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Primes in almost all short intervals

by

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1. Introduction. The object of this paper is to extend the range of validity of a well-known result of prime number theory. We deal with the Selberg integral

$$J(x,h) := \int_x^{2x} \left| \pi(t) - \pi(t-h) - \frac{h}{\log t} \right|^2 dt.$$

The Prime Number Theorem suggests that $J(x,h)$ should be of lower order of magnitude than $xh^2(\log x)^{-2}$, at least when h is not too small with respect to x, and the Brun–Titchmarsh inequality trivially implies $J(x,h) \ll xh^2(\log x)^{-2}$ provided only that $h \ge x^{\varepsilon}$ for some fixed $\varepsilon > 0$.

We prove the following

THEOREM. *We have*

$$J(x,h) \ll \frac{xh^2}{(\log x)^2} \left(\varepsilon(x) + \frac{\log\log x}{\log x} \right)^2$$

provided that $x^{1/6-\varepsilon(x)} \le h \le x$, *where* $0 \le \varepsilon(x) \le 1/6$ *and* $\varepsilon(x) \to 0$ *as* $x \to \infty$.

It is well known that Huxley's density estimates [5] for the zeros of the Riemann zeta-function yield $J(x,h) = o(xh^2(\log x)^{-2})$, but only for $h \ge x^{1/6}(\log x)^C$, for some $C > 0$. The weaker result with $h \ge x^{1/6+\varepsilon}$ is proved in Saffari and Vaughan [8], Lemma 5, and in [13], where an identity of Heath-Brown (Lemma 1 of [3]) is used.

This paper is inspired by Heath-Brown's extension [4] of Huxley's Theorem [5] that

$$\pi(x) - \pi(x-h) \sim h(\log x)^{-1}$$

to the range $h \ge x^{7/12-\varepsilon(x)}$. This was achieved by means of another identity (see (2.2) of [4], or Lemma 2 below), thereby avoiding a direct appeal to the

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properties of the zeros of the Riemann zeta-function, besides Vinogradov's zero-free region. We extend this approach to the above integral.

An immediate consequence of this result is that if $x^{1/6-\varepsilon(x)} \le h \le x$ then for "almost all" $n \in [x, 2x] \cap \mathbb{N}$ we have $\pi(n) - \pi(n-h) \sim h(\log n)^{-1}$. Here "almost all" means that the above asymptotic equality fails for at most $o(x)$ values of $n \in [x, 2x] \cap \mathbb{N}$. Relaxing our demand to $\pi(n) - \pi(n-h) \gg h(\log n)^{-1}$ for almost all n's, one can take h even smaller, and the best result up to date is due to Jia [6] who showed that $h \ge x^{1/20+\varepsilon}$ is acceptable, provided that x is large enough.

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2. Preliminaries. We assume throughout that x is sufficiently large. For the sake of brevity we set $\mathcal{L} := \log x$. Our estimates will be uniform with respect to all parameters but k_0, which will eventually be chosen as 4. For ease of reference, our notation is consistent, as far as possible, with the notation in [4], and will be introduced at appropriate places. A few comments on the proof are collected at the end of the paper.

LEMMA 1. *The Theorem follows from the estimate*

$$J'(x,\theta) := \int_x^{2x} \left|\pi(t) - \pi(t-\theta t) - \frac{\theta t}{\log t}\right|^2 dt \ll \frac{x^3\theta^2}{\mathcal{L}^2}\left(\varepsilon(x) + \frac{\log\log x}{\log x}\right)^2,$$

uniformly for $x^{-5/6-\varepsilon(x)} \le \theta \le 1$.

LEMMA 2 (Linnik–Heath-Brown's identity). *For* $z > 1$ *we have*

$$\log(\zeta(s)\Pi(s)) = \sum_{k\ge 1} \frac{(-1)^{k-1}}{k}(\zeta(s)\Pi(s) - 1)^k = \sum_{k\ge 1}\sum_{p\ge z} \frac{1}{kp^{ks}}, \tag{2.1}$$

where

$$\Pi(s) := \prod_{p<z}\left(1 - \frac{1}{p^s}\right).$$

For Lemma 1 see the proof of Lemma 6 of [8]. Lemma 2 follows from (2.2)–(2.3) of [4].

For $t \in [x, 2x]$ we use the interval $\mathcal{I} = \mathcal{I}(t,\theta) = (t - \theta t, t]$, and a parameter z satisfying

$$x^{1/k_0} < z \le x^{1/3}.$$

We pick out the coefficients in the above identity for the terms with $n \in \mathcal{I}$.

We have

$$(2.2)\qquad \sum_{k\ge1}\frac{1}{k}|\{p: p^k\in\mathcal{I}, p\ge z\}| = \pi(t)-\pi(t-\theta t)+O(\theta x^{1/2}+\log x),$$

the contribution from prime powers being negligible. Now the Dirichlet series for $\zeta(s)\Pi(s)-1$ is $\sum_{n\ge z} a(n)n^{-s}$ where $a(1)=0$ and $a(n)=0$ unless all prime factors of n are $\ge z$, in which case $a(n)=1$. Furthermore, the Dirichlet series for $(\zeta(s)\Pi(s)-1)^k$ is $\sum_{n\ge z} a_k(n)n^{-s}$, a_k being the k-fold Dirichlet convolution of a with itself. This means that $a_k(n)=0$ unless $n\ge z^k$ and $p\ge z$ for all $p\,|\,n$. Hence there are no terms n^{-s} with $n\in\mathcal{I}$ and $k\ge k_0$, and we may consider only the values $k<k_0$.

