{B}), \tag{2.4}$$

$$L(s,\overline{\mathcal{A}}) = \prod_{\mathfrak{p}} \prod_{j=1}^{M(\mathcal{A})} (1 - \overline{\chi_j(\mathfrak{p})}N\mathfrak{p}^{-s})^{-1}. \tag{2.5}$$

By Lemma 1 below we have $L(s,\mathcal{A}+\mathcal{B}), L(s,\overline{\mathcal{A}}) \in \mathcal{L}_K$, so that $\mathcal{L}_K$ is closed under the "addition" and "conjugation" operations. In Section 7 we will briefly discuss some problems connected with the "tensor product" operation, defined in general as

$$L(s,\mathcal{A}\otimes\mathcal{B}) = \prod_{\mathfrak{p}} \prod_{i=1}^{M(\mathcal{A})} \prod_{j=1}^{M(\mathcal{B})} (1 - \chi_i(\mathfrak{p})\psi_j(\mathfrak{p})N\mathfrak{p}^{-s})^{-1},$$

where $\chi_i(\mathfrak{p})$ are the eigenvalues of $A_{\mathfrak{p}}$ and $\psi_j(\mathfrak{p})$ are the eigenvalues of $B_{\mathfrak{p}}$.

(ii) It is well known from the decomposition law in algebraic number fields that, writing $(p) = \prod_{i=1}^{s} \mathfrak{p}_i^{e_i}$ and $N\mathfrak{p}_i = p^{f_i}$ one has $\sum_{i=1}^{s} e_i f_i = n$. Moreover, as p runs over the prime numbers, only a finite number of $\mathfrak{p}_i$ have $e_i > 1$. Since

$$(1 - \chi X^k) = \prod_{j=1}^{k} (1 - \chi_j X),$$

where $\chi \neq 0$ and χ_j are the k-roots of χ, given $L(s,\mathcal{A}) \in \mathcal{L}_K$ we may write, in $\sigma > 1$,

$$L(s,\mathcal{A}) = \prod_p \prod_{j=1}^{nM} (1 - \chi_j(p)p^{-s})^{-1}, \tag{2.6}$$

where the $\chi_j(p)$ are obtained from the $\chi_j(\mathfrak{p})$ by taking suitable k-roots with $k \leq n$ and satisfy $|\chi_j(p)| = 1$, $j = 1, \ldots, nM$ if $p \in \mathcal{P}_{\mathbb{Q}} \setminus \mathcal{P}_{\mathbb{Q}}^0$, $|\chi_j(p)| \leq 1$, $j = 1, \ldots, nM$ if $p \in \mathcal{P}_{\mathbb{Q}}^0$, with a suitable finite subset $\mathcal{P}_{\mathbb{Q}}^0$ of $\mathcal{P}_{\mathbb{Q}}$. It is not difficult to describe the $\chi_j(p)$'s and $\mathcal{P}_{\mathbb{Q}}^0$ in terms of the $\chi_j(\mathfrak{p})$'s, $\mathcal{P}_K^0$, n, M and the discriminant of K. It follows from (2.6) that any $L(s,\mathcal{A}) \in \mathcal{L}_K$ can be viewed as a general L-function over $\mathbb{Q}$. The above observation holds true, with suitable modifications, for any subfield H of K, hence *if* $L(s,\mathcal{A}) \in \mathcal{L}_K$ *then* $L(s,\mathcal{A}) \in \mathcal{L}_H$ *for every subfield* H *of* K.

(iii) We give a brief heuristical discussion of the concept of irreducibility. Call two general L-functions $L(s,\mathcal{A})$ and $L(s,\mathcal{B})$ *compatible* if $L(s,\mathcal{A} \otimes \overline{\mathcal{B}})$ defined above has good analytic properties, i.e. it satisfies properties similar to those of $L(s,\mathcal{A} \otimes \overline{\mathcal{A}})$ stated in (A5). Then, heuristically, an irreducible general L-function is not the product of pairwise compatible L-functions. Indeed, if

$$L(s,\mathcal{A}) = \prod_{i=1}^{r} L(s,\mathcal{A}_i)$$

then

$$L(s,\mathcal{A} \otimes \overline{\mathcal{A}}) = \prod_{i=1}^{r} L(s,\mathcal{A}_i \otimes \overline{\mathcal{A}}_i) \prod_{\substack{i,j=1 \\ i \neq j}}^{r} L(s,\mathcal{A}_i \otimes \overline{\mathcal{A}}_j).$$

Hence $L(s,\mathcal{A} \otimes \overline{\mathcal{A}})$ has a pole at $s = 1$ of order $\geq r > 1$ since $L(1,\mathcal{A}_i \otimes \overline{\mathcal{A}}_j) \neq 0$ and $L(s,\mathcal{A}_i \otimes \overline{\mathcal{A}}_i)$ are positive, and this contrasts with the definition of irreducibility.

We remark that *the irreducibility condition depends on the base field*. A typical example is the Dedekind zeta function of a quadratic field K: $\zeta_K(s)$ is primitive over K but not over $\mathbb{Q}$, as is easy to check. This is the motivation for stating the axioms over general algebraic number fields.

(iv) The axiom (A5) is clearly inspired by the Rankin–Selberg convolution in the theory of modular forms. At the present state of knowledge it appears that some assumptions of the type (A5) are needed in order to obtain zero-free regions in the general case, for instance when $M \geq 2$. From the theory of automorphic forms one sees that a perhaps more natural object would be the symmetric product rather than the tensor product. The introduction of the finite Euler products $P(s,\mathcal{A} \otimes \mathcal{A})$ and $P(s,\mathcal{A} \otimes \overline{\mathcal{A}})$, and the assumptions on them, are connected with facts like: if χ is a primitive

Dirichlet character then χ^2 is not necessarily primitive. The estimates (2.2) and (2.3) are consistent with the classical case.

(v) Definition 4 is inspired by the classical subdivision into real and complex characters. Since, by Lemma 1 below, $L(s,\mathcal{A}\otimes\overline{\mathcal{A}}) = L(s,\mathcal{A}\otimes\mathcal{A})$ if $\mathcal{A}$ is totally real, it is clear that $\mathcal{A}$ totally real implies $\mathcal{A}$ real and $c(\mathfrak{a},\mathcal{A}) \in \mathbb{R}$. Moreover, if $\mathcal{A}$ is complex there exist $\mathfrak{a} \in \mathcal{I}_K$ such that $c(\mathfrak{a},\mathcal{A}) \in \mathbb{C}\setminus\mathbb{R}$. However, if $\mathcal{A}$ is real we have no control over the nature of the Dirichlet coefficients of $L(s,\mathcal{A})$, in our general case.

(vi) A more natural definition of positivity for $L(s,\mathcal{A})$ would be $c(\mathfrak{a},\mathcal{A}) \ge 0$. Lemma 1 implies that $c(\mathfrak{a},\mathcal{A}) \ge 0$ if $\Lambda(\mathfrak{a},\mathcal{A}) \ge 0$, but the vice-versa seems not to be true if $M \ge 3$, at least in the general situation.

Remark 2 (Relaxation of the axioms). The stated axioms are not the weakest required in order to obtain results of the kind presented below. We list here some possible relaxations of the axioms. We have chosen the above axioms for sake of simplicity. We also remark that we will not use the full force of the axioms in each of our results. Every single result could be stated under a "minimal" set of assumptions.

(i) Here we present the general L-functions in a normalized form. Given $k > 0$, one could consider L-functions satisfying a functional equation of the form

$$\Phi(s,\mathcal{A}) = w_{\mathcal{A}}\Phi(k-s,\mathcal{A}^*)$$

with eigenvalues satisfying a condition of the form $|\chi_i(\mathfrak{p})| \le N\mathfrak{p}^{(k-1)/2}$. In this case the absolute convergence of the Euler product is in $\sigma > (k+1)/2$ and the possible pole is at $s = (k+1)/2$. A suitable renormalization would be needed in (A5) in the definition of $L(s,\mathcal{A}\otimes\mathcal{A})$ and $L(s,\mathcal{A}\otimes\overline{\mathcal{A}})$, and the growth condition should be assumed for $(k-1)/2-\delta \le \sigma \le (k+1)/2+\delta$.

(ii) One could consider more general functional equations, of the form

$$\Phi(s,\mathcal{A}) = w_{\mathcal{A}}\Psi(1-s,\mathcal{A}^*)$$

with

$$\Psi(s,\mathcal{A}^*) = Q^s_{\mathcal{A}^*}\prod_{i=1}^{N(\mathcal{A}^*)}\Gamma(\alpha_i(\mathcal{A}^*)s+\beta_i(\mathcal{A}^*))L(s,\mathcal{A}^*)\,.$$

Other poles at points s with $\sigma \le 1$ could be allowed. In this case problems could arise in the proof of the zero-free regions, if such poles are "close" to $s = 1$. Moreover, one could consider nonlinear Γ-factors of the form $\prod_{i=1}^{N}\Gamma(Q_i(s))$, $Q_i \in \mathbb{C}[X]$, in the functional equation. In this case one would be faced with functions of order greater than 1, and related additional problems.

(iii) One could define $L(s,\mathcal{A})$ by means of a sequence $\mathcal{A} = (A_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{P}_K}$ of complex square matrices of order $M_{\mathfrak{p}} \le M$ for every prime $\mathfrak{p}$, with eigenvalues satisfying $|\chi_i(\mathfrak{p})| \le 1$ for every $j = 1,\dots,M_{\mathfrak{p}}$ and $\mathfrak{p} \in \mathcal{P}_K$. We have chosen to use (A1) in order to give a genuine meaning of "dimension of the Euler product" to the parameter M, which would be lost in this more general case.

(iv) One can show that the set of axioms (A1)–(A4) is equivalent to the set of axioms obtained replacing ε by $\pi/2$ in (A3). The proof is based on the Phragmén–Lindelöf theorem. We have chosen to state (A3) as it is since it applies directly in order to show that $\mathcal{L}_K$ is closed under addition.

Since the Euler product (2.1) converges absolutely in $\sigma > 1$, we have $L(s, \mathcal{A}) \neq 0$ in $\sigma > 1$. The functional equation allows one to define, in the usual way, the *trivial zeros* of $L(s, \mathcal{A})$, located at poles of $\Delta(s)$, the *critical strip* $0 \leq \sigma \leq 1$, which contains all the other zeros of $L(s, \mathcal{A})$, and the *critical line* $\sigma = 1/2$. By comparing (2.1) with $\zeta_K(s)^M$ one sees that $m(\mathcal{A}) \leq M$, and from (2.6) we get

$$|c(m, \mathcal{A})| \leq d_{nM}(m)\,. \tag{2.7}$$

3. Examples. Classical examples of general L-functions are the Riemann zeta function, the Dirichlet L-functions formed with primitive characters, the Dedekind zeta function of an algebraic number field and the Hecke L-series associated with suitable characters. Other examples, when suitably normalized, are the zeta functions associated with the eigenfunction modular forms for $SL(2, \mathbb{Z})$ and its congruence subgroups, which satisfy (A1) by Deligne's proof of Ramanujan's conjecture. It is well known that all the above examples satisfy axiom (A5). More generally, suitable classes of L-functions associated with automorphic functions are indeed classes of general L-functions (see Gelbart–Shahidi [3]), although examples by N. Kurokawa cast doubts on the validity, in general, of axiom (A1)(i).

