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Bounds for the solutions of unit equations

by

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1. Introduction. Many diophantine problems can be reduced to (ordinary) unit equations and S-unit equations in two unknowns (for references, see e.g. [15], [24], [11], [16], [25]). Several effective bounds have been established for the heights of the solutions of such equations (see e.g. [24], [11], [25], [3] and the references given there). Except in [3], their proofs involved Baker's method and its p-adic analogue as well as certain quantitative results concerning independent units. The best known estimates for S-unit equations are due to Győry [13] and, for (ordinary) unit equations, to Schmidt [23], Sprindžuk [25] (with not completely explicit constants) and Győry [14] (with explicit constants). These led to a lot of applications.

The purpose of the present paper is to considerably improve (in completely explicit form) the above-mentioned estimates in terms of the cardinality of S and of the parameters involved (degree, unit rank, regulator, class number) of the ground field. To obtain these improvements we use, among other things, some recent improvements of Waldschmidt [26] and Kunrui Yu [27] concerning linear forms in logarithms, some recent estimates of Brindza [5] and Hajdu [18] for fundamental systems of S-units, some upper and lower bounds for S-regulators (cf. Lemma 3 of this paper) and an idea of Schmidt [23]. Further, in our arguments we pay a particular attention to the dependence on the parameters in question. As a consequence of our result, we derive explicit bounds for the solutions of homogeneous linear equations of three terms in S-integers of bounded S-norm. These improve some earlier estimates of Győry [13], [14].

An application of our improvements is given in [17] to decomposable form equations (including Thue equations, norm form equations and discriminant form equations) in S-integers of a number field. Some other applications will be published in two further works.

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2. Bounds for the solutions of S-unit equations. We shall use throughout this paper the following standard notation. Let $\mathbb{K}$ be an algebraic number field of degree d with regulator $R_{\mathbb{K}}$, class number $h_{\mathbb{K}}$ and unit rank r. Denote by $O_{\mathbb{K}}$ the ring of integers of $\mathbb{K}$, and by $O_{\mathbb{K}}^*$ the unit group of $O_{\mathbb{K}}$. Let S be a finite set of places on $\mathbb{K}$ containing the set of infinite places S_∞. Denote by s the cardinality of S, by t the number of finite places in S, and by P the largest of the rational primes lying below the finite places of S, with the convention that $P = 1$ if $S = S_\infty$, i.e. if $t = 0$. Further, denote by O_S the ring of S-integers, and by O_S^* the group of S-units in $\mathbb{K}$. Then $s - 1 = r + t$ is the rank of O_S^*. The case $s = 1$ being trivial, we assume throughout the paper that $s \geq 2$. We denote by R_S the S-regulator of $\mathbb{K}$ (for its definition, see Section 3). We note that for $S = S_\infty$ (i.e. $t = 0$), we have $O_S = O_{\mathbb{K}}$ and $R_S = R_{\mathbb{K}}$.

For any algebraic number α, we denote by $h(\alpha)$ the (absolute) height of α (cf. Section 3). There exists a $\delta_{\mathbb{K}} > 0$, depending only on $\mathbb{K}$, such that $d \log h(\alpha) \geq \delta_{\mathbb{K}}$ for any $\alpha \in \mathbb{K} \setminus \{0\}$ which is not a root of unity (cf. Section 3).

Throughout this paper, we use the notation $\log^* a$ for $\max\{\log a, 1\}$.

Let α, β be non-zero elements of $\mathbb{K}$ with

$$\max\{h(\alpha), h(\beta)\} \leq H \quad (H \geq e).$$

Consider the S-unit equation

$$\alpha x + \beta y = 1 \quad \text{in } x, y \in O_S^*. \tag{1}$$

When $S = S_\infty$ (i.e. $t = 0$) then (1) is an (ordinary) unit equation.

Theorem. *All solutions x, y of* (1) *satisfy*

$$\max\{h(x), h(y)\} < \exp\{c_1 P^d R_S (\log^* R_S)(\log^*(PR_S)/\log^* P) \log H\}, \tag{2}$$

where

$$c_1 = c_1(d, s, \mathbb{K}) = 3^{25}(9d^2/\delta_{\mathbb{K}})^{s+1} s^{5s+10}.$$

Further, if in particular $S = S_\infty$ (*i.e. $t = 0$*), *then the bound in* (2) *can be replaced by*

$$\exp\{c_2 R_{\mathbb{K}} (\log^* R_{\mathbb{K}}) \log H\} \tag{3}$$

where

$$c_2 = c_2(d, r, \mathbb{K}) = 3^{r+27}(r+1)^{5r+17} d^3 \delta_{\mathbb{K}}^{-(r+1)}.$$