As pointed out in Section 2 of [4], the above identity does not give suitable Dirichlet polynomials at once, and we first need to approximate the above Dirichlet series by manageable Dirichlet polynomials. We set

$$\zeta_t(s) := \sum_{n\le t}\frac{1}{n^s}.$$

We introduce parameters $z_1\in[3,z)$ and $z_2 := z_1^\delta$, where $\delta\ge2$ and define v_n by means of

$$\Pi_0(s) := \prod_{p<z_1}\left(1-\frac{1}{p^s}\right) = \sum_{n\ge1}\frac{\mu(n)v_n}{n^s}.$$

Then define $\Pi_1(s) := \Pi(s)\Pi_0(s)^{-1}$, L to be the integer such that $z_1^L \le 2x < z_1^{L+1}$ and

$$\Pi_2(s) := \sum_{n<z_2}\frac{\mu(n)v_n}{n^s},\qquad \Sigma_m(s) := \sum_{z_1\le p<z}\frac{1}{p^{ms}},$$

for $m=1,\ldots,L$. Finally, we set

$$\Pi^*(s) := \prod_{m=1}^{L}\Pi_m^*(s)\quad\text{where}\quad \Pi_m^*(s) := \sum_{l=0}^{L/m}\frac{(-1)^l}{l!m^l}\Sigma_m(s)^l.$$

We remark that our choice of the parameters ensures that the coefficient of n^{-s} in $\Pi_1(s)$ is the same as the coefficient of n^{-s} in $\Pi^*(s)$. We now introduce the Dirichlet polynomials we shall work with. Let B, C, and D be integers such that

$$t/2<2^B\le t,\qquad z_2/2<2^C\le z_2,\qquad z/2\le 2^D<z,$$

and set

$$(2.3)\qquad \zeta_t(s)=\sum_{b=0}^{B}X_b(s),\qquad X_b(s) := \sum_{2^{-1-b}t<n\le 2^{-b}t} n^{-s},$$

$$(2.4)\qquad \varPi_2(s)=\sum_{c=0}^{C}Y_c(s),\qquad Y_c(s):=\sum_{2^{-1-c}z_2<n\le 2^{-c}z_2}\mu(n)v_n n^{-s},$$

$$(2.5)\qquad \varSigma_m(s)=\sum_{d=0}^{D}Z_d^{(m)}(s),\qquad Z_d^{(m)}(s):=\sum_{\substack{2^{-1-d}z<p\le 2^{-d}z\\ p\ge z_1}}p^{-ms}.$$

Hence, for suitable coefficients $c_{m,h}$, we have

$$(2.6)\qquad (\zeta_t(s)\varPi_2(s)\varPi^*(s))^h=\sum_{m=1}^{M(h)}c_{m,h}W(s;m,h),$$

where the Dirichlet polynomials W have the form

$$(2.7)\qquad W(s;m,h)=W_X(s;m,h)W_Y(s;m,h)W_Z(s;m,h),$$

with

$$(2.8)\qquad \begin{aligned} W_X(s)&:=\prod_{i=1}^{h}X_{b_i}(s),\qquad W_Y(s):=\prod_{i=1}^{h}Y_{c_i}(s),\\ W_Z(s)&:=\prod_{m=1}^{L}\prod_{i=1}^{I_m}Z_{d_i}^{(m)}(s),\end{aligned}$$

where each I_m is $\le hL/m$, and we dropped m and h for brevity. Writing

$$(2.9)\qquad X_i:=2^{-1-b_i}t,\qquad Y_i:=2^{-1-c_i}z_2,\qquad Z_i:=2^{-1-d_i}z,$$

and $I=\sum_m I_m$, we have

$$(2.10)\qquad W(s;m,h)=\sum_{N_1<n\le N_2}\frac{e_{m,h}(n)}{n^s},$$

where

$$(2.11)\qquad N_1:=\prod_{i=1}^{h}X_iY_i\cdot\prod_{m=1}^{L}\prod_{i=1}^{I_m}Z_i\quad\text{and}\quad N_2:=2^{2h+I}N_1.$$

Since we are interested in the coefficients of the terms n^{-s} with $n\in\mathcal{I}(t,\theta)$, we may obviously discard those sums $W(s)$ with $N_1\ge t$ or $N_2\le t/2$, leaving, after relabeling,

$$\sum_{m=1}^{N(h)}c_{m,h}W(s;m,h),$$

say. As usual, we denote by $d_m(n)$ the coefficient of n^{-s} in $\zeta^m(s)$. We now state the following results, the first being a consequence of Theorem 2 of Shiu [9].

Lemma 3. *For fixed $\varepsilon > 0$ and $m, h \in \mathbb{N}$ we have*

$$\sum_{x \le n \le x+y} d_m^h(n) \ll_{\varepsilon,m,h} y(\log x)^{m^h-1},$$

uniformly for $x^\varepsilon \le y \le x$.

Lemma 4. *For $t \in [x, 2x]$ there exist Dirichlet polynomials $W(s; m, h)$ satisfying* (2.3)–(2.11) *such that*

$$\sum_{n \in \mathcal{I}(t,\theta)} a_k(n) = \sum_{h=0}^{k} (-1)^{k-h} \binom{k}{h} \sum_{m=1}^{N(h)} c_{m,h} \sum_{n \in \mathcal{I}(t,\theta)} e_{m,h}(n) + O(x\theta \mathcal{L}^{3k} \delta^{-\delta/3})$$

when $z_1 z_2 \le x^{1/8}$ and $\delta \ge (\log\log z_1)^2$.