Several axiomatic treatments of L-functions exist in the literature. Here we list some of them. First we point out the recent papers of Selberg [17] and Duke–Iwaniec [2], the first dealing mainly with the distribution of values of L-functions and the second with estimates of the coefficients. Kurokawa [7] studies a very general class of Euler products. His results show that the Ramanujan–Petersson type condition is a natural one to assume in order to have analytic continuation of the Euler product. The problem of the zero-free regions is treated in great generality by Moreno [13], who develops the ideas in Chapter 2 of Deligne [1]. An axiomatic treatment of problems concerning the distribution of zeros and values of L-functions is given by Joyner [6]. We point out that his axioms essentially follow from ours, hence the results stated in his book also hold for a fairly wide subclass of general L-functions. Several properties of the coefficients of Dirichlet series with a functional equation have been studied in a well known series of papers by Chandrasekharan–Narasimhan and others; for recent results in this direction see Hafner [5]. Redmond [16] deals with explicit formulae, and Goldfeld–Viola [4] treat the problem of averages of special values of L-functions.

Further examples of general L-functions are contained in the references of Perelli [14].

4. The results. Before stating our results, we make two more remarks.

Remark 3 (Uniformity of the results). Our results will be *uniform only in the main parameter* of the general L-functions which come into play. This means that the $O, \ll$ and o symbols and the other constants may depend, unless explicitly stated, *only on the parameters* (see Definition 2) and not on the other quantities involved in the definition of a general L-function. This situation reflects the q-uniformity problem in the theory of Dirichlet L-functions. The problem of the dependence on the base field K, which gives rise to well-known problems even in the case of $\zeta_K(s)$, deserves some further comments. All the well-known quantities connected with K, such as the degree and the discriminant, may a priori be contained in all the quantities used in the definition of the general L-functions. However, in most concrete examples this is not the case. For instance, the influence of the structure of K on the Dirichlet coefficients of $L(s,\mathcal{A})$ is controlled by n. Also, N usually depends only on n, and $\alpha_i, |\beta_i|,\ i=1,\ldots,N, |w_{\mathcal{A}}|$ are often bounded by absolute constants, although applications of L-functions are known in which uniformity in the α_i's and $|\beta_i|$'s is important. The most interesting quantities connected with K are usually embodied in the main parameter, and this is the quantity which will be controlled explicitly. Hence we can assert that the dependence on K of our constants is restricted to those quantities which are involved in the parameters.

In our general case there is no a priori connection between the main parameters $Q_{\mathcal{A}}, Q_{\mathcal{A}\otimes\mathcal{A}}$ and $Q_{\mathcal{A}\otimes\bar{\mathcal{A}}}$. However, in most examples there is a polynomial dependence between the above quantities, of the form

$$(Q_{\mathcal{A}})^a \ll Q_{\mathcal{A}\otimes\mathcal{A}}, \qquad Q_{\mathcal{A}\otimes\bar{\mathcal{A}}} \ll (Q_{\mathcal{A}})^b\,,$$

where a and b are suitable constants depending at most on M.

Remark 4. When stating a result on $L(s,\mathcal{A})$ we will use the following convention.

(i) If (A5) is not used, then all the constants explicitly or implicitly contained in the statement of the result and in its proof may depend, unless explicitly stated, at most on the parameters of $\mathcal{A}$. The uniformity is only in $Q_{\mathcal{A}}$.

(ii) If (A5) is used, then such constants may depend, unless explicitly stated, at most on the parameters of $\mathcal{A}$, $\mathcal{A}\otimes\mathcal{A}$ and $\mathcal{A}\otimes\overline{\mathcal{A}}$, and the uniformity is only in $Q_{\mathcal{A}}$, $Q_{\mathcal{A}\otimes\mathcal{A}}$, $Q_{\mathcal{A}\otimes\bar{\mathcal{A}}}$ and Q_K.

All the constants are *effectively computable*.

THEOREM 1. *Let* $L(s,\mathcal{A}) \in \mathcal{L}_K$.

(i) *If* $\mathcal{A}$ *is positive and* $m(\mathcal{A})=1$, *then* $L(s,\mathcal{A}) \neq 0$ *in the region*

$$\sigma > 1 - \frac{C_1}{\log(Q+2)(|t|+2)}, \qquad t \in \mathbb{R}, \tag{4.1}$$

except for a possible simple real zero $\beta_{\mathcal{A}} < 1$; *here* $Q = Q_{\mathcal{A}}$ *and* $C_1 > 0$ *is a suitable constant.*

(ii) *If* $\mathcal{A}$ *satisfies* (A5), *is irreducible over* K, *complex and* $m(\mathcal{A}) = 0$, *then* $L(s,\mathcal{A}) \neq 0$ *in the region* (4.1), *where* $C_1 > 0$ *is a suitable constant.*

(iii) *If* $\mathcal{A}$ *satisfies* (A5), *is irreducible over* K, *real and* $m(\mathcal{A}) = 0$, *then* $L(s,\mathcal{A}) \neq 0$ *in the region* (4.1) *except for a possible simple zero* $\varrho_{\mathcal{A}} = \beta_{\mathcal{A}} + i\gamma_{\mathcal{A}}$ *satisfying*

$$\beta_{\mathcal{A}} < 1, \qquad |\gamma_{\mathcal{A}}| < \frac{20C_1}{\log(Q+2)},$$

where $C_1 > 0$ *is a suitable constant. If*, *in addition*, $\mathcal{A}$ *is totally real*, *then* $\varrho_{\mathcal{A}}$, *if it exists*, *is real.*

DEFINITION 6. The possible zero $\varrho_{\mathcal{A}}$ (resp. $\beta_{\mathcal{A}}$) is called the *Siegel zero*. In those cases in which the Siegel zero is defined we write

$$\delta(\mathcal{A}) = \begin{cases} 1 & \text{if the Siegel zero exists}, \\ 0 & \text{otherwise}. \end{cases}$$

Theorem 1 clearly reflects the zero-free region for the Dirichlet L-functions. It contains the classical zero-free region for $\zeta_K(s)$ and a zero-free region for $L(s,\mathcal{A}\otimes\overline{\mathcal{A}})$ if $\mathcal{A}$ is irreducible. Vinogradov's type zero-free regions appear to be very difficult to obtain in a general setting, although they have been found in some cases, see e.g. Sokolovskiĭ [18] and Mitsui [12].

THEOREM 2. *Let* $L(s,\mathcal{A}) \in \mathcal{L}_K$ *and*

$$R_1(T) = \left\{ s \in \mathbb{C} : |s-1| \le \frac{1}{\log(T+20)} \right\}.$$

(i) *If* $m(\mathcal{A}) = 0$ *then*

$$\frac{L^{(j)}(s,\mathcal{A})}{j!} \ll \frac{\log^{nM+j}(Q_{\mathcal{A}}+2)}{j!} + \frac{1}{2^j(Q_{\mathcal{A}}+2)}$$

uniformly for $j \ge 0$ *and* $s \in R_1(Q_{\mathcal{A}})$.

(ii) *If* $m(\mathcal{A}) = 1$, *let* $g(s) = (s-1)L(s,\mathcal{A})$. *Then*

$$\frac{g^{(j)}(s)}{j!} \ll \frac{\log^{nM+j-1}(Q_{\mathcal{A}}+2)}{2^j}$$

uniformly for $j \ge 0$ *and* $s \in R_1(Q_{\mathcal{A}})$.

From (ii) we get, in particular,

$$r_j(\mathcal{A}) \ll \frac{\log^{nM+j}(Q_{\mathcal{A}}+2)}{2^j} \tag{4.2}$$

uniformly for $j \geq -1$.

We observe that Theorem 2, when suitably specialized, gives the classical estimate

$$L(1,\chi) \ll \log q$$

in the case of Dirichlet L-functions. We also remark that the dependence on j in Theorem 2 can, if necessary, be improved.

THEOREM 3. *Let* $L(s,\mathcal{A}) \in \mathcal{L}_K$ *and*

$$R_2(T) = \left\{ s \in \mathbb{C} : 1 - \frac{C_1}{5\log(T+2)} \leq \sigma \leq 1,\ |t| \leq \frac{40C_1}{\log(T+2)} \right\},$$

where C_1 *is the constant which appears in Theorem* 1.

(i) *If* $\mathcal{A}$ *satisfies* (A5), *is irreducible over* K *and* $m(\mathcal{A}) = 0$, *then*

$$\frac{L'}{L}(s,\mathcal{A}) = \frac{\delta(\mathcal{A})}{s-\varrho_{\mathcal{A}}} + O(\log(Q+2)),$$

where $s \in R_2(Q)$.

(ii) *If* $\mathcal{A}$ *is positive and* $m(\mathcal{A}) = 1$, *let* $g(s) = (s-1)L(s,\mathcal{A})$. *Then*

$$\frac{g'}{g}(s) = \frac{\delta(\mathcal{A})}{s-\beta_{\mathcal{A}}} + O(\log(Q_{\mathcal{A}}+2)),$$

where $s \in R_2(Q_{\mathcal{A}})$.

From (ii) we get, in particular,

$$\frac{L'}{L}(s,\mathcal{A}) = -\frac{1}{s-1} + \frac{\delta(\mathcal{A})}{s-\beta_{\mathcal{A}}} + O(\log(Q_{\mathcal{A}}+2)) \tag{4.3}$$

with $s \in R_2(Q_{\mathcal{A}}) \setminus \{1\}$ and

$$\frac{r_0(\mathcal{A})}{r_{-1}(\mathcal{A})} = \frac{\delta(\mathcal{A})}{1-\beta_{\mathcal{A}}} + O(\log(Q_{\mathcal{A}}+2)). \tag{4.4}$$

We have the following

COROLLARY. *Let* $L(s,\mathcal{A}) \in \mathcal{L}_K$ *and*

$$R_3(T) = \left\{ s \in \mathbb{C} : 1 - \frac{C_1}{6\log(T+2)} \leq \sigma \leq 1,\ |t| \leq \frac{30C_1}{\log(T+2)} \right\},$$

where C_1 *is the constant which appears in Theorem* 1.

(i) *If* $\mathcal{A}$ *satisfies* (A5), *is irreducible over* K *and* $m(\mathcal{A}) = 0$, *we have*

$$\frac{L^{(j)}}{L}(s,\mathcal{A}) \ll_j \delta(\mathcal{A})\frac{\log(Q+2)}{|s-\varrho_{\mathcal{A}}|^{j-1}} + \log^j(Q+2)$$

for $s \in R_3(Q)$ *and* $j \geq 2$.

(ii) *If* $\mathcal{A}$ *is positive and* $m(\mathcal{A}) = 1$, *let* $g(s) = (s-1)L(s,\mathcal{A})$. *Then*

$$\frac{g^{(j)}}{g}(s) \ll_j \delta(\mathcal{A})\frac{\log(Q_{\mathcal{A}}+2)}{|s-\beta_{\mathcal{A}}|^{j-1}} + \log^j(Q_{\mathcal{A}}+2)$$

for $s \in R_3(Q_{\mathcal{A}})$ *and* $j \geq 2$.

Here the $\ll$ *symbols may also depend on* j.

From (ii) we obtain, in particular,

$$\frac{r_j(\mathcal{A})}{r_{-1}(\mathcal{A})} \ll \delta(\mathcal{A})\frac{\log(Q_{\mathcal{A}}+2)}{(1-\beta_{\mathcal{A}})^j} + \log^{j+1}(Q_{\mathcal{A}}+2)\,. \tag{4.5}$$

Theorem 3 allows one to save a $\log(Q+2)$ in the result which follows from a direct application of Lemma 4 in Section 5. This is often relevant in applications.

Now we state a generalization of Hecke's theorem.

THEOREM 4. *Let* $L(s,\mathcal{A}) \in \mathcal{L}_K$.