Remark 1. It is clear that the factor $(\log^*(PR_S)/\log^* P)$ in (2) does not exceed $2 \log^* R_S$, and if $\log^* R_S \leq \log^* P$, then it is at most 2. Further, by Lemma 3 (cf. Section 3), we have

$$R_S \leq R_{\mathbb{K}} h_{\mathbb{K}} (d \log^* P)^t. \tag{4}$$

R e m a r k 2. As is known, $R_{\mathbb{K}} h_{\mathbb{K}}$ can be estimated from above in terms of d and $D_{\mathbb{K}}$, the discriminant of $\mathbb{K}$. Denote by q the number of complex places of $\mathbb{K}$, and put $\Delta = (2/\pi)^q |D_{\mathbb{K}}|^{1/2}$. If $d \geq 2$, then we have e.g. (cf. [21])

$$R_{\mathbb{K}} h_{\mathbb{K}} \leq \Delta (\log \Delta)^{d-1-q} (d-1+\log \Delta)^q/(d-1)!. \tag{5}$$

Our theorem provides a considerable improvement of earlier estimates of Kotov and Trelina [19], Győry [13], [14], Schmidt [23] and Sprindžuk [25] for S-unit equations.

For $\alpha \in \mathbb{K} \setminus \{0\}$, the ideal generated by α can be uniquely written in the form $\mathfrak{a}_1 \cdot \mathfrak{a}_2$ where the ideal $\mathfrak{a}_1$ (resp. $\mathfrak{a}_2$) is composed of prime ideals outside (resp. inside) S. Then the S-norm of α, denoted by $N_S(\alpha)$, is defined as $N(\mathfrak{a}_1)$. In the particular case $S = S_\infty$, we have $N_{S_\infty}(\alpha) = |N_{\mathbb{K}/\mathbb{Q}}(\alpha)|$. Further, $N_S(\alpha)$ is a positive integer for every $\alpha \in O_S \setminus \{0\}$.

In some applications, it is more convenient to consider the following equation instead of (1):

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \qquad \text{in } x_i \in O_S \setminus \{0\} \text{ with } N_S(x_i) \leq N \text{ for } i = 1,2,3, \tag{6}$$

where $\alpha_1,\ \alpha_2,\ \alpha_3 \in \mathbb{K} \setminus \{0\}$ with $\max_{1 \leq i \leq 3} h(\alpha_i) \leq H$ $(H \geq e)$.

Let $c_3 = c_3(d, r, \mathbb{K}) = r^{r+1}\, \delta_{\mathbb{K}}^{-(r-1)}/2$ and let $c_1 = c_1(d, s, \mathbb{K})$, $c_2 = c_2(d, r, \mathbb{K})$ denote the numbers specified in the Theorem. Then we have

COROLLARY. *For every solution* $x_1,\ x_2,\ x_3$ *of* (6) *there is an* $\varepsilon \in O_S^*$ *such that*

$$\max_{1 \leq i \leq 3} h(\varepsilon x_i) < \exp\{3c_1 c_3 P^d R_S (\log^* R_S)(\log^*(PR_S)/\log^* P) \times (R_{\mathbb{K}} + t h_{\mathbb{K}} \log^* P + \log(HN))\}. \tag{7}$$

Further, *if* $S = S_\infty$, *then the bound in* (7) *can be replaced by*

$$\exp\{3c_2 c_3 R_{\mathbb{K}} (\log^* R_{\mathbb{K}})(R_{\mathbb{K}} + \log(HN))\}.$$

Our Corollary considerably improves the earlier bounds of Győry [13], [14] concerning equation (6).

3. Bounds for S-units and S-regulators. Keeping the notations of Section 2, denote by $M_{\mathbb{K}}$ the set of places on $\mathbb{K}$. In every place v we choose a valuation $|\cdot|_v$ in the following way: if v is infinite and corresponds to an embedding $\sigma : \mathbb{K} \to \mathbb{C}$ then we put, for every $\alpha \in \mathbb{K}$,

$$|\alpha|_v = |\sigma(\alpha)|^{d_v},$$

where $d_v = 1$ or 2 according as $\sigma(\mathbb{K})$ is contained in $\mathbb{R}$ or not; if v is a finite place corresponding to the prime ideal $\mathfrak{p}$ in $\mathbb{K}$ then we put $|0|_v = 0$ and, for

$\alpha \in \mathbb{K} \setminus \{0\}$,

$$|\alpha|_v = N(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(\alpha)}.$$

The (absolute) *height* of an algebraic number α contained in $\mathbb{K}$ is defined by

$$h(\alpha) = \Big(\prod_{v \in M_{\mathbb{K}}} \max(1, |\alpha|_v)\Big)^{1/d}.$$

This height is independent of the choice of $\mathbb{K}$. If the algebraic number α is of degree n with minimal polynomial $a_0(X-\alpha_1)\ldots(X-\alpha_n) \in \mathbb{Z}[X]$ over $\mathbb{Z}$, then, by ([20], p. 54), we have

$$h(\alpha) = \Big(|a_0| \prod_{i=1}^{n} \max(1, |\alpha_i|)\Big)^{1/n}. \tag{8}$$

There is a positive constant $\delta_{\mathbb{K}}$, depending only on $\mathbb{K}$, such that for every non-zero algebraic number $\alpha \in \mathbb{K}$ which is not a root of unity we have $\log h(\alpha) \ge \delta_{\mathbb{K}}/d$ (we recall that d denotes the degree of $\mathbb{K}$). Further, if α is not an algebraic integer then (8) implies that $\log h(\alpha) \ge \log 2/d$. Hence we have $\delta_{\mathbb{K}} \le \log 2$.