The proof is quite similar to the proof of Lemma 3 of [4], using Lemma 3 above. We omit it for brevity. Set

$$\Sigma(h,t,\theta) := \sum_{m=1}^{N(h)} c_{m,h} \sum_{n \in \mathcal{I}(t,\theta)} e_{m,h}(n)$$

(here a minor clash with the notation of [4] occurs). Then

$$S(t,\theta) := \pi(t) - \pi(t-\theta t) = \sum_{1 \le k < k_0} \sum_{h=0}^{k} \alpha(h,k) \Sigma(h,t,\theta) + O(E(t,\theta,\delta)),$$

say, where $\alpha(h,k) \ll 1$ and $E(t,\theta,\delta) \ll \theta(x^{1/2} + x\mathcal{L}^{3k}\delta^{-\delta/3})$ by (2.1), (2.2) and Lemma 4. Our aim is to prove that each Σ can be written as

$$\Sigma(h,t,\theta) = \theta \mathfrak{M}(h,t) + \mathfrak{R}(h,t,\theta), \tag{2.12}$$

where $\mathfrak{M}(h,t)$ is independent of θ and $\mathfrak{R}(h,t,\theta)$ is small in L^2 norm over $[x, 2x]$. In fact, assume that (2.12) holds for suitable $\mathfrak{M}$ and $\mathfrak{R}$, and let

$$\mathfrak{M}(t) := \sum_{1 \le k < k_0} \sum_{h=0}^{k} \alpha(h,k) \mathfrak{M}(h,t),$$

$$\mathfrak{R}(t,\theta) := \sum_{1 \le k < k_0} \sum_{h=0}^{k} \alpha(h,k) \mathfrak{R}(h,t,\theta),$$

so that $S(t,\theta) = \theta\mathfrak{M}(t) + \mathfrak{R}(t,\theta) + O(E(t,\theta,\delta))$. Since $(a+b+c)^2 \ll a^2 + b^2 + c^2$ we have

$$J'(x,\theta) \ll \int_x^{2x} \left\{ \theta^2 \left(\mathfrak{M}(t) - \frac{t}{\log t} \right)^2 + \mathfrak{R}(t,\theta)^2 \right\} dt + \theta^2 x^3 \mathcal{L}^{3k-2} (\delta^{-\delta/3} + \mathcal{L}^{3k} \delta^{-2\delta/3}). \tag{2.13}$$

The error term is $\ll_A x^3\theta^2\mathcal{L}^{-A}$ for any fixed A, provided that $\delta \ge \log\mathcal{L}$, which we assume. Hence by Lemma 1 and (2.13) we have proved

LEMMA 5. *The Theorem follows from the estimates*

$$\int_x^{2x}\left(\mathfrak{M}(t)-\frac{t}{\log t}\right)^2 dt \ll \frac{x^3}{\mathcal{L}^2}\left(\varepsilon(x)+\frac{\log\log x}{\log x}\right)^2, \tag{2.14}$$

$$\int_x^{2x}|\mathfrak{R}(t,\theta)|^2\,dt \ll \frac{x^3\theta^2}{\mathcal{L}^2}\left(\varepsilon(x)+\frac{\log\log x}{\log x}\right)^2 \tag{2.15}$$

uniformly for $x^{-5/6-\varepsilon(x)} \le \theta \le 1$, *provided that* $\delta \ge \max(\log\mathcal{L}, (\log\log z_1)^2)$.

We shall prove the first part of Lemma 5 in Section 5 by taking θ "large", whereas the proof of the other estimate is achieved by means of mean-value bounds as described below.

3. The case $k \le 2$: reduction to mean-value estimates. For brevity we write $s = s(\tau) = 1/2 + i\tau$ throughout this section. By Perron's formula (see Lemma 3.12 of [10]) we have

$$\begin{aligned}\Sigma(h,t,\theta) &= \frac{1}{2\pi i}\sum_{m=1}^{N(h)} c_{m,h}\int_{-T_0}^{T_0} W(s;m,h)\frac{t^s-(t-\theta t)^s}{s}\,d\tau \\ &\quad + O\biggl(\sum_{j=0}^{1}\sum_{m=1}^{N(h)}|c_{m,h}|\sum_{n=N_1(m)+1}^{N_2(m)}|e_{m,h}(n)|\left(\frac{x}{n}\right)^{1/2} \\ &\quad \times \min\left(1, T_0^{-1}\left|\log\frac{t-j\theta t}{n}\right|^{-1}\right)\biggr).\end{aligned} \tag{3.1}$$

The error term is estimated in Section 6 where we prove that

$$\begin{aligned}\Sigma(h,t,\theta) &= \frac{1}{2\pi i}\sum_{m=1}^{N(h)} c_{m,h}\int_{-T_0}^{T_0} W(s;m,h)\frac{t^s-(t-\theta t)^s}{s}\,d\tau \\ &\quad + O\left(\frac{x}{T_0}e^{2I}(\log N_7)^{3h}\right),\end{aligned} \tag{3.2}$$

where

$$N_7 := \max_{1\le m\le N(h)} N_2(m).$$

The main term of Σ will come from a short interval: for $|\tau| \le T_1$ we have

$$\frac{t^s-(t-\theta t)^s}{s} = \theta t^s + O(|s|\theta^2 t^{1/2}). \tag{3.3}$$

Hence, setting $S_0 = S_0(h) := \sum_{m=1}^{N(h)} |c_{m,h}|$,

$$(3.4)\qquad \begin{aligned} \mathfrak{M}(h,t) &:= \frac{1}{2\pi i}\sum_{m=1}^{N(h)} c_{m,h}\int_{-T_1}^{T_1} W(s(\tau);m,h)t^s\,d\tau,\\ J_0 = J_0(h) &:= \max_{1\le m\le N(h)}\int_{-T_1}^{T_1} |W(s(\tau);m,h)|\,d\tau, \end{aligned}$$

we have

$$(3.5)\qquad \frac{1}{2\pi i}\sum_{m=1}^{N(h)} c_{m,h}\int_{-T_1}^{T_1} W(s;m,h)\frac{t^s-(t-\theta t)^s}{s}\,d\tau = \theta\mathfrak{M}(h,t) + O(T_1J_0S_0\theta^2x^{1/2}).$$