(i) *If* $\mathcal{A}$ *is positive and* $m(\mathcal{A}) = 1$, *then*

$$r_{-1}(\mathcal{A}) \gg \delta(\mathcal{A})(1-\beta_{\mathcal{A}}) + \frac{1-\delta(\mathcal{A})}{\log(Q_{\mathcal{A}}+2)}\,.$$

(ii) *If* $\mathcal{A}$ *satisfies* (A5), *is irreducible over* K *and* $m(\mathcal{A}) = 0$, *then if* $\delta(\mathcal{A}) = 1$ *we have*

$$|L(1+i\gamma_{\mathcal{A}},\mathcal{A})| \gg \min\Bigg(\frac{|1-\beta_{\mathcal{A}}|}{|r_{-1}(K)r_{-1}(\mathcal{A}\otimes\overline{\mathcal{A}})|^{1/2}},\ \left|\frac{1-\beta_{\mathcal{A}}}{r_0(K)r_{-1}(\mathcal{A}\otimes\overline{\mathcal{A}})}\right|^{1/2}, \left|\frac{1-\beta_{\mathcal{A}}}{r_{-1}(K)r_0(\mathcal{A}\otimes\overline{\mathcal{A}})}\right|^{1/2}\Bigg),$$

and if $\delta(\mathcal{A}) = 0$ *we have*

$$|L(1+it,\mathcal{A})| \gg \min\Bigg(\frac{\log^{-1}(Q+2)}{|r_{-1}(K)r_{-1}(\mathcal{A}\otimes\overline{\mathcal{A}})|^{1/2}}, \frac{\log^{-1/2}(Q+2)}{|r_0(K)r_{-1}(\mathcal{A}\otimes\overline{\mathcal{A}})|^{1/2}},$$
$$\frac{\log^{-1/2}(Q+2)}{|r_{-1}(K)r_0(\mathcal{A}\otimes\overline{\mathcal{A}})|^{1/2}}, \frac{1}{|r_0(K)r_0(\mathcal{A}\otimes\overline{\mathcal{A}})|^{1/2}},$$
$$\frac{1}{|r_{-1}(K)r_1(\mathcal{A}\otimes\overline{\mathcal{A}})|^{1/2}}, \frac{1}{|r_1(K)r_{-1}(\mathcal{A}\otimes\overline{\mathcal{A}})|^{1/2}}\Bigg),$$

uniformly for $|t| \leq 20C_1/\log(Q+2)$.

From Theorem 2 and (ii) of Theorem 4 we get, in particular,

$$|L(1+i\delta(\mathcal{A})\gamma_{\mathcal{A}},\mathcal{A})| \gg \delta(\mathcal{A})\frac{|1-\beta_{\mathcal{A}}|}{\log^{nM-1}(Q+2)} + \frac{1-\delta(\mathcal{A})}{\log^{nM}(Q+2)}\,. \tag{4.6}$$

This result should be compared with Theorem 1 in Perelli–Puglisi [15]. We also observe that (4.6) gives the classical result of Hecke in the case of Dirichlet L-functions with real characters, and the estimate

$$L(1,\chi) \gg \frac{1}{\log q}$$

in the case of complex characters.

Before stating our last result we need one more assumption.

(A6) (Symmetry of the Siegel zero). Let $L(s,\mathcal{A}) \in \mathcal{L}_K$. If $L(\varrho_{\mathcal{A}},\mathcal{A}) = 0$ for some $\varrho_{\mathcal{A}} = \beta_{\mathcal{A}} + i\gamma_{\mathcal{A}}$ satisfying

$$1 - \frac{C_1}{\log(Q+2)} < \beta_{\mathcal{A}} < 1, \qquad |\gamma_{\mathcal{A}}| < \frac{20C_1}{\log(Q+2)}\,,$$

where C_1 is the constant which appears in Theorem 1, then $L(\widetilde{\varrho}_{\mathcal{A}},\mathcal{A}) = 0$ for some $\widetilde{\varrho}_{\mathcal{A}} = \widetilde{\beta}_{\mathcal{A}} + i\gamma_{\mathcal{A}}$ satisfying $0 < \widetilde{\beta}_{\mathcal{A}} < 1/\log(Q+2)$, where $Q = Q_{\mathcal{A}}$ if $\mathcal{A}$ is positive.

Assumption (A6) is a trivial consequence of the functional equation in most concrete cases. In our general case we can at most deduce that $L(1-\varrho_{\mathcal{A}},\mathcal{A}^*) = 0$, but at any rate we have no direct connection between $\mathcal{A}$ and $\mathcal{A}^*$.

THEOREM 5. *Let* $L(s,\mathcal{A}) \in \mathcal{L}_K$ *satisfy* (A6), *and let* $\varepsilon > 0$ *be any constant.*

(i) *If* $\mathcal{A}$ *is positive and* $m(\mathcal{A}) = 1$, *then*

$$r_{-1}(\mathcal{A}) \gg_\varepsilon (Q_{\mathcal{A}}+2)^{-2-\varepsilon}\,.$$

(ii) *If* $\mathcal{A}$ *satisfies* (A5), *is irreducible over* K, $m(\mathcal{A}) = 0$ *and* $L(s,\mathcal{A}\otimes\overline{\mathcal{A}}) \in \mathcal{L}_K$ *then*

$$|L(1+i\gamma_{\mathcal{A}},\mathcal{A})| \gg_\varepsilon ((Q_K+2)(Q_{\mathcal{A}}+2)^2(Q_{\mathcal{A}\otimes\overline{\mathcal{A}}}+2))^{-2-\varepsilon}\,.$$

Here the constant implicit in the $\gg$ *symbol may also depend on* ε *and is effectively computable.*

A direct application of Theorem 5 to the Dedekind zeta function of a quadratic field gives the poor estimate

$$L(1,\chi) \gg_\varepsilon \frac{1}{q^{1+\varepsilon}}\,.$$

However, the method of proof of Theorem 5 gives better estimates in many concrete cases, due to a better estimation of the left hand side of (6.41) and

(6.44) below. For instance, it is not difficult to prove that, if $[K:\mathbb{Q}]=n$, then

$$r_{-1}(K) \gg_\varepsilon (Q_K+2)^{-2(1-1/n)-\varepsilon} \quad \text{if } n\ge 3 \tag{4.7}$$

and

$$r_{-1}(K) \gg (Q_K+2)^{-1} \quad \text{if } n=2\,. \tag{4.8}$$

Hence (4.8) recovers the classical estimate

$$L(1,\chi) \gg q^{1/2}\,,$$

by an essentially analytic method. The estimates (4.7) and (4.8) should be compared with Theorem 1 of Stark [20]. The dependence on n of the $\gg$ symbol in (4.7) is not easy to make explicit. As usual, from the results of Theorem 5 one can obtain analogous lower bounds on $1-\beta_{\mathcal{A}}$. Such results should be compared with Corollary 5.2 of Lagarias–Montgomery–Odlyzko [8].

If $L(s,\mathcal{A}\otimes\overline{\mathcal{A}})$ is not a general L-function, i.e. the factor $P(s,\mathcal{A}\otimes\overline{\mathcal{A}})$ is non-trivial, the method of proof of Theorem 5 only gives

$$|L(1+i\gamma_{\mathcal{A}},\mathcal{A})| \gg (Q+2)^{C_3}\,,$$

provided $\mathcal{A}$ satisfies (A5) and (A6), is irreducible over K and $m(\mathcal{A})=0$, where $C_3>0$ is a suitable constant depending on the parameters of $\mathcal{A}\otimes\overline{\mathcal{A}}$. This is due to the fact that the estimate (2.3) only implies a poor estimate of the form

$$P(s,\mathcal{A}\otimes\overline{\mathcal{A}}) \ll (Q_{\mathcal{A}\otimes\bar{\mathcal{A}}}+2)^{c(\sigma)}$$

where $c(\sigma)$ may depend on the parameters of $\mathcal{A}\otimes\overline{\mathcal{A}}$.

Our last result concerns the structure of $L(s,\mathcal{A})$ with $\mathcal{A}$ complex. Let $\mathcal{A}$ be irreducible over K. If $\mathcal{A}$ is real it is quite possible that $m(\mathcal{A})>0$, since a positive $\mathcal{A}$ is in particular real. If $\mathcal{A}$ is complex one would expect that $m(\mathcal{A})=0$. In fact, we have

THEOREM 6. *Let $L(s,\mathcal{A})\in\mathcal{L}_K$ be irreducible. If $\mathcal{A}$ is complex then $m(\mathcal{A})=0$.*

The proof is inspired by Hadamard's proof of $\zeta(1+it)\ne 0$.

5. Some lemmas. Let $L(s,\mathcal{A})$ satisfy (A1). By the absolute convergence of (2.1) we have, in $\sigma>1$,

$$\log L(s,\mathcal{A}) = \sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{1}{mN\mathfrak{p}^{ms}}\Big(\sum_{j=1}^{M}\chi_j(\mathfrak{p})^m\Big)\,, \tag{5.1}$$

hence

$$\Lambda(\mathfrak{a},\mathcal{A}) = \begin{cases} \sum_{j=1}^{M}\chi_j(\mathfrak{p})^m\log N\mathfrak{p} & \text{if } \mathfrak{a}=\mathfrak{p}^m\,,\\ 0 & \text{otherwise}\,.\end{cases} \tag{5.2}$$

By (5.1) we have, in $\sigma > 1$,

$$L(s,\mathcal{A}) = \prod_{\mathfrak{p}} \prod_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{m} \sum_{j=1}^{M} \chi_j(\mathfrak{p})^m N\mathfrak{p}^{-ms} \right)^k , \tag{5.3}$$

and from (2.1) we get

$$c(\mathfrak{p}^m,\mathcal{A}) = \sum_{\substack{m_1+\ldots+m_M=m \\ m_j \ge 0}} \chi_1(\mathfrak{p})^{m_1} \ldots \chi_M(\mathfrak{p})^{m_M} . \tag{5.4}$$

LEMMA 1. *Let* $L(s,\mathcal{A}), L(s,\mathcal{B}) \in \mathcal{L}_K$.

(a) $L(s,\mathcal{A}+\mathcal{B}), L(s,\overline{\mathcal{A}}) \in \mathcal{L}_K$.

(b) *The following conditions are equivalent*:

(i) $P_{\mathfrak{p}} \in \mathbb{R}[X]$ *for every* $\mathfrak{p} \in \mathcal{P}_K$.

(ii) $\Lambda(\mathfrak{a},\mathcal{A}) \in \mathbb{R}$ *for every* $\mathfrak{a} \in \mathcal{I}_K$.

(iii) $c(\mathfrak{a},\mathcal{A}) \in \mathbb{R}$ *for every* $\mathfrak{a} \in \mathcal{I}_K$.

Hence if either (i), (ii) *or* (iii) *holds we have* $c(\mathfrak{a},\mathcal{A}) = c(\mathfrak{a},\overline{\mathcal{A}}) \in \mathbb{R}$, $\Lambda(\mathfrak{a},\mathcal{A}) = \Lambda(\mathfrak{a},\overline{\mathcal{A}}) \in \mathbb{R}$ *and also* $L(s,\mathcal{A}) = L(s,\overline{\mathcal{A}})$.

(c) (i) *If* $P_{\mathfrak{p}} \in \mathbb{R}[X]$ *for every* $\mathfrak{p} \in \mathcal{P}_K$ *then* $c(\mathfrak{a},\mathcal{A}\otimes\mathcal{A}) = c(\mathfrak{a},\mathcal{A}\otimes\overline{\mathcal{A}}) \in \mathbb{R}$ *and* $\Lambda(\mathfrak{a},\mathcal{A}\otimes\mathcal{A}) = \Lambda(\mathfrak{a},\mathcal{A}\otimes\overline{\mathcal{A}}) \in \mathbb{R}$ *for every* $\mathfrak{a} \in \mathcal{I}_K$. *In particular*, $L(s,\mathcal{A}\otimes\mathcal{A}) = L(s,\mathcal{A}\otimes\overline{\mathcal{A}})$.

(ii) $\Lambda(\mathfrak{a},\mathcal{A}\otimes\overline{\mathcal{A}}) \ge 0$ *for every* $\mathfrak{a} \in \mathcal{I}_K$.