It is easy to see that we can take

$$\delta_{\mathbb{K}} = \frac{\log 2}{r+1} \quad \text{for } d = 1, 2,$$

where r denotes the unit rank of $\mathbb{K}$. Further, it follows from results of Blanksby and Montgomery [2] and of Dobrowolski [7], [8] that both

$$\delta_{\mathbb{K}} = \frac{1}{53d \log 6d} \quad \text{and} \quad \delta_{\mathbb{K}} = \frac{1}{1201}\Big(\frac{\log\log d}{\log d}\Big)^3 \ (^1)$$

are appropriate choices for $d \ge 3$. For large d, the factor $1/1201$ can be replaced by a larger one (see e.g. [9]).

We recall that s denotes the cardinality of S. For $v \in S$, denote by $|\cdot|_v$ the corresponding valuation normalized as above. Let $v_1, \ldots, v_{s-1}$ be a subset of S, and let $\{\varepsilon_1, \ldots, \varepsilon_{s-1}\}$ be a fundamental system of S-units in $\mathbb{K}$. Denote by R_S the absolute value of the determinant of the matrix $(\log|\varepsilon_i|_{v_j})_{i,j=1,\ldots,s-1}$. It is easy to verify that R_S is a positive number which is independent of the choice of $v_1, \ldots, v_{s-1}$ and of the fundamental system of S-units $\{\varepsilon_1, \ldots, \varepsilon_{s-1}\}$. R_S is called the *S-regulator* of $\mathbb{K}$. If in particular $S = S_\infty$, then we have $R_S = R_{\mathbb{K}}$.

There are several quantitative results in the literature for units and S-units of small height; for references, see e.g. [24], [5] and [18]. The following lemma is in fact due to Hajdu [18]. It is an extended version of an earlier

(1) Added in proof. By a recent result of P. M. Voutier (see this issue), one can take here $1/4$ instead of $1/1201$.

theorem of Brindza [5]. For convenience of the reader, we give here a proof for Lemma 1 with a slightly better value for c_4 than in [18].

Put

$$c_4 = c_4(d,s) = ((s-1)!)^2/(2^{s-2}d^{s-1})$$

and

$$c_5 = c_5(d,s,\mathbb{K}) = c_4\left(\frac{\delta_{\mathbb{K}}}{d}\right)^{2-s}, \qquad c_6 = c_6(d,s,\mathbb{K}) = c_4 d^{s-1}\delta_{\mathbb{K}}^{-1}.$$

LEMMA 1. *There exists in* $\mathbb{K}$ *a fundamental system* $\{\varepsilon_1,\ldots,\varepsilon_{s-1}\}$ *of* S*-units with the following properties*:

(i) $$\prod_{i=1}^{s-1}\log h(\varepsilon_i) \le c_4 R_S;$$

(ii) $$\log h(\varepsilon_i) \le c_5 R_S, \quad i=1,\ldots,s-1;$$

(iii) *the absolute values of the entries of the inverse matrix of* $(\log|\varepsilon_i|_{v_j})_{i,j=1,\ldots,s-1}$ *do not exceed* c_6.

Proof. We shall combine some arguments from the proofs of [5] and [18]. For $\alpha\in\mathbb{K}\setminus\{0\}$ put

$$\mathbf{v}(\alpha) = (\log|\alpha|_{v_1},\ldots,\log|\alpha|_{v_{s-1}}).$$

The lattice Λ in $\mathbb{R}^{s-1}$ spanned by the vectors $\mathbf{v}(\eta)$ with $\eta\in O_S^*$ has determinant R_S.

The function $F:\mathbb{R}^{s-1}\to\mathbb{R}$ defined by

$$F(\mathbf{x}) = |x_1| + \ldots + |x_{s-1}|$$

for $\mathbf{x}=(x_1,\ldots,x_{s-1})\in\mathbb{R}^{s-1}$ is a symmetric convex distance function (cf. [6], Ch. IV), i.e. it is non-negative, continuous, $F(\alpha\mathbf{x})=\alpha F(\mathbf{x})$ ($\alpha\ge 0$ real) and $F(\mathbf{x}+\mathbf{y})\le F(\mathbf{x})+F(\mathbf{y})$ for $\mathbf{x}$, $\mathbf{y}\in\mathbb{R}^{s-1}$. Denote by V_F the volume of the bounded set $\{\mathbf{x}\in\mathbb{R}^{s-1}\mid F(\mathbf{x})<1\}$. It is easy to check that $V_F = 2^{s-1}/(s-1)!$. By a theorem of Minkowski (cf. [6], Ch. VIII) the successive minima $\lambda_1,\ldots,\lambda_{s-1}$ of Λ with respect to F have the property