Summing up, from (3.1)–(3.5) we have

$$(3.6)\qquad \begin{aligned} \Sigma(h,t,\theta) &= \theta\mathfrak{M}(h,t) + \mathfrak{R}_1(h,t,\theta)\\ &\quad + \frac{1}{2\pi i}\sum_{m=1}^{N(h)} c_{m,h}\Big\{\int_{-T_0}^{-T_1} + \int_{T_1}^{T_0}\Big\}W(s;m,h)\frac{t^s-(t-\theta t)^s}{s}\,d\tau\\ &= \theta\mathfrak{M}(h,t) + \mathfrak{R}_1(h,t,\theta) + \mathfrak{R}_2(h,t,\theta) \end{aligned}$$

say, where $\mathfrak{M}(h,t)$ is independent of θ. The ranges $[-T_0,-T_1]$ and $[T_1,T_0]$ are dealt with by means of the following mean-value bound, which will be proved in Section 7.

LEMMA 6. *There is a constant $C_0 > 0$ with the following property. Let*

$$(3.7)\qquad \eta = \eta(T) := C_0(\log T)^{-2/3}(\log\log T)^{-1/3}$$

and

$$\mathcal{E} := \exp\Big\{\Big(\frac{\mathcal{L}}{\log z_1}\Big)^2\log\log z_1\Big\}$$

and assume that $z_1 = z_1(x)$ and $\delta = \delta(x)$ are functions of x such that $\delta \ge (\log\log z_1)^2$, $\log z_1 \ge \mathcal{L}^{2/3}$, $z_2 = z_1^{\delta} = x^{o(1)}$ and $\mathcal{E} = x^{o(1)}$. Then for each fixed $\alpha \in (0,1/12)$ there exists $\beta = \beta(\alpha)$ with $\beta \in (0,1/42)$ with the following property. Let

$$x^{1/4} < z \le x^{1/3-\alpha} \quad \textit{and} \quad 3 \le T \le T_0 = x^{5/6+\beta}.$$

Then for $t \in [x,2x]$ and $h \le 2$ we have

$$\int_T^{2T} |W(s(\tau);m,h)|^2\,d\tau \ll x\mathcal{E}^{2h^2}(z_1^{-\eta/6} + T^{-1/6}).$$

We obviously have

$$\mathfrak{R}_2(h,t,\theta) \ll \sum_{m=1}^{N(h)} |c_{m,h}| \Bigl| \int_{T_1}^{T_0} W(s;m,h) \frac{t^s - (t-\theta t)^s}{s} \, d\tau \Bigr|$$

and this means that

$$\int_x^{2x} |\mathfrak{R}_2(h,t,\theta)|^2 \, dt \ll S_0^2 \max_{1 \le m \le N(h)} \int_x^{2x} \Bigl| \int_{T_1}^{T_0} W(s;m,h) \frac{t^s - (t-\theta t)^s}{s} \, d\tau \Bigr|^2 \, dt. \tag{3.8}$$

The next lemma is needed to invert the order of integration.

Lemma 7. *Let $F(s)$ be a continuous complex-valued function. Then for $1 \le T_1 \le T_0 \le x$ and $s = 1/2 + i\tau$ we have*

$$\int_x^{2x} \Bigl| \int_{T_1}^{T_0} F(s) \frac{t^s - (t-\theta t)^s}{s} \, d\tau \Bigr|^2 \, dt \ll x^2 \theta^2 \mathcal{L}^2 \max_{T_1 \le T \le T_0} \int_T^{2T} |F(s)|^2 \, d\tau.$$

A proof can be easily given by squaring out the integral, performing the integration with respect to t first and then using the elementary inequality $|ab| \le |a|^2 + |b|^2$ on the remaining double integral. A form of this result appears as Lemma 2 in Harman [2] and elsewhere. We omit the details for brevity.

We remark that $\mathcal{L}^A \ll_A \mathcal{E}$ for any fixed A, that $N_7 \ll 2^{2h+I} x \ll \mathcal{E} x$ and that the definition of W easily implies $J_0 \ll T_1 x^{1/2}$. The next lemma is proved in Section 6.

Lemma 8. *For large enough x we have*

$$|S_0| \ll \exp\Bigl\{ h \frac{\mathcal{L}}{\log z_1} (\log \mathcal{L})^2 \Bigr\}.$$

Hence $\mathcal{L}^2 S_0^2 \ll \mathcal{E}$. We now choose $k_0 := 4$ and set

$$\mathfrak{M}_1(t) := \sum_{k=1}^{2} \sum_{h=0}^{k} \alpha(h,k) \mathfrak{M}(h,t),$$

$$\mathfrak{R}_j(t,\theta) := \sum_{k=1}^{2} \sum_{h=0}^{k} \alpha(h,k) \mathfrak{R}_j(h,t,\theta),$$

for $j = 1,2$. Summing up, from Lemmas 4, 6–8, and from (3.2), (3.5)–(3.8) we have

$$\pi(t) - \pi(t-\theta t) - \frac{1}{3} \sum_{n \in \mathcal{I}(t,\theta)} a_3(n) = \theta \mathfrak{M}_1(t) + \mathfrak{R}_1(t,\theta) + \mathfrak{R}_2(t,\theta), \tag{3.9}$$

where

$$\mathfrak{R}_1(t,\theta) \ll x\mathcal{E}T_0^{-1} + x\theta^2\mathcal{E}T_1^2, \tag{3.10}$$

$$\int_x^{2x} |\mathfrak{R}_2(t,\theta)|^2\,dt \ll x^3\theta^2\mathcal{E}^9(z_1^{-\xi/6} + T_1^{-1/6}), \tag{3.11}$$

and $\xi := \eta(T_1)$. We finally choose our parameters as follows. First we choose $\delta := (\log\mathcal{L})^2$ so that $\delta \ge \max(\log\mathcal{L}, (\log\log z_1)^2)$ if $z_1 \le x$, and $z_2 = x^{o(1)}$ provided that $\log z_1 = o(\mathcal{L}(\log\mathcal{L})^{-2})$. Next, we choose $T_1 := \mathcal{E}^{55}$ and observe that T_1 tends to infinity with x. The choice

$$z_1 := \exp\{\mathcal{L}^{8/9}\log\mathcal{L}\}$$

implies

$$z_1^{-\xi} \ll_A \mathcal{E}^{-A},$$

for any fixed A. We now see that the hypotheses of Lemma 6 are satisfied and (3.9)–(3.11) finally yield