(d) (i) $\Lambda(\mathfrak{a},\mathcal{A}) \ge 0$ *for every* $\mathfrak{a} \in \mathcal{I}_K$ *implies* $c(\mathfrak{a},\mathcal{A}) \ge 0$ *for every* $\mathfrak{a} \in \mathcal{I}_K$.

(ii) *If* $c(\mathfrak{a},\mathcal{A}) \ge 0$ *for every* $\mathfrak{a} \in \mathcal{I}_K$ *then* $c(m,\mathcal{A}) \ge 0$ *for every* $m \in \mathbb{N}$.

(e) *If* $L(s,\mathcal{A})$ *satisfies* (A5) *and is irreducible over* K, *then* $L(s,\overline{\mathcal{A}})$ *satisfies* (A5) *and is irreducible over* K.

P r o o f. (a) We define $(\mathcal{A}+\mathcal{B})_{\mathfrak{p}}$ to be the diagonal matrix of order $2M$ having the eigenvalues of $\mathcal{A}_{\mathfrak{p}}$ and $\mathcal{B}_{\mathfrak{p}}$ on the diagonal, and (A1) is satisfied. Axiom (A2) is clearly satisfied, and since ε can be chosen arbitrarily small (A3) is also satisfied. Axiom (A4) follows by multiplying the functional equations of $L(s,\mathcal{A})$ and $L(s,\mathcal{B})$.

Analogously, we define $\overline{\mathcal{A}}_{\mathfrak{p}}$ to be the diagonal matrix of order M having the conjugates of the eigenvalues of $\mathcal{A}_{\mathfrak{p}}$ on the diagonal, and (A1) is satisfied. In $\sigma > 1$ we have

$$L(s,\overline{\mathcal{A}}) = \overline{L(\overline{s},\mathcal{A})}, \qquad c(\mathfrak{a},\overline{\mathcal{A}}) = \overline{c(\mathfrak{a},\mathcal{A})}\,,$$

hence (A2) is satisfied. Axiom (A3) is clearly satisfied, and the functional equation of $L(s,\overline{\mathcal{A}})$ is

$$\Phi(s,\overline{\mathcal{A}}) = \overline{w}_{\mathcal{A}} \Phi(1-s,\overline{\mathcal{A}}^*)\,,$$

where $\overline{\mathcal{A}}^*$ is defined similarly to $\overline{\mathcal{A}}$ and

$$\Phi(s,\overline{\mathcal{A}}) = Q_{\mathcal{A}}^s \prod_{i=1}^{N} \Gamma(\alpha_i s + \overline{\beta}_i) L(s,\overline{\mathcal{A}}),$$

and similarly for $\Phi(s,\overline{\mathcal{A}}^*)$.

(b) By (5.2), (5.4) and definition of $P_{\mathfrak{p}}(X)$ in (A1) it is clear that $e_i, \Lambda(\mathfrak{a},\mathcal{A})$ and $c(\mathfrak{a},\mathcal{A})$ are symmetric polynomials in $\chi_1(\mathfrak{p}),\ldots,\chi_M(\mathfrak{p})$. Moreover, it is well known (see Macdonald [11]) that each such set of polynomials generates over $\mathbb{R}$ a symmetric polynomial. Hence (i)–(iii) are equivalent.

(c) If $P_{\mathfrak{p}} \in \mathbb{R}[X]$ then $\{\chi_j(\mathfrak{p})\}_{j=1,\ldots,M} = \{\overline{\chi_j(\mathfrak{p})}\}_{j=1,\ldots,M}$, hence

$$\{\chi_i(\mathfrak{p})\overline{\chi_j(\mathfrak{p})}\}_{i,j=1,\ldots,M} = \{\chi_i(\mathfrak{p})\chi_j(\mathfrak{p})\}_{i,j=1,\ldots,M}$$

and (i) follows by arguments similar to those in (b). From (5.2) we see that (ii) follows easily.

(d) We observe that (i) follows from (5.3) and (ii) is clear since $c(m,\mathcal{A})$ is a suitable linear combination of $c(\mathfrak{a},\mathcal{A})$ with positive coefficients.

(e) This is again clear since $L(s,\overline{\mathcal{A}}\otimes\overline{\mathcal{A}}) = L(s,\overline{\mathcal{A}\otimes\mathcal{A}})$ and $L(s,\overline{\mathcal{A}}\otimes\overline{\overline{\mathcal{A}}}) = L(s,\mathcal{A}\otimes\overline{\mathcal{A}})$, and hence

$$L(s,\overline{\mathcal{A}}\otimes\overline{\mathcal{A}}) = \prod_{\mathfrak{p}\in\mathcal{P}}\prod_{j=1}^{M^2}(1-\overline{\lambda_j(\mathfrak{p})}N\mathfrak{p}^{-s})L(s,\overline{\widetilde{\mathcal{A}\otimes\mathcal{A}}})$$

where, by (a), $L(s,\overline{\widetilde{\mathcal{A}\otimes\mathcal{A}}}) \in \mathcal{L}_K$. ∎

Remark 5. If $K=\mathbb{Q}$ we have that $L(s,\mathcal{A}) = L(s,\overline{\mathcal{A}})$ implies $P_{\mathfrak{p}} \in \mathbb{R}[X]$. Indeed, by the identity principle, we have $c(m,\mathcal{A}) = c(m,\overline{\mathcal{A}})$ and hence $P_{\mathfrak{p}} \in \mathbb{R}[X]$. Also, $L(s,\mathcal{A}\otimes\mathcal{A}) = L(s,\mathcal{A}\otimes\overline{\mathcal{A}})$ implies

$$\sum_{i,j=1}^{M}(\chi_i(\mathfrak{p})\chi_j(\mathfrak{p}))^m = \sum_{i,j=1}^{M}(\chi_i(\mathfrak{p})\overline{\chi_j(\mathfrak{p})})^m,$$

hence $\sum_{j=1}^{M}\chi_j(\mathfrak{p})^m \in \mathbb{R}$ and so $P_{\mathfrak{p}} \in \mathbb{R}[X]$. If $K \neq \mathbb{Q}$ the identity principle does not hold, due to the possibility of ideals with the same norm. In particular, a Dirichlet series over K which is real on the real axis does not necessarily have real coefficients. Hence, for instance, $L(s,\mathcal{A}) = L(s,\overline{\mathcal{A}})$ does not imply $c(\mathfrak{a},\mathcal{A}) \in \mathbb{R}$.

Lemma 2. *Let $L(s,\mathcal{A}) \in \mathcal{L}_K$ and $a,b \in \mathbb{R}$, $a<b$. There exists a constant $C_2 = C_2(a,b) > 0$ such that*

$$(s-1)^{m(\mathcal{A})}L(s,\mathcal{A}) \ll_{a,b} ((Q_{\mathcal{A}}+2)(|t|+2))^{C_2}$$

uniformly for $a \le \sigma \le b$ and $t \in \mathbb{R}$.

Proof. Since $(s-1)^{m(\mathcal{A})}L(s,\mathcal{A})$ is entire and $m(\mathcal{A}) \le M$, Lemma 2 follows from the Phragmén–Lindelöf theorem, using (A3), (2.7), the functional equation and well known properties of the Γ-function. ∎

Let

$$N(T,\mathcal{A}) = |\{\varrho = \beta + i\gamma : L(\varrho,\mathcal{A}) = 0,\ \beta \ge 0,\ |\gamma| \le T\}|\,.$$

LEMMA 3. *Let $L(s,\mathcal{A}) \in \mathcal{L}_K$. Then*

$$N(T,\mathcal{A}) = \frac{2A}{\pi} T\log T + \frac{2}{\pi}\Big(\sum_{i=1}^{N}\alpha_i(\log\alpha_i - 1) + \log Q_{\mathcal{A}}\Big)T + O(\log T(Q_{\mathcal{A}}+2))$$

for $T \ge 2$. Hence for $0 \le H \le T$,

$$N(T+H,\mathcal{A}) - N(T,\mathcal{A}) \ll (H+1)\log T(Q_{\mathcal{A}}+2)\,.$$

Proof. Lemma 3 follows by classical arguments (see e.g. Perelli [14], Theorem 1). ∎

LEMMA 4. *Let $L(s,\mathcal{A}) \in \mathcal{L}_K$. Then*

$$-\frac{L'}{L}(s,\mathcal{A}) = \frac{m(\mathcal{A})}{s-1} - \sum_{|t-\gamma|\le 1}\frac{1}{s-\varrho} + O(\log(Q_{\mathcal{A}}+2)(|t|+2))\,,$$

where $\varrho = \beta + i\gamma$ runs over the zeros of $L(s,\mathcal{A})$, uniformly for $-1 \le \sigma \le 2$ and $t \in \mathbb{R}$.

Proof. We use Lemma α of Titchmarsh [21], Ch. 3. Note that the constants involved in Lemmas α,β and γ are absolute. Let $T \in \mathbb{R}$ and

$$f(s) = (s-1)^{m(\mathcal{A})}L(s,\mathcal{A})\,,$$

and let $s_0 = \sigma_0 + iT$ with suitable $\sigma_0 > 1$ such that

$$\left|\frac{1}{f(s_0)}\right| \le \frac{\zeta_K(\sigma_0)^M}{|s_0-1|^{m(\mathcal{A})}} \le 2\,.$$

We choose $r = 4(\sigma_0+2)$ in Lemma α. Then by Lemma 2 we have

$$\left|\frac{f(s)}{f(s_0)}\right| \ll ((Q_{\mathcal{A}}+2)(|T|+2))^{C_2} \qquad \text{for } |s-s_0| \le r\,.$$

Lemma 4 follows from Lemma 3 and Lemma α, taking $T = t$. ∎

LEMMA 5. *Let $k \ge 1$ and $h \ge 0$ be integers. Let $a_0, a_k, \ldots, a_{k+h}$ be defined by*

$$\frac{1}{s(s+k)\ldots(s+k+h)} = \frac{a_0}{s} + \frac{a_k}{s+k} + \ldots + \frac{a_{k+h}}{s+k+h} \tag{5.5}$$

and let $P_{k,h}(X) = a_0 + a_kX^k + \ldots + a_{k+h}X^{k+h}$. Then $P_{k,h}(x) \ge 0$ for $x \in [0,1]$ and $a_0 > 0$.

Proof. Computing the residue of (5.5) at $s = 0, -k, \ldots, -(k+h)$ we get

$$P_{k,h}(x) = \frac{(k-1)!}{(k+h)!}(1-x)^{h+1}Q_{k,h}(x)\,,$$

where $Q_{k,h}(x) = \sum_{i=0}^{k-1}\binom{h+i}{i}x^i$, and Lemma 5 follows, since clearly $a_0 = \frac{1}{k(k+1)\ldots(k+h)}$. ■

We wish to thank Roberto Dvornicich for pointing out the above elegant proof of Lemma 5.

6. Proof of the results. We will denote by $c_1, c_2, \ldots$ suitable positive constants satisfying the convention of Remark 4. We will begin with c_1 in the proof of each theorem.

Proof of Theorem 1. (i) We use the classical inequality $3+4\cos\theta+\cos 2\theta \geq 0$ which gives, since $\Lambda(\mathfrak{a},\mathcal{A}) \geq 0$,

$$-3\frac{L'}{L}(\sigma,\mathcal{A}) - 4\,\mathrm{Re}\,\frac{L'}{L}(\sigma+it,\mathcal{A}) - \mathrm{Re}\,\frac{L'}{L}(\sigma+2it,\mathcal{A}) \geq 0 \tag{6.1}$$

for $\sigma > 1$. Lemma 4 and (6.1) give

$$\begin{aligned}\frac{3}{\sigma-1} &+ 4\,\mathrm{Re}\,\frac{1}{\sigma-1+it} + \mathrm{Re}\,\frac{1}{\sigma-1+2it} - 3\,\mathrm{Re}\sum_{|\mathrm{Im}\,\varrho|\leq 1}\frac{1}{\sigma-\varrho} \\ &- 4\,\mathrm{Re}\sum_{|t-\mathrm{Im}\,\varrho|\leq 1}\frac{1}{\sigma+it-\varrho} - \mathrm{Re}\sum_{|2t-\mathrm{Im}\,\varrho|\leq 1}\frac{1}{\sigma+2it-\varrho} \\ &+ O(\log(Q_{\mathcal{A}}+2)(|t|+2)) \geq 0\,,\end{aligned} \tag{6.2}$$

where ϱ runs over the zeros of $L(s,\mathcal{A})$.