(9) $$\lambda_1\ldots\lambda_{s-1} \le 2^{s-1}R_S/V_F = (s-1)!R_S.$$

Further, there are multiplicatively independent S-units $\eta_1,\ldots,\eta_{s-1}$ for which

(10) $$F(\mathbf{v}(\eta_i)) = \lambda_i, \quad i=1,\ldots,s-1.$$

It follows (cf. [6], p. 135, Lemma 8) that there exists a fundamental system $\{\varepsilon_1,\ldots,\varepsilon_{s-1}\}$ of S-units such that

(11) $$F(\mathbf{v}(\varepsilon_i)) \le \max\{1,i/2\}F(\mathbf{v}(\eta_i)), \quad i=1,\ldots,s-1.$$

However, for every $\eta \in O_S^*$, we have $\prod_{v\in S} |\eta|_v = 1$, hence

$$\log h(\eta) = \frac{1}{d}\sum_{v\in S}\max\{0, \log|\eta|_v\} = \frac{1}{2d}\sum_{v\in S}|\log|\eta|_v|,$$

which implies that

$$\frac{1}{2d}F(\mathbf{v}(\eta)) \le \log h(\eta) \le \frac{1}{d}F(\mathbf{v}(\eta)). \tag{12}$$

Hence, by (12), (11), (10) and (9), we have

$$\prod_{i=1}^{s-1}\log h(\varepsilon_i) \le \frac{1}{d^{s-1}}\prod_{i=1}^{s-1}F(\mathbf{v}(\varepsilon_i)) \le \frac{(s-1)!}{2^{s-2}d^{s-1}}\prod_{i=1}^{s-1}F(\mathbf{v}(\eta_i)) \le ((s-1)!)^2 R_S/(2^{s-2}d^{s-1}), \tag{13}$$

which proves (i).

(ii) follows immediately from (i) and $\log h(\varepsilon_i) \ge \delta_{\mathbb{K}}/d$ for $i = 1, \ldots, s-1$.

To prove (iii), let $E = (\log|\varepsilon_i|_{v_j})_{i,j=1,\ldots,s-1}$ and $e_{ij} = \det(E_{ij})/\det(E)$, where E_{ij} denotes the matrix obtained from E by omitting the ith row and jth column. It follows from (13) and Hadamard's inequality that

$$|\det(E_{ij})| \le \prod_{\substack{p=1\\p\ne i}}^{s-1}\sqrt{\sum_{\substack{q=1\\q\ne j}}^{s-1}(\log|\varepsilon_p|_{v_q})^2} \le \prod_{\substack{p=1\\p\ne i}}^{s-1}F(\mathbf{v}(\varepsilon_p)) \le c_4 R_S/F(\mathbf{v}(\varepsilon_i)).$$

Together with (12), $|\det(E)| = R_S$ and $\log h(\varepsilon_i) \ge \delta_{\mathbb{K}}/d$ this implies $|e_{ij}| \le c_4\delta_{\mathbb{K}}^{-1}d^{s-1}$, which completes the proof. ∎

The next lemma has various versions in the literature (for references, see e.g. [15], [24], [10], [18]). Our lemma is an explicit version of Lemma 10 of [10].

Let $c_3 = c_3(d, r, \mathbb{K})$ denote the constant specified in the Corollary.

Lemma 2. *For every $\alpha \in O_S \setminus \{0\}$ and every integer $n \ge 1$ there exists an S-unit ε such that*

$$h(\varepsilon^n\alpha) \le N_S(\alpha)^{1/d}\exp\{n(c_3R_{\mathbb{K}} + th_{\mathbb{K}}\log^* P)\}. \tag{14}$$

Proof. First consider the case when $S = S_\infty$. So let $\alpha \in O_{\mathbb{K}}\setminus\{0\}$ and put $M = |N_{\mathbb{K}/\mathbb{Q}}(\alpha)|$. Let $S_\infty = \{v_1, \ldots, v_{r+1}\}$ and $L(\alpha) = \max_{1\le i\le r}|\log|\alpha|_{v_i}|$. Then there are multiplicatively independent units $\eta_1, \ldots, \eta_r$ in $O_{\mathbb{K}}$ such that $L(\eta_1)\ldots L(\eta_r) \le R_{\mathbb{K}}$ (cf. [14]). On the other hand, we have $L(\eta_j) \ge (d/r)\log h(\eta_j) \ge \delta_{\mathbb{K}}/r$, whence $L(\eta_j) \le r^{r-1}\delta_{\mathbb{K}}^{-(r-1)}R_{\mathbb{K}}$ for each j.