LEMMA 9. *Let α, β and z be as in Lemma 6. For $t \in [x,2x]$ there exist functions $\mathfrak{M}_1(t)$ and $\mathfrak{R}'(t,\theta)$ such that*

$$\pi(t) - \pi(t-\theta t) - \frac{1}{3}\sum_{n\in\mathcal{I}(t,\theta)} a_3(n) = \theta\mathfrak{M}_1(t) + \mathfrak{R}'(t,\theta),$$

where $\mathfrak{M}_1(t)$ is independent of θ and

$$\int_x^{2x} |\mathfrak{R}'(t,\theta)|^2\,dt \ll_A x^3\theta^2\mathcal{L}^{-A},$$

for any fixed A, provided that

$$x^{-5/6-\beta} \le \theta \le \exp\{-100\mathcal{L}^{2/9}\}. \tag{3.12}$$

4. The case $k=3$: reduction to mean-value estimates. The analysis of the case $k=3$ is quite similar to the previous one, but we have to be slightly more careful in order to obtain a good error term. We exploit the fact that each Dirichlet polynomial we use is the product of only 3 factors, as opposed to Section 3 where the number of factors was $2h+I$. Define

$$P(s) := \sum_{z\le p\le 2x}\frac{1}{p^s} \quad\text{and}\quad P^*(s) := \sum_{z_3\le p\le 2x}\frac{1}{p^s},$$

where z_3 is a new parameter satisfying $z \le z_3 \le x^{1/3}$. Note that if $n \le 2x$ then $a_3(n)$ is precisely the coefficient of n^{-s} in $P(s)^3$. Let $b_3(n)$ be the coefficient of n^{-s} in $P^*(s)^3$. We write $P_1(s) = P(s) - P^*(s)$ so that $a_3(n) -$

$b_3(n)$ is the coefficient of n^{-s} in

$$P(s)^3 - P^*(s)^3 = \sum_{j=1}^{3} \binom{3}{j} P_1(s)^j P^*(s)^{3-j}$$

if $n \le t$. We write

$$P_1(s) = \sum_{-E \le e \le 0} P_e(s) \quad \text{and} \quad P^*(s) = \sum_{1 \le e \le F} P_e(s),$$

where E and F are integers satisfying $2^{-E-1} z_3 \le z < 2^{-E} z_3$ and $2^{F-1} z_3 \le 2x < 2^F z_3$, and

$$P_e(s) := \sum_{\substack{2^{e-1} z_3 \le p < 2^e z_3 \\ z \le p \le 2x}} \frac{1}{p^s}.$$

Since $E, F \ll \mathcal{L}$, for some $M \ll \mathcal{L}^3$ and $c_m \ll 1$ we have

$$P(s)^3 - P^*(s)^3 = \sum_{m=1}^{M} c_m P(s;m) \quad \text{where} \quad P(s;m) := \prod_{j=1}^{3} P_{e_j}(s)$$

with $e_1 \le 0$. Write $V_j := 2^{e_j - 1} z_3$ so that

$$P(s;m) = \sum_{N_8 \le n \le N_9} \frac{f_m(n)}{n^s},$$

say, where $N_8 := \prod_j V_j$ and $N_9 := 2^3 N_8$. As above, we discard those $P(s;m)$ having either $N_8 \ge t$ or $N_9 \le t/2$ and relabel the remaining ones so that for some $N \le M$ we have

$$\sum_{n \in \mathcal{I}(t,\theta)} a_3(n) = \sum_{n \in \mathcal{I}(t,\theta)} b_3(n) + \sum_{m=1}^{N} \sum_{n \in \mathcal{I}(t,\theta)} f_m(n). \tag{4.1}$$

The same analysis of Section 3, with the bound $|f_m(n)| \le 3!$, yields

$$\sum_{n \in \mathcal{I}(t,\theta)} f_m(n) = \frac{1}{2\pi i} \int_{1/2 - iT_2}^{1/2 + iT_2} P(s;m) \frac{t^s - (t - \theta t)^s}{s} \, ds + O\left(\frac{x\mathcal{L}}{T_2}\right),$$

for $T_2 \le x$. The ranges $[-T_2, -T_3]$ and $[T_3, T_2]$ are treated by means of the following mean-value bound, which will be proved in Section 8.

Lemma 10. *Let $x^{19/60} \le z \le x^{1/3}$ and $x^{5/6} \le T_2 \le x^{11/12}$. Then, if $P(s;m)$ is as above with $V_3 \ge V_2 \ge V_1 \ge z/2$, we have*

$$\int_T^{2T} \left| P\left(\frac{1}{2} + i\tau; m\right) \right|^2 d\tau \ll x\mathcal{L}^{62} (z_1^{-\eta/6} + T^{-1/6} + (T_2 V_3^{-5/2})^{1/9})$$

uniformly for $3 \le T \le T_2$, where η is given by (3.7).