Let $\varrho = \beta + i\gamma$ be a zero of $L(s,\mathcal{A})$ and

$$\sigma = 1 + \frac{c_1}{\log(Q_{\mathcal{A}}+2)(|\gamma|+2)}\,,$$

with $c_1 > 0$ to be chosen later.

If $|\gamma| \geq 1/\log(Q_{\mathcal{A}}+2)$, from (6.2) we get, choosing $t = \gamma$ and neglecting some negative terms,

$$\frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + O(\log(Q_{\mathcal{A}}+2)(|\gamma|+2)) \geq 0$$

and (i) follows in this case, by choosing c_1 suitably small.

If $0 < |\gamma| < 1/\log(Q_{\mathcal{A}}+2)$ we take into account the conjugate zero $\overline{\varrho} = \beta - i\gamma$, and from (6.2), choosing $t = \gamma$ and neglecting some negative terms, we get

$$\frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + \frac{5(\sigma-1)}{(\sigma-1)^2+\gamma^2} - \frac{7(\sigma-\beta)}{(\sigma-\beta)^2+\gamma^2} + O(\log(Q_{\mathcal{A}}+2)) \geq 0$$

and (i) follows in this case too, by choosing c_1 suitably small.

If $\gamma = 0$ we denote by $\beta \le \beta_{\mathcal{A}}$ the two real simple zeros with largest real part of $L(s, \mathcal{A})$, if they exist, with the convention that a double zero coincides with two simple zeros. Choosing $t = 0$ and neglecting some negative terms, from (6.2) we get

$$\frac{8}{\sigma - 1} - \frac{16}{\sigma - \beta} + O(\log(Q_{\mathcal{A}} + 2)) \ge 0$$

and again (i) follows in this case by choosing c_1 suitably small.

(ii) In this case we use the inequality

$$0 \le 2(1 + \eta \cos\theta)^2 = 2 + \eta^2 + 4\eta\cos\theta + \eta^2 \cos 2\theta\,.$$

Writing

$$\sum_{j=1}^{M} \chi_j(\mathfrak{p})^m = \eta(\mathfrak{p}, m)\exp(i\theta(\mathfrak{p}, m)) = \eta e^{i\theta}\,,$$

we clearly have, in $\sigma > 1$,

$$\begin{aligned}(6.3)\qquad -2\frac{\zeta_K'}{\zeta_K}(\sigma) - \frac{L'}{L}(\sigma, \mathcal{A}\otimes\overline{\mathcal{A}}) - 4\,\mathrm{Re}\,\frac{L'}{L}(\sigma + it, \mathcal{A}) - \mathrm{Re}\,\frac{L'}{L}(\sigma + 2it, \mathcal{A}\otimes\mathcal{A})\\ = \sum_{\mathfrak{p}}\sum_{m=1}^{\infty} \log N\mathfrak{p} N\mathfrak{p}^{-\sigma m}(2 + \eta^2 + 4\eta\cos(\theta - mt\log N\mathfrak{p})\\ + \eta^2\cos(2\theta - 2mt\log N\mathfrak{p})) \ge 0\,.\end{aligned}$$

Let $\varrho = \beta + i\gamma$ be a zero of $L(s, \mathcal{A})$. We observe that

$$-\frac{L'}{L}(s, \mathcal{A}\otimes\mathcal{A}) = -\frac{P'}{P}(s, \mathcal{A}\otimes\mathcal{A}) - \frac{L'}{L}(s, \widetilde{\mathcal{A}\otimes\mathcal{A}})$$

and, by the assumption in (A5), in $\sigma \ge 1/2$,

$$\begin{aligned}(6.4)\qquad -\frac{P'}{P}(s, \mathcal{A}\otimes\mathcal{A}) &\ll \sum_{\mathfrak{p}}\sum_{m=1}^{\infty} \log N\mathfrak{p} N\mathfrak{p}^{-\sigma m}\Big|\sum_{j=1}^{M^2} \lambda_j(\mathfrak{p})^m\Big|\\ &\ll \sum_{\mathfrak{p}} \log N\mathfrak{p} \ll \log(Q_{\mathcal{A}\otimes\mathcal{A}} + 2)\,.\end{aligned}$$

A similar result holds for $\mathcal{A}\otimes\overline{\mathcal{A}}$. Hence a result completely analogous to Lemma 4 holds, for $1/2 \le \sigma \le 2$, for $L(s, \mathcal{A}\otimes\mathcal{A})$ and $L(s, \mathcal{A}\otimes\overline{\mathcal{A}})$.

When $|\gamma| > 1/\log(Q + 2)$ the proof of (ii) is similar to the one of (i). When $0 \le |\gamma| \le 1/\log(Q+2)$ we see from (6.3) that there is only a pole with residue 3 at $s = 1$ to play against a pole with residue ≤ -4 at $s = \beta + i\gamma$ since $L(s, \mathcal{A} \otimes \mathcal{A})$ is holomorphic at $s = 1$, and (ii) follows as before from Lemma 4, (6.3) and (6.4).

(iii) We use again (6.3). Let $\varrho = \beta + i\gamma$ be a zero of $L(s, \mathcal{A})$. When $|\gamma| > 1/\log(Q+2)$ the proof of (iii) runs exactly as in (i). When $0 \le |\gamma| \le 1/\log(Q+2)$ the situation is a bit more complicated, due to the fact that we cannot assume the existence of the conjugate zero $\overline{\varrho} = \beta - i\gamma$. Let $0 < c_2, c_3 < 1$ to be chosen later,

$$\sigma = 1 + \frac{c_2}{\log(Q+2)} \quad \text{and} \quad \frac{c_3}{\log(Q+2)} \le |\gamma| \le \frac{1}{\log(Q+2)} .$$

By Lemma 4, (6.3) and (6.4), choosing $t = \gamma$ and neglecting some negative terms we get

$$\frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta} + \frac{\sigma - 1}{(\sigma-1)^2 + \gamma^2} + O(\log(Q+2)) \ge 0 . \tag{6.5}$$

A computation shows that (6.5) implies

$$\beta < 1 - \frac{c_4}{\log(Q+2)} \tag{6.6}$$

provided

$$0 < c_4 < \frac{c_2}{2} \frac{(c_3/c_2)^2}{1 + 3(c_3/c_2)^2} \tag{6.7}$$

and c_2 is sufficiently small.

Let now $0 \le |\gamma| \le c_3/\log(Q+2)$ and suppose there exists another zero $\varrho_{\mathcal{A}} = \beta_{\mathcal{A}} + i\gamma_{\mathcal{A}}$ of $L(s, \mathcal{A})$ such that $\beta \le \beta_{\mathcal{A}}$ and $0 \le |\gamma_{\mathcal{A}}| \le c_3/\log(Q+2)$. In this case from (6.3) with $t = \gamma$ we obtain, in the usual way,

$$\begin{aligned} \frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta} + \frac{\sigma - 1}{(\sigma-1)^2 + 4\gamma^2} \\ - \frac{4(\sigma - \beta_{\mathcal{A}})}{(\sigma - \beta_{\mathcal{A}})^2 + (\gamma - \gamma_{\mathcal{A}})^2} + O(\log(Q+2)) \ge 0 . \end{aligned} \tag{6.8}$$

Choosing

$$c_3 = 20 c_4 \quad \text{and} \quad c_3 = c_2/5 , \tag{6.9}$$

a computation shows that (6.8) implies

$$\beta < 1 - \frac{c_4}{\log(Q+2)} , \tag{6.10}$$

provided c_2 is sufficiently small. Now, (6.7) is satisfied with the choices in (6.9), hence (6.6) and (6.10) give (iii), except for $\beta_{\mathcal{A}} < 1$.

In order to show that $\beta_{\mathcal{A}} < 1$, we consider

$$H(s) = \zeta_K(s)L(s + i\gamma_{\mathcal{A}}, \mathcal{A})L(s - i\gamma_{\mathcal{A}}, \overline{\mathcal{A}})L(s, \mathcal{A} \otimes \overline{\mathcal{A}}) = \sum_{\mathfrak{a}} a(\mathfrak{a})N\mathfrak{a}^{-s}$$

in $\sigma > 1$. It is easy to see, computing the Dirichlet coefficients of $-\frac{H'}{H}(s)$, that $a(\mathfrak{a}) \ge 0$ for every $\mathfrak{a} \in \mathcal{I}_K$. If $\beta_{\mathcal{A}} = 1$ then

$$L(1 + i\gamma_{\mathcal{A}}, \mathcal{A}) = L(1 - i\gamma_{\mathcal{A}}, \overline{\mathcal{A}}) = 0\,,$$

hence $H(s)$ would be holomorphic at $s = 1$, and $H(1) \ne 0$ since $\varrho_{\mathcal{A}}$ is simple. Let $\beta_0 < 1$ be the largest real zero of $H(s)$. In $\sigma > 1$ we have

$$\log H(s) = \sum_{\mathfrak{a}} b(\mathfrak{a})N\mathfrak{a}^{-s}$$

with $b(\mathfrak{a}) \ge 0$ for every $\mathfrak{a} \in \mathcal{I}_K$, and $\log H(s)$ is holomorphic in $\sigma > \beta_0$. Hence by Landau's theorem we see that $\sum_{\mathfrak{a}} b(\mathfrak{a})N\mathfrak{a}^{-s}$ is convergent in $\sigma > \beta_0$. Thus $\log H(\sigma) \ge 0$ in $\sigma > \beta_0$, hence $H(\sigma) \ge 1$ in $\sigma > \beta_0$, which contradicts the fact that β_0 is a zero of $H(s)$. Since $H(s)$ does indeed have real zeros < 1, due to the trivial zeros of $\zeta_K(s)$, we obtain a contradiction, hence $\beta_{\mathcal{A}} < 1$.