Consider the system of linear equations

$$\sum_{j=1}^{r}X_j\log|\eta_j|_{v_i} = -\log(M^{-d_{v_i}/d}|\alpha|_{v_i}), \quad i = 1, \ldots, r+1,$$

in $X_1, \ldots, X_r$. It has a unique solution $x_1, \ldots, x_r$ in $\mathbb{R}$. For $1 \le j \le r$, there exist $b_j \in \mathbb{Z}$ and $\varrho_j \in \mathbb{R}$ with $|\varrho_j| \le n/2$ such that $x_j = nb_j + \varrho_j$. Putting $\eta_1^{b_1} \ldots \eta_r^{b_r} = \varepsilon$, we infer that

$$\begin{aligned}(15)\quad |\log(M^{-d_{v_i}/d}|\alpha\varepsilon^n|_{v_i})| &= \Big|\sum_{j=1}^{r} \varrho_j \log|\eta_j|_{v_i}\Big| \\ &\le \frac{nr}{2} \max_{1\le j\le r} |\log|\eta_j|_{v_i}| \le \frac{nr}{2}\cdot r \max_{1\le j\le r} L(\eta_j) \\ &\le nc_3 R_{\mathbb{K}}, \qquad i = 1, \ldots, r+1,\end{aligned}$$

which implies (14).

The general case of our lemma follows from the case $S = S_\infty$ as in the proof of Lemma 10 of [10]. ∎

Denote by $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ the prime ideals in $\mathbb{K}$ corresponding to the finite places in S. We recall that P denotes the largest of the rational primes lying below of these prime ideals.

The following lemma is an improvement of some estimates of Pethő [22] and Hajdu [18] for R_S. It should, however, be remarked that Pethő's estimate was established in a more general situation, for some S-orders instead of O_S.

LEMMA 3. *If* $t > 0$, *then*

$$R_S \le R_{\mathbb{K}} h_{\mathbb{K}} \prod_{i=1}^{t} \log N(\mathfrak{p}_i) \le R_{\mathbb{K}} h_{\mathbb{K}} (d \log^* P)^t \tag{16}$$

and

$$R_S \ge R_{\mathbb{K}} \prod_{i=1}^{t} \log N(\mathfrak{p}_i) \ge c_7 (\log 2)(\log^* P), \tag{17}$$

where $c_7 = 0.2052$.

Proof. $O_S^*/O_{\mathbb{K}}^*$ is a free abelian group of rank t which is isomorphic to the multiplicative group of principal ideals in $\mathbb{K}$ generated by the elements of O_S^*. This latter group is a subgroup of finite index, say i_S, of the multiplicative group generated by $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ and we have $i_S \le h_{\mathbb{K}}$. Hence, as is known (see e.g. [4], pp. 85 and 125), this subgroup has a basis of the form

$$(\varepsilon_i) = \mathfrak{p}_i^{a_{ii}} \mathfrak{p}_{i+1}^{a_{i,i+1}} \ldots \mathfrak{p}_t^{a_{it}}, \qquad i = 1, \ldots, t,$$

with rational integers a_{ij} such that $a_{ii} > 0$ for $i = 1, \ldots, t$ and that $a_{11} \ldots a_{tt} = i_S$. It now follows that if $\{\varepsilon_{t+1}, \ldots, \varepsilon_{t+r}\}$ is a fundamental system of units in $O_{\mathbb{K}}$ then $\{\varepsilon_1, \ldots, \varepsilon_t, \ldots, \varepsilon_{t+r}\}$ is a fundamental system of S-units in $\mathbb{K}$. Consequently, it is easy to see that

$$R_S = |\det(\log|\varepsilon_i|_{v_j})_{i,j=1,\ldots,r+t}| = R_{\mathbb{K}} i_S \prod_{i=1}^{t} \log N(\mathfrak{p}_i), \tag{18}$$

which gives (16). Inequalities (17) follow from (18) and the estimate $R_{\mathbb{K}} \ge c_7$ of Friedman [12]. ■

We remark that, in our Theorem and its Corollary, the improvements of the previous bounds in terms of $R_{\mathbb{K}}$, $h_{\mathbb{K}}$ and P are mainly due to the use of fundamental systems of S-units, S-regulators as well as Lemmas 1 to 3.

4. Estimates for linear forms in logarithms. In our proofs, we shall use the best known estimates, due to Waldschmidt [26] and Kunrui Yu [27] respectively, for linear forms in logarithms in the complex and in the p-adic case. We shall formulate them in a more convenient form for our purpose. These estimates enable us to considerably improve the previous bounds for the solutions of equation (1) in terms of d, r and s.

Let $\alpha_1, \ldots, \alpha_n$ $(n \ge 2)$ be non-zero algebraic numbers and let $\mathbb{K} = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$. Put $[\mathbb{K} : \mathbb{Q}] = d$. Let $A_1, \ldots, A_n$ be real numbers such that

$$\log A_i \ge \max\left\{\log h(\alpha_i), \frac{|\log \alpha_i|}{3.3d}, \frac{1}{d}\right\}, \qquad i = 1, \ldots, n, \tag{19}$$

where log denotes the principal value of the logarithm. Let $b_1, \ldots, b_n$ be rational integers and put $B = \max\{|b_1|, \ldots, |b_n|, 3\}$. Further, set

$$\Lambda = \alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1.$$

In Proposition 1, it will be convenient to make the following technical assumptions:

$$B \ge \log A_n \exp\{4(n+1)(7 + 3\log(n+1))\} \tag{20}$$

and

$$7 + 3\log(n+1) \ge \log d. \tag{21}$$

Proposition 1 is a consequence of Corollary 10.1 of Waldschmidt [26].