We proceed precisely as in Section 3, using Lemma 7 again with $F(s) = P(s;m)$ and (3.3) for the range $[-T_3, T_3]$, obtaining

$$\sum_{n\in\mathcal{I}(t,\theta)} f_m(n) = \theta\frac{1}{2\pi i}\int_{1/2-iT_3}^{1/2+iT_3} P(s;m)t^s\,ds + \mathfrak{R}_1(3,t,\theta) + \mathfrak{R}_2(3,t,\theta), \tag{4.2}$$

where

$$\mathfrak{R}_1(3,t,\theta) \ll x\mathcal{L}T_2^{-1} + x\theta^2T_3^2, \tag{4.3}$$

$$\int_x^{2x} |\mathfrak{R}_2(3,t,\theta)|^2\,dt \ll x^3\theta^2(z_1^{-\varrho/3} + T_3^{-1/3} + (T_2V_3^{-5/2})^{1/9})\mathcal{L}^{62}, \tag{4.4}$$

and $\varrho = \eta(T_2)$. Since $V_3^2 \geq xz_3^{-1}$ we have $T_2V_3^{-5/2} \ll T_2z_3^{5/4}x^{-5/4}$. We finally choose the parameters: Let ν be a sufficiently large positive constant and set $T_2 := \mathcal{L}^\nu\max(\theta^{-1}, x^{5/6})$, $T_3 := \mathcal{L}^\nu$ and also $x^{19/60} \leq z_3 \leq \mathcal{L}^{-\nu}\min(\theta^{4/5}x, x^{1/3})$. Then (4.1)–(4.4) imply

$$\sum_{n\in\mathcal{I}(t,\theta)} a_3(n) = \sum_{n\in\mathcal{I}(t,\theta)} b_3(n) + \theta\mathfrak{M}_3(t,z_3) + \mathfrak{R}''(t,\theta,z_3), \tag{4.5}$$

say, where $\mathfrak{M}_3(t,z_3)$ is independent of θ and

$$\int_x^{2x} |\mathfrak{R}''(t,\theta,z_3)|^2\,dt \ll x^3\theta^2\mathcal{L}^{60-\nu/18}, \tag{4.6}$$

provided that θ satisfies (3.12). Now choose $z := x^{19/60}$, so that the hypotheses of both Lemmas 6 and 10 are satisfied, and take $\nu := 1500$. Hence, from Lemma 9, (4.5) and (4.6) we deduce

Lemma 11. *There exists a small positive constant λ such that if*

$$x^{-5/6-\lambda} \leq \theta \leq \exp\{-100\mathcal{L}^{2/9}\}$$

and

$$x^{19/60} \leq w \leq \mathcal{L}^{-1500}\min(\theta^{4/5}x, x^{1/3}) \tag{4.7}$$

then for $t \in [x, 2x]$ there exists a function $\mathfrak{M}(t,w)$ independent of θ such that

$$\pi(t) - \pi(t-\theta t) - \frac{1}{3}\sum_{n\in\mathcal{I}(t,\theta)} b_3(n) = \theta\mathfrak{M}(t,w) + \mathfrak{R}(t,\theta,w) \tag{4.8}$$

where

$$\int_x^{2x} |\mathfrak{R}(t,\theta,w)|^2\,dt \ll x^3\theta^2\mathcal{L}^{-20}.$$

It now remains to estimate the contribution of $b_3(n)$. First we remark that

$$(4.9)\qquad \int_x^{2x}\Big|\sum_{n\in\mathcal{I}(t,\theta)} b_3(n)\Big|^2\,dt \ll \Big(\sup_{t\in[x,2x]}\sum_{n\in\mathcal{I}(t,\theta)} b_3(n)\Big)\int_x^{2x}\sum_{n\in\mathcal{I}(t,\theta)} b_3(n)\,dt,$$

and that a simple argument based on the Brun–Titchmarsh inequality gives

$$\begin{aligned}(4.10)\qquad \int_x^{2x}\sum_{n\in\mathcal{I}(t,\theta)} b_3(n)\,dt &\ll \sum_{x-\theta x<n\le 2x} b_3(n)\int_{\max(x,n)}^{\min(2x,n(1-\theta)^{-1})} dt\\ &\ll \theta x\sum_{n\le 2x} b_3(n) \ll \theta x\sum_{w\le p,q\le 2x/w^2}\ \sum_{r\le 2x/(pq)} 1\\ &\ll \frac{\theta x^2}{\mathcal{L}}\Big(\sum_{w\le p\le 2x/w^2}\frac{1}{p}\Big)^2 \ll \frac{\theta x^2}{\mathcal{L}}\Big(\frac{\log(xw^{-3})}{\mathcal{L}}\Big)^2.\end{aligned}$$

The same argument leading to (4.10) shows that the expected order of magnitude for the supremum over t in (4.9) is $\theta x\mathcal{L}^{-1}(\log(xw^{-3})/\mathcal{L})^2$, and this would imply the Theorem with the exponent 2 attached to the last factor replaced by 4. But we are unable to prove such a good bound. By Theorem 3.4 of Halberstam–Richert [1] we find

$$\sup_{t\in[x,2x]}\sum_{n\in\mathcal{I}(t,\theta)} b_3(n) \ll \frac{\theta x}{\mathcal{L}},$$

the lower bound in (4.7) ensuring that we save a log factor over the trivial estimate. We collect these results in the form of

LEMMA 12. *Let θ and w be as in the statement of Lemma* 11. *Then*

$$\int_x^{2x}\Big|\sum_{n\in\mathcal{I}(t,\theta)} b_3(n)\Big|^2\,dt \ll \frac{\theta^2x^3}{\mathcal{L}^2}\Big(\frac{\log(xw^{-3})}{\mathcal{L}}\Big)^2.$$

5. Conclusion of the proof: the main term. Here we choose θ as large as possible, i.e. $\theta=\theta_0 := \exp(-100\mathcal{L}^{2/9})$, and any w satisfying (4.7). The Prime Number Theorem gives

$$\pi(t)-\pi(t-\theta_0 t) = \frac{\theta_0 t}{\log t} + O\Big(\frac{x\theta_0^2}{\mathcal{L}^2}\Big).$$