Finally, if $\mathcal{A}$ is totally real we can exploit the existence of the conjugate zero $\overline{\varrho} = \beta - i\gamma$ as in the proof of (i), thus proving that $\varrho_{\mathcal{A}}$, if it exists, is real. ∎

Proof of Theorem 2. (i) Writing $L^{(j)}(s, \mathcal{A})$ over $\mathbb{Q}$ we see by (2.7) that, in $\sigma > 1$,

$$L^{(j)}(s, \mathcal{A}) = \sum_{m=1}^{\infty} c_j(m, \mathcal{A})m^{-s}, \qquad |c_j(m, \mathcal{A})| \le d_{nM}(m) \log^j m\,.$$

Let $z = x + iy$, $1 + z \in R_1(Q_{\mathcal{A}})$ and

$$F_j(s, z) = \zeta(s)L^{(j)}(s + z, \mathcal{A}) = \sum_{m=1}^{\infty} a_j(m, z)m^{-s}, \qquad \sigma > 1\,.$$

Clearly

$$|a_j(m, z)| \le d_{nM+1}(m) \log^j(m)m^{|x|}\,. \tag{6.11}$$

First we need an estimate for $F_j(s, z)$. By Cauchy's integral formula,

$$L^{(j)}(s + z, \mathcal{A}) = \frac{j!}{2\pi i} \int_C \frac{L(w, \mathcal{A})}{(w - (s + z))^{j+1}}\, dw\,,$$

where $C = \{w \in \mathbb{C} : |w - (s+z)| = 2\}$. Hence by Lemma 2 we get, uniformly for $-1 \le \sigma \le 3, j \ge 0$ and $1 + z \in R_1(Q_{\mathcal{A}})$,

$$F_j(s, z) \ll \frac{j!}{2^j}((Q_{\mathcal{A}} + 2)(|t| + 2))^{c_1}, \qquad |s - 1| \ge 1/4\,. \tag{6.12}$$

Now, by Perron's formula we have

$$\begin{aligned}(6.13)\qquad \sum_{m=1}^{\infty} a_j(m,z)e^{-m/X} &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F_j(s,z)\Gamma(s)X^s\,ds \\ &= L^{(j)}(1+z,\mathcal{A})\Gamma(1)X + \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} F_j(s,z)\Gamma(s)X^s\,ds\,.\end{aligned}$$

From (6.11) and well known estimates for the average order of $d_k(m)$ (see Linnik [10], Ch. 1), by partial summation we get

$$(6.14)\qquad \sum_{m=1}^{\infty} a_j(m,z)e^{-m/X} \ll X^{1+1/\log(Q_{\mathcal{A}}+2)}\log^{nM+j}(X)$$

uniformly for $j \ge 0$ and $1+z \in R_1(Q_{\mathcal{A}})$. From (6.12) we have

$$(6.15)\qquad \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} F_j(s,z)\Gamma(s)X^s\,ds \ll \frac{j!}{2^j}X^{1/2}(Q_{\mathcal{A}}+2)^{c_1}\,.$$

Hence from (6.13)–(6.15) we obtain

$$L^{(j)}(1+z,\mathcal{A}) \ll X^{1/\log(Q_{\mathcal{A}}+2)}\log^{nM+j}(X) + \frac{j!}{2^j}X^{-1/2}(Q_{\mathcal{A}}+2)^{c_1}\,,$$

and (i) follows by choosing $X = (Q_{\mathcal{A}}+2)^{2(c_1+1)}$.

(ii) Although it would be possible to prove (ii) in a way similar to (i), we give here a slight variation which is simpler in this case. Using

$$\begin{aligned}\sum_{m=1}^{\infty} c(m,\mathcal{A})e^{-m/X} &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} L(s,\mathcal{A})\Gamma(s)X^s\,ds \\ &= r_{-1}(\mathcal{A})\Gamma(1)X + \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} L(s,\mathcal{A})\Gamma(s)X^s\,ds\end{aligned}$$

we obtain, as in (i),

$$(6.16)\qquad r_{-1}(\mathcal{A}) \ll \log^{nM-1}(Q_{\mathcal{A}}+2)\,.$$

Now we apply again Perron's formula in the form

$$\begin{aligned}(6.17)\qquad \sum_{m=1}^{\infty} c(m,\mathcal{A})m^{-s}e^{-m/X} &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} L(s+w,\mathcal{A})\Gamma(w)X^w\,dw\end{aligned}$$

$$= r_{-1}(\mathcal{A})\Gamma(1-s)X^{1-s} + L(s,\mathcal{A}) + \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} L(s+w,\mathcal{A})\Gamma(w)X^w\,dw$$

where s satisfies

$$\frac{1}{4\log(Q_{\mathcal{A}}+2)} \le |s-1| \le \frac{4}{\log(Q_{\mathcal{A}}+2)}\,.$$

Choosing $X = (Q_{\mathcal{A}}+2)^{c_2}$, c_2 suitable, and arguing as in (i), from (6.16) and (6.17) we get

$$L(s,\mathcal{A}) \ll \log^{nM}(Q_{\mathcal{A}}+2) \tag{6.18}$$

provided

$$\frac{1}{4\log(Q_{\mathcal{A}}+2)} \le |s-1| \le \frac{4}{\log(Q_{\mathcal{A}}+2)}\,. \tag{6.19}$$

Let $g(s) = (s-1)L(s,\mathcal{A})$. For $j \ge 0$, we have

$$\frac{g^{(j)}(s)}{j!} = \frac{1}{2\pi i}\int_C \frac{(w-1)L(w,\mathcal{A})}{(w-s)^{j+1}}\,dw \tag{6.20}$$

where

$$s \in R_1(Q_{\mathcal{A}}) \quad \text{and} \quad C = \left\{ w \in \mathbb{C} : |w-s| = \frac{c_3}{\log(Q_{\mathcal{A}}+2)} \right\}$$

with suitable $c_3 = c_3(s) \ge 2$, such that C is contained in the region (6.19). From (6.18)–(6.20) we get

$$\frac{g^{(j)}(s)}{j!} \ll 2^{-j}\log^{nM+j-1}(Q_{\mathcal{A}}+2)$$

uniformly for $j \ge 0$ and $s \in R_1(Q_{\mathcal{A}})$. Since $g^{(j)}(s)/j! = r_{j-1}(\mathcal{A})$, (ii) follows. ∎

Proof of Theorem 3 and Corollary. We will assume that $\delta(\mathcal{A}) = 1$ in the proof of Theorem 3, otherwise the reasoning is much the same and even simpler. Moreover, we may clearly assume that $C_1 < 1/2$, where C_1 is the constant which appears in Theorem 1, so that the range in which Theorem 2 is applied below is contained in $R_1(Q)$.

(i) We make the following choice in Lemma γ of Titchmarsh [21], Ch. 3, no 9:

$$r = \frac{1}{4},\quad r' = \frac{C_1}{3\log(Q+2)},\quad s_0 = 1 + \frac{C_1}{100\log(Q+2)} + it,\quad |t| \le \frac{40C_1}{\log(Q+2)}$$

and

$$f(s) = \frac{L(s,\mathcal{A})}{s-\varrho_{\mathcal{A}}}\,.$$

By (i) of Theorem 2, in the circle $|s-s_0| \le 1/(2\log(Q+2))$ we have

$$f(s) = \sum_{j=1}^{\infty} \frac{L^{(j)}(\varrho_{\mathcal{A}},\mathcal{A})}{j!}(s-\varrho_{\mathcal{A}})^{j-1} \tag{6.21}$$
$$\ll \sum_{j=1}^{\infty}\biggl(\frac{\log^{nM+1}(Q+2)}{j!} + 2^{-j}\log^{1-j}(Q+2)\biggr)$$
$$\ll \log^{nM+1}(Q+2)\,.$$

By Lemma 2, in the region

$$\frac{1}{2\log(Q+2)} \le |s-s_0| \le \frac{1}{4}$$

we have

$$f(s) \ll (Q+2)^{c_1} \tag{6.22}$$

for suitable $c_1 > 0$. Moreover,

$$\frac{1}{f(s_0)} = (s_0-\varrho_{\mathcal{A}})\prod_{\mathfrak{p}}\prod_{j=1}^{M}(1-\chi_j(\mathfrak{p})N\mathfrak{p}^{-s_0}) \tag{6.23}$$
$$\ll \zeta\biggl(1+\frac{C_1}{100\log(Q+2)}\biggr)^{nM} \ll \log^{nM}(Q+2)\,.$$

Hence from (6.21)–(6.23) we get, in $|s-s_0| \le 1/4$,

$$\frac{f(s)}{f(s_0)} \ll (Q+2)^{c_2} \tag{6.24}$$

for suitable $c_2 > 0$. Since

$$\frac{f'}{f}(s) = \frac{L'}{L}(s,\mathcal{A}) - \frac{1}{s-\varrho_{\mathcal{A}}}\,, \tag{6.25}$$

we have

$$\frac{f'}{f}(s_0) \ll -\frac{\zeta'}{\zeta}\biggl(1+\frac{C_1}{100\log(Q+2)}\biggr) + \log(Q+2) \ll \log(Q+2)\,. \tag{6.26}$$

From the zero-free region of Theorem 1 we see that $f(s) \ne 0$ in the part $\sigma \ge \sigma_0 - 2r'$ of the circle $|s-s_0| \le r$, and (i) follows from Lemma γ and (6.24)–(6.26).

(ii) In this case we choose $f(s) = g(s)/(s-\beta_{\mathcal{A}})$ and the proof runs exactly as in case (i). ∎

Now we turn to the proof of the Corollary.

(i) Let

$$f(s) = (s-\varrho_{\mathcal{A}})^{\delta(\mathcal{A})}\frac{L'}{L}(s,\mathcal{A})\,,$$

which is holomorphic in the region (4.1). By Cauchy's integral formula, for $j \ge 1$ we have

$$f^{(j)}(s) = \frac{j!}{2\pi i} \int\limits_{C} \frac{f(w)}{(w-s)^{j+1}}\, dw, \qquad s \in R_3(Q)\,, \tag{6.27}$$

where

$$C = \left\{ w \in \mathbb{C} : |w - s| = \frac{C_1}{100 \log(Q+2)} \right\}.$$

Hence $w \in R_2(Q)$ so that by Theorem 3 we get

$$f(w) \ll \log^{1-\delta(\mathcal{A})}(Q+2)\,,$$

and hence by (6.27),

$$f^{(j)}(s) \ll \log^{j+1-\delta(\mathcal{A})}(Q+2)\,. \tag{6.28}$$

At this point one may use a recursive expression for $\frac{L^{(j)}}{L}(s,\mathcal{A})$ involving $f^{(i)}(s)$, with $i \le j-1$, and suitable powers of $s - \varrho_{\mathcal{A}}$ and $\frac{L'}{L}(s,\mathcal{A})$. An application of Theorem 3 shows that the "main terms", of order $1/(s-\varrho_{\mathcal{A}})^j$, cancel and using (6.28) we get (i).

(ii) The proof is the same as in (i). ∎

Proof of Theorem 4. (i) Suppose $\delta(\mathcal{A}) = 1$. Writing $L(s,\mathcal{A})$ over $\mathbb{Q}$, by Perron's formula, for $9/10 \le s_0 < 1$ we have

$$\begin{aligned}
\sum_{m=1}^{\infty} c(m,\mathcal{A}) m^{-s_0} e^{-m/X} &= \frac{1}{2\pi i} \int\limits_{1-i\infty}^{1+i\infty} L(s_0 + w, \mathcal{A}) \Gamma(w) X^w\, dw \\
&= r_{-1}(\mathcal{A}) \Gamma(1-s_0) X^{1-s_0} + L(s_0,\mathcal{A}) \\
&\quad + \frac{1}{2\pi i} \int\limits_{-1/2-i\infty}^{-1/2+i\infty} L(s_0 + w, \mathcal{A}) \Gamma(w) X^w\, dw\,.
\end{aligned} \tag{6.29}$$

Since, by Lemma 1(d), $c(1,\mathcal{A}) = 1$ and $c(m,\mathcal{A}) \ge 0$ for $m \in \mathbb{N}$, choosing $s_0 = \beta_{\mathcal{A}}$ we deduce from Lemma 2 that

$$\begin{aligned}
\frac{1}{2} &\le \sum_{m=1}^{\infty} c(m,\mathcal{A}) m^{-\beta_{\mathcal{A}}} e^{-m/X} \\
&= \frac{r_{-1}(\mathcal{A})}{1-\beta_{\mathcal{A}}} (1 + o(1)) X^{1-\beta_{\mathcal{A}}} + O(X^{-1/2}(Q_{\mathcal{A}}+2)^{c_1})
\end{aligned} \tag{6.30}$$

for suitable $c_1 > 0$. Now we choose $X = c_2(Q_{\mathcal{A}}+2)^{c_3}$, with suitable $c_2, c_3 > 0$, so that (6.29) reduces to

$$1 \ll \frac{r_{-1}(\mathcal{A})}{1-\beta_{\mathcal{A}}} \tag{6.31}$$

since $X^{1-\beta_{\mathcal{A}}} \ll 1$.