Proposition 1 (M. Waldschmidt [26]). *If* $\Lambda \ne 0$, $b_n = 1$ *and* (20), (21) *hold*, *then*

$$|\Lambda| \ge \exp\left\{-c_8(n) d^{n+2} \log A_1 \ldots \log A_n \log\left(\frac{2nB}{\log A_n}\right)\right\}, \tag{22}$$

where $c_8(n) = 1500 \cdot 38^{n+1}(n+1)^{3n+9}$.

We remark that a recent explicit estimate of Baker and Wüstholz [1] for linear forms in logarithms would give here a smaller value for $c_8(n)$ in terms of n. However, the lower bound in (22) is better in terms of A_n, which is essential for our present applications.

Proof of Proposition 1. We denote by log the principal value of the logarithm. Setting $\alpha_0 = -1$, there is a $b_0 \in \mathbb{Z}$ such that $|b_0| \le$

$|b_1| + \ldots + |b_{n-1}| + 2 \le nB$ and that

$$\log(\alpha_1^{b_1} \ldots \alpha_n^{b_n}) = \sum_{j=1}^{n} b_j \log \alpha_j + b_0 \log \alpha_0 := \Omega,$$

where $b_n = 1$. It suffices to deal with the case when $|\Lambda| \le 1/3$. Since $|\log z| \le 2|z-1|$ for any $z \in \mathbb{C}$ with $|z-1| \le 1/3$, we get

$$|\Lambda| \ge |\Omega|/2. \tag{23}$$

After some calculations and under the conditions (20), (21), Corollary 10.1 of [26] implies the following inequality with the choice $E = e$, $f = 1/(3.3d)$ and $g = 2$:

$$|\Omega| \ge 2 \exp\left\{ - c_8(n) d^{n+2} \log A_1 \ldots \log A_n \log\left(\frac{2nB}{\log A_n}\right)\right\}.$$

Together with (23) this implies (22). ∎

In Proposition 2, let $v = v_{\mathfrak{p}}$ be a finite place on $\mathbb{K}$, corresponding to the prime ideal $\mathfrak{p}$ of $\mathbb{K}$. Let p denote the rational prime lying below $\mathfrak{p}$, and denote by $|\cdot|_v$ the non-archimedean valuation normalized as in Section 3. Instead of (19), assume now that $A_1, \ldots, A_n$ are real numbers such that

$$\log A_i \ge \max\{\log h(\alpha_i), |\log \alpha_i|/(10d), \log p\}, \qquad i = 1, \ldots, n. \tag{24}$$

The following proposition is a simple consequence of the main result of Kunrui Yu [27].

PROPOSITION 2 (Kunrui Yu [27]). *Let*

$$\Phi = c_9(n)(d/\sqrt{\log p})^{2(n+1)} p^d \log A_1 \ldots \log A_n \log(10nd \log A),$$

where $c_9(n) = 22000(9.5(n+1))^{2(n+1)}$ *and* $A = \max\{A_1, \ldots, A_n, e\}$. *If* $\Lambda \ne 0$ *then*

$$|\Lambda|_v \ge \exp\{-d(\log p)\Phi \log(dB)\}.$$

Further, if $b_n = 1$ *and* $A_n \ge A_i$ *for* $i = 1, \ldots, n-1$, *then* A *can be replaced by* $\max\{A_1, \ldots, A_{n-1}, e\}$ *and for any* δ *with* $0 < \delta \le 1$, *we have*

$$|\Lambda|_v \ge \exp\{-d(\log p) \max\{\Phi \log(\delta^{-1}\Phi/\log A_n), \delta B\}\}.$$

P r o o f. This is a reformulation of the result presented in the introduction of Kunrui Yu [27]. ∎

R e m a r k 6. We remark that, in Propositions 1 and 2, the condition $\mathbb{K} = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ can be removed. It is enough to assume that $\mathbb{K}$ is an algebraic number field of degree d which contains $\alpha_1, \ldots, \alpha_n$. This observation will be needed in Section 5.

5. Proofs of the Theorem and the Corollary

Proof of the Theorem. Let x, y be an arbitrary but fixed solution of

$$\alpha x + \beta y = 1 \quad \text{in } x, y \in O_S^*. \tag{1}$$

We assume that $h(x) \ge h(y)$. Let $\{\varepsilon_1, \dots, \varepsilon_{s-1}\}$ be a fundamental system of S-units in $\mathbb{K}$ with the properties specified in Lemma 1. Then we can write

$$y = \zeta \varepsilon_1^{b_1} \dots \varepsilon_{s-1}^{b_{s-1}} \tag{25}$$

with a root of unity ζ in $\mathbb{K}$ and with rational integers $b_1, \dots, b_{s-1}$. Put $B = \max\{|b_1|, \dots, |b_{s-1}|, 3\}$ and $S = \{v_1, \dots, v_s\}$. Then (25) implies

$$\log|y|_{v_j} = \sum_{i=1}^{s-1} b_i \log|\varepsilon_i|_{v_j}, \qquad j = 1, \dots, s-1,$$

whence, by (iii) of Lemma 1 and (12), we get

$$B \le c_6 \sum_{j=1}^{s-1} |\log|y|_{v_j}| \le 2dc_6 \log h(y) \le 2dc_6 \log h(x) \tag{26}$$

with the $c_6 = c_6(d, s, \mathbb{K})$ specified in Lemma 1.