Hence (4.8) yields

$$\theta_0\Big(\mathfrak{M}(t,w)-\frac{t}{\log t}\Big) = -\frac{1}{3}\sum_{n\in\mathcal{I}(t,\theta_0)} b_3(n) - \mathfrak{R}(t,\theta_0,w) + O\Big(\frac{x\theta_0^2}{\mathcal{L}^2}\Big),$$

so that by Lemmas 11 and 12 we have

$$\theta_0^2 \int_x^{2x} \left(\mathfrak{M}(t,w) - \frac{t}{\log t}\right)^2 dt \ll \frac{x^3\theta_0^2}{\mathcal{L}^2}\left(\frac{\log(xw^{-3})}{\mathcal{L}}\right)^2 + \frac{x^3\theta_0^2}{\mathcal{L}^{20}} + \frac{x^3\theta_0^4}{\mathcal{L}^4}. \tag{5.1}$$

We finally take

$$w := \mathcal{L}^{-1500}\min(\theta^{4/5}x, x^{1/3}).$$

This choice of w implies that the left hand side of (5.1) is

$$\ll \frac{x^3\theta_0^2}{\mathcal{L}^2}\left(\varepsilon(x) + \frac{\log\log x}{\log x}\right)^2$$

and the first estimate of Lemma 5 follows. The second part of Lemma 5 is a consequence of Lemmas 11 and 12 and our choice of w. The proof of the Theorem is therefore complete.

6. Proofs of (3.2) and Lemma 8. In order to prove (3.2) we first need the bound

$$\sum_m |c_{m,h}|\cdot|e_{m,h}(n)| \le d_{3h}(n).$$

By (2.6) this sum is bounded by the coefficient of n^{-s} occurring in

$$\zeta(s)^{2h}\prod_{m=1}^{L}\exp\left(\frac{h}{m}\Sigma_m(s)\right),$$

which, in its turn, is bounded by the one in

$$\zeta(s)^{2h}\prod_{m\ge 1}\exp\left(\frac{h}{m}\Sigma_m(s)\right)$$

and the latter is a partial product of $\zeta(s)^h$.

We recall that we chose $N_2 \ge t/2$ and that $N_1 = 2^{-2h-I}N_2$ by (2.11). Setting

$$N_7' := \min_{1\le m\le N(h)} N_1(m),$$

the error term with $j=0$ in (3.1) is

$$\ll 2^{I/2}\sum_{N_7'<n\le N_7} d_{3h}(n)\min\left(1, T_0^{-1}\left|\log\frac{t}{n}\right|^{-1}\right), \tag{6.1}$$

since each n counted in (3.1) is $\ge N_1(m) \ge N_7' \gg x2^{-I}$. For the sake of brevity, for $r\in\mathbb{N}$ let

$$H_r = \{n \in (N_7', N_7] : rT_0^{-1} \le |\log(t/n)| < (r+1)T_0^{-1}\}.$$

Observe that $H_r \ne \emptyset$ only for $0\le r\le M$, say, with $M \ll IT_0$. Then the sum in (6.1) is

$$\begin{aligned}
&\ll \sum_{n\in H_0} d_{3h}(n) + \sum_{r=1}^{M}\sum_{n\in H_r} T_0^{-1} d_{3h}(n)\left|\log\frac{t}{n}\right|^{-1} \\
&\ll \sum_{n\in H_0} d_{3h}(n) + \sum_{r=1}^{M}\sum_{n\in H_r} T_0^{-1} d_{3h}(n)(rT_0^{-1})^{-1} \\
&\ll \sum_{r=0}^{M}\frac{1}{r+1}\sum_{n\in H_r} d_{3h}(n).
\end{aligned}$$

Furthermore $tT_0^{-1}\exp(-rT_0^{-1}) \ll |H_r| \ll tT_0^{-1}\exp(rT_0^{-1})$ for all $r \le M$, and (3.2) follows using Lemma 3. The term with $j = 1$ in (3.1) is dealt with in the same way.

For Lemma 8 we need the following elementary inequality which is easily proved by induction: For any integer $A \ge 2$ and real number $B \ge 3$ we have

$$\sum_{n=0}^{A}\frac{B^n}{n!} \le B^A.$$

Arguing as in Section 5 of [4] we find, after a simple computation,

$$\begin{aligned}
S_0 &\le (B+1)^h(C+1)^h \exp\left\{h\sum_{m=1}^{L/2}\frac{L}{m}\log\frac{D+1}{m} + h\frac{L}{2}\log\frac{2D}{L}\right\} \\
&\le \exp\left\{h\frac{\mathcal{L}}{\log z_1}(\log\mathcal{L})^2\right\},
\end{aligned}$$

for large enough x, since B, C, $D \ll \mathcal{L}$ and $z_1 = x^{o(1)}$, and Lemma 8 follows.

7. Proof of Lemma 6

Preliminaries. The proof is quite similar to the proof of Lemma 8 in [4]. For the sake of brevity we do not duplicate the whole argument, but merely give the needed modifications. We say that a set $\mathcal{S}$ of points $\tau_n \in [T, 2T]$ is *well spaced* if $|\tau_m - \tau_n| \ge 1$ for every $\tau_m, \tau_n \in \mathcal{S}$ with $n \ne m$. We write $s = 1/2 + i\tau$ and $s_n = 1/2 + i\tau_n$ throughout this section. We need an estimate for

$$J_1(T) := \int_T^{2T} |W(s)|^2\,d\tau.$$

We first write W as the product of W_1, W_2 and W_3, where

$$W_1(s) := \prod_{X_i\ge z_1} X_{b_i}(s)\prod_{i=1}^{I_1} Z_{d_i}^{(1)}(s), \qquad W_2(s) := \prod_{X_i< z_1} X_{b_i}(s)\prod_{i=1}^{h} Y_{c_i}(s),$$

$$W_3(s) := W(s)(W_1(s)W_2(s))^{-1}.$$

We also set

$$x_1 := \prod_{X_i \ge z_1} X_i \prod_{i=1}^{I_1} Z_i, \quad x_2 := \prod_{X_i < z_1} X_i \prod_{i=1}^{h} Y_i, \quad x_3 := \prod_{m=2}^{L} \prod_{i=1}^{I_m} Z_i,$$

so that $x_1x_2x_3 = N_1 \le x$. We observe that $|Z_{d_i}^{(m)}(s)| \le Z_i^{1-m/2}$ for $m \ge 2$ and large enough x, whence $|W_3(s)| \le 1$.