If $\delta(\mathcal{A})=0$ we choose

$$s_0 = 1 - \frac{C_1}{2\log(Q_{\mathcal{A}}+2)}\,,$$

where C_1 is the constant which appears in Theorem 1, in (6.29). Noting that $L(s_0,\mathcal{A})<0$, the previous argument gives in this case

$$1 \ll \frac{r_{-1}(\mathcal{A})}{\log(Q_{\mathcal{A}}+2)}\,, \tag{6.32}$$

and (i) follows from (6.31) and (6.32).

(ii) Let $\delta(\mathcal{A})=1$, $9/10 \le s_0 < 1$, $|t| \le 20C_1/\log(Q+2)$ and

$$\begin{aligned} F_t(s) &= \zeta_K(s)L(s+it,\mathcal{A})L(s-it,\overline{\mathcal{A}})L(s,\mathcal{A}\otimes\overline{\mathcal{A}}) \\ &= \sum_{\mathfrak{a}} b_t(\mathfrak{a})N\mathfrak{a}^{-s}, \quad \sigma>1\,. \end{aligned}$$

In $\sigma>1$ we have

$$-\frac{F_t'}{F_t}(s) = \sum_{\mathfrak{p}}\sum_{m=1}^{\infty} \log N\mathfrak{p}\ N\mathfrak{p}^{-sm}\Big|1+\sum_{j=1}^{M}(\chi_j(\mathfrak{p})N\mathfrak{p}^{-it})^m\Big|^2\,.$$

Hence, writing $F_t(s)$ over $\mathbb{Q}$, by Lemma 1 we get $b_t(1)=1$ and $b_t(m)\ge 0$ for $m\in\mathbb{N}$, so that as in (i) we have

$$\begin{aligned} \frac{1}{2} &\le \sum_{m=1}^{\infty} b_t(m)m^{-s_0}e^{-m/X} \\ &= R_t(X,s_0) + F_t(s_0) + \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} F_t(s_0+w)\Gamma(w)X^w\,dw\,, \end{aligned} \tag{6.33}$$

where $R_t(X,s_0)$ is the residue of $F_t(s_0+w)\Gamma(w)X^w$ at the double pole in $w=1-s_0$. We recall that, by Lemma 1(a), $L(s,\overline{\mathcal{A}})\in\mathcal{L}_K$.

A computation shows that

$$\begin{aligned} &R_t(X,s_0) \\ &= X^{1-s_0}|L(1+it,\mathcal{A})|^2 \sum_{j_1+\ldots+j_6=-1}\frac{1}{j_2!j_3!j_5!j_6!} r_{j_1}(K)\frac{L^{(j_2)}}{L}(1+it,\mathcal{A}) \\ &\quad\times \frac{L^{(j_3)}}{L}(1-it,\overline{\mathcal{A}})r_{j_4}(\mathcal{A}\otimes\overline{\mathcal{A}})\Gamma^{(j_5)}(1-s_0)\log^{j_6}X\,. \end{aligned} \tag{6.34}$$

Now we choose $t=\gamma_{\mathcal{A}}$, $s_0=\beta_{\mathcal{A}}$ and $X=c_4(Q+2)^{c_5}$, with suitable $c_4,c_5>0$. Since in (6.33) we have $\mathrm{Re}(s_0+w)>0$, we get

$$P(s_0+w,\mathcal{A}\otimes\overline{\mathcal{A}}) \ll (Q_{\mathcal{A}\otimes\bar{\mathcal{A}}}+2)^{c_6}$$

with suitable $c_6 > 0$. Hence, as in (i), we get

$$\frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} F_t(s_0 + w)\Gamma(w)X^w \, dw \ll (Q+2)^{c_7} X^{-1/2}$$

for suitable $c_7 > 0$, so that (6.33) gives

$$1 \ll |R_{\gamma_{\mathcal{A}}}(X, s_0)| \,. \tag{6.35}$$

Since $\log X \ll 1/|1-\beta_{\mathcal{A}}|$, from (6.34), Lemma 1(e), Theorem 3 and its Corollary we see that the worst cases are given by the following choices of $(j_1, \ldots, j_6)$:

$$(-1,0,0,-1,1,0),\ (-1,0,0,0,0,0),\ (0,0,0,-1,0,0)\,.$$

Accordingly, from (6.34) and (6.35) we get

$$1 \ll |L(1+i\gamma_{\mathcal{A}}, \mathcal{A})|^2 \max\left(\frac{|r_{-1}(K) r_{-1}(\mathcal{A}\otimes\overline{\mathcal{A}})|}{|1-\beta_{\mathcal{A}}|^2}, \frac{|r_{-1}(K) r_0(\mathcal{A}\otimes\overline{\mathcal{A}})|}{|1-\beta_{\mathcal{A}}|}, \frac{|r_0(K) r_{-1}(\mathcal{A}\otimes\overline{\mathcal{A}})|}{|1-\beta_{\mathcal{A}}|} \right). \tag{6.36}$$

Let now $\delta(\mathcal{A}) = 0$. Choose

$$s_0 = 1 - \frac{C_1}{2\log(Q+2)} \quad \text{and} \quad |t| \le \frac{20C_1}{\log(Q+2)}\,.$$

Let

$$\eta(\mathcal{A}) = \begin{cases} 0 & \text{if } \zeta_K(s_0) L(s_0, \mathcal{A}\otimes\overline{\mathcal{A}}) \le 0, \\ 1 & \text{if } \zeta_K(s_0) L(s_0, \mathcal{A}\otimes\overline{\mathcal{A}}) > 0. \end{cases}$$

Consider

$$G_t(s) = \zeta(s)^{\eta(\mathcal{A})} F_t(s)$$

and denote by $c_t(m)$ its Dirichlet coefficients over $\mathbb{Q}$. Clearly

$$G_t(s_0) \le 0\,. \tag{6.37}$$

Suppose $\eta(\mathcal{A}) = 1$. Then, by Perron's formula and (6.37), choosing $X = c_8(Q+2)^{c_9}$ with suitable $c_8, c_9 > 0$, as before we obtain

$$\sum_{m \le X} c_t(m) m^{-s_0} e^{-m/X} \ll |\widetilde{R}_t(X, s_0)| \tag{6.38}$$

where $\widetilde{R}_t(X, s_0)$ is the residue of $G_t(s_0 + w)\Gamma(w)X^w$ at the triple pole at $w = 1 - s_0$. A computation similar to the one leading to (6.34), Theorem 3 and its Corollary, and the relation

$$\frac{1}{1-s_0} \ll \log X \ll \frac{1}{1-s_0}$$

give in this case

$$(6.39)\qquad |\widetilde{R}_t(X,s_0)| \ll |L(1+it,\mathcal{A})|^2 \times \max(|r_{-1}(K)r_{-1}(\mathcal{A}\otimes\overline{\mathcal{A}})|\log^3(Q+2),\; |r_0(K)r_{-1}(\mathcal{A}\otimes\overline{\mathcal{A}})|\log^2(Q+2),\; |r_{-1}(K)r_0(\mathcal{A}\otimes\overline{\mathcal{A}})|\log^2(Q+2),\; |r_0(K)r_0(\mathcal{A}\otimes\overline{\mathcal{A}})|\log(Q+2),\; |r_1(K)r_{-1}(\mathcal{A}\otimes\overline{\mathcal{A}})|\log(Q+2),\; |r_{-1}(K)r_1(\mathcal{A}\otimes\overline{\mathcal{A}})|\log(Q+2)).$$

But

$$c_t(m)=\sum_{d\mid m} b_t(d),$$

hence

$$(6.40)\qquad \sum_{m\le X} c_t(m)m^{-s_0}e^{-m/X} \gg \sum_{m\le X}\sum_{d\mid m} b_t(d)\frac{1}{m} \gg \sum_{d\le X}\frac{b_t(d)}{d}\sum_{m'\le X/d}\frac{1}{m'} \gg \sum_{m\le X}\frac{1}{m} \gg \log(Q+2).$$

The result of (ii) now follows, in the case $\eta(\mathcal{A})=1$, from (6.36), (6.38)–(6.40). When $\eta(\mathcal{A})=0$ we have $G_t(s)=F_t(s)$. In this case the proof is simpler, and the result a bit stronger, since the last three terms in (6.39) do not appear. ∎

Proof of Theorem 5. (i) We may clearly suppose $\delta(\mathcal{A})=1$, otherwise Theorem 4 gives a much better result. Using the notation of Lemma 5, by Perron's formula we have

$$\sum_{m\le X} c(m,\mathcal{A})m^{-\widetilde{\beta}_{\mathcal{A}}}P_{k,h}(m/X) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{L(\widetilde{\beta}_{\mathcal{A}}+w,\mathcal{A})X^w}{w(w+k)\ldots(w+k+h)}\,dw,$$

where h and k are sufficiently large. Shifting the line of integration to $\mathrm{Re}(w)=-k+1/2$ and computing the residue of the integrand at $w=1-\widetilde{\beta}_{\mathcal{A}}$ we get, since the integrand is holomorphic at $w=0$ by (A6),

$$(6.41)\qquad \sum_{m\le X} c(m,\mathcal{A})m^{-\widetilde{\beta}_{\mathcal{A}}}P_{k,h}(m/X) = r_{-1}(\mathcal{A})X^{1-\widetilde{\beta}_{\mathcal{A}}}(1-\widetilde{\beta}_{\mathcal{A}})^{-1}\prod_{j=k}^{k+h}(1-\widetilde{\beta}_{\mathcal{A}}+j)^{-1}$$

$$+\frac{1}{2\pi i}\int_{-k+1/2-i\infty}^{-k+1/2+i\infty}\frac{L(\widetilde{\beta}_{\mathcal{A}}+w,\mathcal{A})X^w}{w(w+k)\dots(w+k+h)}\,dw\,.$$

By the functional equation, on the line $\sigma=\widetilde{\beta}_{\mathcal{A}}-k+1/2$ we have

$$L(s,\mathcal{A})\ll_k (Q_{\mathcal{A}}+2)^{1-2(\widetilde{\beta}_{\mathcal{A}}-k+1/2)}(|t|+2)^{c_1} \tag{6.42}$$

for suitable $c_1>0$, depending also on k. Choosing h sufficiently large as a function of k, from (6.41), (6.42) and Lemma 5 we get, recalling that $0<\widetilde{\beta}_{\mathcal{A}}<1/\log(Q_{\mathcal{A}}+2)$,

$$1\ll_k r_{-1}(\mathcal{A})X^{1-\widetilde{\beta}_{\mathcal{A}}}+(Q_{\mathcal{A}}+2)^{2k}X^{-k+1/2}\,. \tag{6.43}$$

Choosing $X=c_2(Q_{\mathcal{A}}+2)^{2+1/(k-1/2)}$, from (6.43) we obtain

$$r_{-1}(\mathcal{A})\gg_k (Q_{\mathcal{A}}+2)^{-2-1/(k-1/2)}$$

and (i) follows by taking k sufficiently large.