Let $v \in S$ for which $|x|_v$ is minimal. Setting $\alpha_s = \zeta\beta$ and $b_s = 1$, we deduce from (1) that

$$|\alpha x|_v = |\varepsilon_1^{b_1} \dots \varepsilon_{s-1}^{b_{s-1}} \alpha_s^{b_s} - 1|_v. \tag{27}$$

We shall derive a lower bound for $|\alpha x|_v$.

First assume that v is infinite. In order to apply Proposition 1, put

$$\begin{aligned} \log A_i &= \delta_{\mathbb{K}}^{-1} \log h(\varepsilon_i), \qquad i = 1, \dots, s-1, \\ \log A_s &= \delta_{\mathbb{K}}^{-1} \log H. \end{aligned} \tag{28}$$

It is easy to check that $7 + 3\log(s+1) \ge \log d$. Further, we may assume that

$$B \ge \log A_s \exp\{4(s+1)(7 + 3\log(s+1))\}. \tag{29}$$

Indeed, (1) implies that

$$h(x) \le 2H^2 h(y). \tag{30}$$

Further, it follows from (25) and (ii) of Lemma 1 that

$$h(y) \le \prod_{i=1}^{s-1} h(\varepsilon_i)^{|b_i|} \le \exp\{(s-1)c_5 R_S B\}. \tag{31}$$

Hence, if (29) does not hold, we get at once a bound for $h(x)$ which is better than that in the Theorem.

We have $|\cdot|_v = |\sigma(\cdot)|^{d_v}$ for some $\sigma : \mathbb{K} \to \mathbb{C}$. Applying σ to equation (1) and then omitting σ everywhere, we may assume that $|\cdot|_v = |\cdot|^{d_v}$. On applying now Proposition 1 to (27) and using (i) of Lemma 1, we derive that

$$|\alpha x|_v \geq \exp\left\{ - c_{10} R_S \log H \log\left(\frac{c_{11} B}{\log H}\right)\right\}, \tag{32}$$

where $c_{10} = d_v c_8(s) c_4 d^{s+2} \delta_{\mathbb{K}}^{-s}$ and $c_{11} = 2s\delta_{\mathbb{K}}$.

Since $|x|_v$ is minimal, we have

$$h(x) = h(1/x) \leq |x|_v^{-(s-1)/d}. \tag{33}$$

Hence it follows from (32), (26) and $|\alpha|_v \leq H^d$ that

$$\frac{\log h(x)}{\log H} \leq \frac{2(s-1)}{d} c_{10} R_S \log\left(\frac{c_{12}\log h(x)}{\log H}\right),$$

where $c_{12} = 2dc_6c_{11}$. This gives ([2])

$$h(x) \leq \exp\{c_{13} R_S (\log^* R_S) \log H\} \tag{34}$$

with

$$c_{13} = 3^{s+26} d^3 \delta_{\mathbb{K}}^{-s} s^{5s+12}.$$

We remark that in the particular case $S = S_\infty$, i.e. when $t = 0$, (34) implies the second part of the Theorem.

Next assume that v is finite. To apply Proposition 2, we put now

$$\begin{aligned} \log A_i &= \delta_{\mathbb{K}}^{-1} \log h(\varepsilon_i) + \log^* P, \qquad i = 1, \ldots, s-1, \\ \log A_s &= \delta_{\mathbb{K}}^{-1} \log H + \log^* P. \end{aligned} \tag{35}$$

Using (i) of Lemma 1, we get

$$\begin{aligned} &\log A_1 \ldots \log A_{s-1} \\ &\quad \leq \prod_{i=1}^{s-1} (\delta_{\mathbb{K}}^{-1} \log h(\varepsilon_i)) \left(\sum_{j=0}^{s-1} \binom{s-1}{j} (d\log^* P)^j - (d\log^* P)^{s-1} \right) \\ &\qquad + (\log^* P)^{s-1} \\ &\quad \leq (\log^* P)^{s-2} (c_{14} R_S + \log^* P) \end{aligned}$$

with $c_{14} = (s/d)((s-1)!)^2 \delta_{\mathbb{K}}^{-(s-1)}$. Together with the second inequality of Lemma 3 this gives

$$\log A_1 \ldots \log A_{s-1} \leq 2c_{14} R_S (\log^* P)^{s-2}. \tag{36}$$

([2]) In certain applications (e.g. in case of practical solutions of S-unit equations), it can be more useful to work with our upper bounds of B, provided by (26), (34) and (43).