(ii) Again we suppose $\delta(\mathcal{A})=1$. Let

$$F_t(s)=\zeta_K(s)L(s+it,\mathcal{A})L(s-it,\overline{\mathcal{A}})L(s,\mathcal{A}\otimes\overline{\mathcal{A}})=\sum_{m=1}^{\infty}b_t(m)m^{-s},$$

where $t=\gamma_{\mathcal{A}}$. Applying Perron's formula and shifting the line of integration as in (i) we get

$$\begin{aligned}\sum_{m\le X}b_t(m)m^{-\widetilde{\beta}_{\mathcal{A}}}P_{k,h}(m/X)\\ =\frac{1}{2\pi i}\int_{-k+1/2-i\infty}^{-k+1/2+i\infty}\frac{F_t(\widetilde{\beta}_{\mathcal{A}}+w)X^w}{w(w+k)\dots(w+k+h)}\,dw+R_t(X,\widetilde{\beta}_{\mathcal{A}})\,,\end{aligned} \tag{6.44}$$

where $R_t(X,\widetilde{\beta}_{\mathcal{A}})$ is the residue of the integrand at the double pole at $w=1-\widetilde{\beta}_{\mathcal{A}}$. By the functional equation, on the line $\sigma=\widetilde{\beta}_{\mathcal{A}}-k+1/2$ we have

$$F_t(s)\ll_k ((Q_K+2)(Q_{\mathcal{A}}+2)^2(Q_{\mathcal{A}\otimes\overline{\mathcal{A}}}+2))^{1-2(\widetilde{\beta}_{\mathcal{A}}-k+1/2)}(|t|+2)^{c_3} \tag{6.45}$$

for suitable $c_3>0$, depending also on k. We have

$$\begin{aligned}R_t(X,\widetilde{\beta}_{\mathcal{A}})\ll_k X^{1-\widetilde{\beta}_{\mathcal{A}}}\sum_{j_1+\dots+j_6=-1}r_{j_1}(K)\frac{L^{(j_2)}(1+it,\mathcal{A})}{j_2!}\frac{L^{(j_3)}(1-it,\overline{\mathcal{A}})}{j_3!}\\ \times r_{j_4}(\mathcal{A}\otimes\overline{\mathcal{A}})\frac{\log^{j_5}X}{j_5!}\,c_{j_6}(h,k)\,,\end{aligned}$$

where $c_{j_6}(h,k)$ is connected with the expansion of $\frac{1}{w(w+k)\dots(w+k+h)}$ at $w=1-\widetilde{\beta}_{\mathcal{A}}$, and $j_6\ge0$. Hence by Theorem 2 we get

$$|R_t(X,\widetilde{\beta}_{\mathcal{A}})|\ll_{h,k}|L(1+it,\mathcal{A})|X^{1-\widetilde{\beta}_{\mathcal{A}}}\log^{c_4}(Q+2) \tag{6.46}$$

for suitable $c_4 > 0$. Now we choose

$$X = c_5((Q_K+2)(Q_{\mathcal{A}}+2)^2(Q_{\mathcal{A}\otimes\bar{\mathcal{A}}}+2))^{2+1/(k-1/2)}$$

and h sufficiently large compared with k, and the conclusion of (ii) follows as before from (6.44)–(6.46). ∎

Proof of Theorem 6. Writing in polar form

$$\sum_{j=1}^{M} \chi_j(\mathfrak{p})^m = \eta(\mathfrak{p}, m)\exp(i\theta(\mathfrak{p}, m)), \qquad -\pi < \theta(\mathfrak{p}, m) \le \pi\,,$$

as $\sigma \to 1^+$ we have

$$\begin{aligned} \operatorname{Re}\log L(\sigma,\mathcal{A}) &= \sum_{\mathfrak{p}}\sum_{m=1}^{\infty} \frac{\eta(\mathfrak{p}, m)\cos\theta(\mathfrak{p}, m)}{mN\mathfrak{p}^{m\sigma}} \\ &= \sum_{\mathfrak{p}} \eta(\mathfrak{p})\cos\theta(\mathfrak{p})N\mathfrak{p}^{-\sigma} + O(1)\,, \end{aligned} \tag{6.47}$$

where $\eta(\mathfrak{p}) = \eta(\mathfrak{p}, 1)$ and $\theta(\mathfrak{p}) = \theta(\mathfrak{p}, 1)$. Since $\eta(\mathfrak{p}) \le M$ for every $\mathfrak{p}$, define, for $\sigma > 1$,

$$R(\sigma,\mathcal{A}) = \sum_{\mathfrak{p}} \eta(\mathfrak{p})\cos\theta(\mathfrak{p})N\mathfrak{p}^{-\sigma}\,,$$

$$R_K(\sigma) = \sum_{\mathfrak{p}} N\mathfrak{p}^{-\sigma}, \qquad R(\sigma,|\mathcal{A}|) = \sum_{\mathfrak{p}} \eta(\mathfrak{p})N\mathfrak{p}^{-\sigma}\,,$$

$$R(\sigma,\mathcal{A}\otimes\mathcal{A}) = \sum_{\mathfrak{p}} \eta(\mathfrak{p})^2\cos 2\theta(\mathfrak{p})N\mathfrak{p}^{-\sigma}, \qquad R(\sigma,\mathcal{A}\otimes\overline{\mathcal{A}}) = \sum_{\mathfrak{p}} \eta(\mathfrak{p})^2 N\mathfrak{p}^{-\sigma}\,.$$

Suppose now that $m(\mathcal{A}) > 0$. By (6.47) we have

$$R(\sigma,\mathcal{A}) \sim m(\mathcal{A})\log\frac{1}{\sigma-1}, \qquad R_K(\sigma) \sim R(\sigma,\mathcal{A}\otimes\overline{\mathcal{A}}) \sim \log\frac{1}{\sigma-1} \tag{6.48}$$

as $\sigma \to 1^+$. By the Cauchy–Schwarz inequality we have

$$R(\sigma,\mathcal{A}) \le R(\sigma,|\mathcal{A}|) \le (R_K(\sigma)R(\sigma,\mathcal{A}\otimes\overline{\mathcal{A}}))^{1/2}$$

so that by (6.48) we have

$$m(\mathcal{A}) = 1 \quad \text{and} \quad R(\sigma,|\mathcal{A}|) \sim \log\frac{1}{\sigma-1} \quad \text{as } \sigma \to 1^+\,. \tag{6.49}$$

Let $0 < \alpha < \pi/4$ and $0 < \delta < 1$ be suitable numbers to be chosen later. Denote by $^{\#}$ the condition that the summation in the above five series is restricted to those $\mathfrak{p}$ for which $|\theta(\mathfrak{p})| \le \alpha$. Analogously $^{\#\#}$ will mean that $|\theta(\mathfrak{p})| > \alpha$, * will mean that $\eta(\mathfrak{p}) \ge 1-\delta$ and ** that $\eta(\mathfrak{p}) < 1-\delta$. Define

$$\lambda = \lambda(\sigma) = \frac{R(\sigma,|\mathcal{A}|)^{\#}}{R(\sigma,|\mathcal{A}|)}\,.$$

From (6.48) and (6.49) we see that given any $\varepsilon > 0$, for $\sigma > 1$ sufficiently close to 1 we have

$$R(\sigma,\mathcal{A})^{\#} \le R(\sigma,|\mathcal{A}|)^{\#} = \lambda R(\sigma,|\mathcal{A}|)\,, \tag{6.50}$$

$$R(\sigma,\mathcal{A})^{\#\#} \le R(\sigma,|\mathcal{A}|)^{\#\#}\cos\alpha = (1-\lambda)R(\sigma,|\mathcal{A}|)\cos\alpha\,, \tag{6.51}$$

$$R(\sigma,\mathcal{A}) \ge (1-\varepsilon)R(\sigma,|\mathcal{A}|)\,. \tag{6.52}$$

From (6.50)–(6.52) we obtain

$$(1-\varepsilon) \ge \lambda + (1-\lambda)\cos\alpha\,,$$

hence

$$(1-\lambda) \le \frac{\varepsilon}{1-\cos\alpha}\,. \tag{6.53}$$

From (6.48), (6.49) and (6.53) we see that for $\sigma \to 1^+$ we have

$$\begin{aligned}
R(\sigma,\mathcal{A}\otimes\mathcal{A}) \qquad\qquad & \\
= R(\sigma,\mathcal{A}\otimes\mathcal{A})^{\#} &+ R(\sigma,\mathcal{A}\otimes\mathcal{A})^{\#\#} \\
\ge R(\sigma,\mathcal{A}\otimes\mathcal{A})^{\#*} &+ R(\sigma,\mathcal{A}\otimes\mathcal{A})^{\#**} - MR(\sigma,|\mathcal{A}|)^{\#\#} \\
\ge (1-\delta)\cos 2\alpha(R(\sigma,|\mathcal{A}|)^{\#*} &- M(1-\lambda)R(\sigma,|\mathcal{A}|) \\
= (1-\delta)\cos 2\alpha(R(\sigma,|\mathcal{A}|)^{\#} &- R(\sigma,|\mathcal{A}|)^{\#**}) - M(1-\lambda)R(\sigma,|\mathcal{A}|) \\
\ge (1-\delta)\cos 2\alpha(\lambda R(\sigma,|\mathcal{A}|) &- R(\sigma,|\mathcal{A}|)^{**}) - M(1-\lambda)R(\sigma,|\mathcal{A}|) \\
\ge (1-\delta)\cos 2\alpha(\lambda(1-\varepsilon)R_K(\sigma) &- (1-\delta)R_K(\sigma)) \\
- M(1-\lambda)(1+\varepsilon)R_K(\sigma) & \\
= ((1-\delta)\cos 2\alpha(\lambda(1-\varepsilon) &- (1-\delta)) - M(1-\lambda)(1+\varepsilon))R_K(\sigma)\,.
\end{aligned} \tag{6.54}$$

Choosing for example $\lambda = 1/2$, $\alpha = 1/100$ and $\varepsilon < 10^{-6}/M$, we see from (6.54) that

$$R(\sigma,\mathcal{A}\otimes\mathcal{A}) \ge \frac{1}{100}R_K(\sigma) \qquad \text{as } \sigma \to 1^+\,,$$

a contradiction since $\mathcal{A}$ is complex. Hence $m(\mathcal{A}) = 0$. ■

7. Further problems. In the present section we list some further problems and remarks concerning general L-functions.

1) First of all we remark that a number of applications follow from our axioms and results, such as prime number theorems, explicit formulae, estimates for the average of the coefficients and so on.

2) The main defect of the present paper is the lack of the generalization of Siegel's theorem. A statement of this sort was made in Perelli–Puglisi [15], but it turns out that the proof contains a serious mistake. In fact, one could get Siegel's theorem assuming suitable properties of functions of the type

$L(s, \mathcal{A} \otimes \mathcal{B})$, when $\mathcal{A}$ and $\mathcal{B}$ are irreducible. The required properties would be in the spirit of the Aramata–Brauer theorem concerning the divisibility of $\zeta_K(s)$ by $\zeta(s)$ if K is normal over $\mathbb{Q}$, and could indeed be interpreted as an abstract and analytic statement of this sort. A proof of a Siegel-type result using only (A1)–(A5) appears to be difficult, since it would provide in particular a proof of the Siegel–Brauer theorem without using Brauer's theory.

3) Here we list some interesting problems.

(i) From Theorem 2 one gets, in particular,

$$L(1, \mathcal{A}) \ll \log^{nM}(Q_{\mathcal{A}} + 2) \tag{7.1}$$

for any entire L-function. This is, in general, the best one can prove at present, since it is not possible to exclude situations like $n = 1$ and $\mathcal{A} = \chi_1 + \ldots + \chi_M$, where χ_j are Dirichlet characters $(\bmod\, q)$. However, we did not assume the irreducibility condition in the proof of (7.1). We expect that if $\mathcal{A}$ is irreducible over K and $m(\mathcal{A}) = 0$, then

$$L(1, \mathcal{A}) \ll \log^{n}(Q + 2)\,.$$

Something similar should happen in the estimate from below, if in addition $\delta(\mathcal{A}) = 0$. In this case we expect

$$L(1, \mathcal{A}) \gg \log^{-n}(Q + 2)\,.$$

(ii) Let $\mathcal{A}$ be irreducible over K and $m(\mathcal{A}) = 0$. From Theorem 1 we get a zero-free region for $L(s, \mathcal{A})$ and $L(s, \mathcal{A} \otimes \overline{\mathcal{A}})$. It would be desirable to have a zero-free region for $L(s, \mathcal{A} \otimes \mathcal{A})$ too, assuming (A1)–(A5). We observe that one cannot expect $\mathcal{A} \otimes \mathcal{A}$ to be irreducible, as the results on the Rankin–Selberg convolution show.

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