We distinguish two cases. First assume that $\log H < c_5 R_S$. Then, by Lemmas 1 and 3, we have

$$\log A := \max_{1\le i\le s} \log A_i \le c_{15} R_S \tag{37}$$

with $c_{15} = c_5 \delta_{\mathbb{K}}^{-1} + (c_7 \log 2)^{-1}$. We now apply to (27) the first part of Proposition 2. Putting

$$\Phi = c_{16} \frac{P^d}{(\log^* P)^{s+1}} \log A_1 \ldots \log A_s \log(10sd \log A)$$

with $c_{16} = c_9(s)(d^2/\log 2)^{s+1}$, we infer that

$$|\alpha x|_v \ge \exp\{-d(\log^* P)\Phi \log(dB)\}, \tag{38}$$

whence, by (33), (26) and $|\alpha|_v \le H^d$,

$$\log h(x) \le 2(s-1)(\log^* P)\Phi \log(c_{17} \log h(x))$$

follows with $c_{17} = 2d^2 c_6$. Together with (36), (37) and $\log H < c_5 R_S$ this gives

$$h(x) \le \exp\{c_{18} P^d R_S (\log^* R_S)(\log^*(PR_S)/\log^* P) \log H\}, \tag{39}$$

where

$$c_{18} = 3^{26}(18d^2/\delta_{\mathbb{K}})^{s+1} s^{4s+7}.$$

Next assume that $\log H \ge c_5 R_S$. Then, by Lemmas 1 and 3, we have $A_s \ge A_i$ for $i = 1, \ldots, s-1$ and

$$\log A := \max_{1\le i\le s-1} \log A_i \le c_{15} R_S. \tag{40}$$

Consider now the above defined Φ with this value of $\log A$. First we give an upper bound for $h(x)$ in terms of Φ.

If $B < \Phi(\log^* P)/(c_5 R_S)$ then (30), (31) and (35) imply that

$$h(x) \le 2H^2 \exp\{(s-1)\Phi \log^* P\} < \exp\{s\Phi \log^* P\}. \tag{41}$$

Assume now that $B \ge \Phi(\log^* P)/(c_5 R_S)$. We apply the second part of Proposition 2 to (27). Putting $\delta = \Phi(\log^* P)/(Bc_5 R_S)$ we obtain

$$|\alpha x|_v \ge \exp\left\{ - d(\log^* P)\Phi \log\left(\frac{Bc_5 R_S}{\log^* P \log A_s}\right)\right\}.$$

Hence, proceeding again as above, we deduce that

$$\frac{\log h(x)}{\log^* P \log A_s} \le 2(s-1)(\Phi/\log A_s) \log\left(\frac{c_{19} R_S \log h(x)}{\log^* P \log A_s}\right)$$

with $c_{19} = 2dc_6 c_5$. From this we infer as above that

$$h(x) \le \exp\{c_{20}\Phi(\log^* P)\log^*(PR_S)\}, \tag{42}$$

where $c_{20} = 19(s-1)\log(c_{16})$.

The right hand side of (42) is greater than that of (41). Lemma 3, (35) and $\log H \geq c_5 R_S$ imply that $\log A_s < c_{21} \log H$ with $c_{21} = (c_5 c_7 \log 2)^{-1} + \delta_{\mathbb{K}}^{-1}$. Hence, estimating from above Φ, we obtain in both cases that

$$h(x) \leq \exp\{c_{18} P^d R_S (\log^* R_S)(\log^*(PR_S)/\log^* P) \log H\}, \tag{43}$$

with the constant c_{18} defined above. However, it is easy to verify that both c_{13} in (34) and c_{18} in (39) and (43) are less than $c_1 = c_1(d, s, \mathbb{K})$ specified in the Theorem. This completes the proof of our assertion. ∎

Proof of the Corollary. Let x_1, x_2, x_3 be a solution of (6). Then, by Lemma 2, there are $\varepsilon_i \in O_S^*$ such that

$$h(\varepsilon_i x_i) \leq N^{1/d} \exp\{c_3 R_{\mathbb{K}} + t h_{\mathbb{K}} \log^* P\} \tag{44}$$

with the constant c_3 specified in Lemma 2. Put

$$\alpha = \frac{\alpha_1(\varepsilon_1 x_1)}{\alpha_3(\varepsilon_3 x_3)}, \qquad \beta = \frac{\alpha_2(\varepsilon_2 x_2)}{\alpha_3(\varepsilon_3 x_3)}.$$

Then $x = -\varepsilon_3/\varepsilon_1$, $y = -\varepsilon_3/\varepsilon_2$ is a solution of equation (1).

We have

$$\max\{h(\alpha), h(\beta)\} \leq \exp\{2c_3(R_{\mathbb{K}} + t h_{\mathbb{K}} \log^* P + \log(HN))\}.$$

Now our Theorem provides an explicit upper bound for $\max\{h(x), h(y)\}$. Together with (44), this implies (7) with the choice $\varepsilon = -\varepsilon_3$. ∎

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