The main tool to obtain mean-value estimates such as our Lemmas 6 and 10 is a combination of Montgomery's mean-value bound (see Theorem 7.3 of [7]) and the Halász method. These are summarized in the following

LEMMA 13. *Let $K(s)$ be the Dirichlet polynomial*

$$K(s) = \sum_{n \le K} \frac{k(n)}{n^s},$$

where $K \ge 2$ and $|k(n)| \le 1$ for every $n \le K$. Assume that $|K(1/2+i\tau_n)| \ge \mathcal{K}$ for a set $\mathcal{S}$ of well-spaced points $\tau_n \in [T, 2T]$. Then, uniformly for $g \in \mathbb{N}$, we have

$$|\mathcal{S}| \ll \{\mathcal{K}^{-2g}K^g + T\min(\mathcal{K}^{-2g}, \mathcal{K}^{-6g}K^g)\}\exp\{6g^2 \log\log K\}(\log TK)^5.$$

This is (8.4) and the following is Lemma 19 of [4].

LEMMA 14. *For every factor $K(s)$ of $W_1(s)$ we have*

$$K(s) \ll K^{1/2}(z_1^{-\eta} + T^{-1})\mathcal{L}^2,$$

uniformly for $\tau \in [T, 2T]$, where $\eta = \eta(T)$ is given by (3.7).

Actually, if x_3 is large enough, $x_3 \ge z_1$, say, we see that Lemma 6 follows directly from Montgomery's mean-value bound. In fact, we have

$$J_1 \ll \sup_{\tau \in [T,2T]} |W_3(s)|^2 \int_T^{2T} |W_1(s)W_2(s)|^2 \, d\tau \ll (T + x_1x_2) \sum_{n \le x_1x_2} \frac{|c_n|^2}{n},$$

for suitable coefficients c_n. The same argument leading to Lemma 13 above implies that the last sum is $\ll \mathcal{E}^{2h^2}$, and the hypothesis on x_3 ensures that $T + x_1x_2 \ll xz_1^{-1}$, which is more than enough for Lemma 6. Hence we may assume in what follows that $x_3 \le z_1$. We remark that from the definitions above and (2.11) we have $x_2 = x^{o(1)}$ and $x_1 = x^{1+o(1)}$. We do not rule out the possibility that W_1 consists of a single factor X_{b_i}. We use Lemma 14 in conjunction with Montgomery's mean-value theorem if W_1 has at least one factor $X_{b_i}(s)$ or $Z_{d_i}^{(1)}(s)$ with $X_i \le x^{1/6-\alpha}$ or $Z_i \le x^{1/6-\alpha}$, respectively. In fact, setting $K(s) = X_{b_i}(s)$, $K = X_i$ (resp. $K(s) = Z_{d_i}^{(1)}(s)$, $K = Z_i$),

$W_1(s) = K(s)W_4(s)$, $x_4 = x_1/K$, in this case we have

$$J_1 \ll \sup_{\tau\in[T,2T]} |W_2(s)W_3(s)|^2 \int_T^{2T} |W_1(s)|^2 \, d\tau \ll x_2 K(z_1^{-2\eta} + T^{-2}) \int_T^{2T} |W_4(s)|^2 \, d\tau,$$

and the last integral is estimated by means of Montgomery's theorem, giving

$$J_1 \ll x_2 K(z_1^{-2\eta} + T^{-2})(T + x_4) \sum_{n\le x_4} \frac{|c'_n|^2}{n},$$

for suitable coefficients c'_n. As above, the last sum is $\ll \mathcal{E}^{2h^2}$, and the hypothesis on K ensures that Lemma 6 follows in this case, with $\beta = \alpha/2$.

From now on we may assume that every factor $K(s)$ of $W_1(s)$ has $K \ge x^{1/6-\alpha}$. Thus we have $I_1 \le 12$ and there exists a set $\mathcal{S}$ of $\ll T$ well-spaced points $\tau_n \in [T, 2T]$ such that

$$J_1 \ll \sum_{\tau_n\in\mathcal{S}} |W(s_n)|^2.$$

The contribution to the sum of the points τ_n for which some factor of W_1 is $\le x^{-1}$ is easily seen to be $\ll T$. We discard these points, and from now on assume that each factor of W_1 is $\ge x^{-1}$. Then we split the range for each factor of $W_1(s)$ into dyadic intervals $[D_j, 2D_j)$ (if the factor is an $X_{b_i}(s)$) or $[E_j, 2E_j)$ (if the factor is a $Z^{(1)}_{d_i}(s)$), where

$$x^{-1} \ll D_j = 2^d \ll X_i^{1/2} \quad \text{and} \quad x^{-1} \ll E_j = 2^e \ll Z_i^{1/2}$$

for some integers d and e. We observe that our hypothesis that each factor of $W_1(s)$ is not too small ensures that the number of ranges (that is, the number of values taken by d and e above) is $\le C_2\mathcal{L}$ in each case, for some absolute constant C_2. For brevity we write $\mathcal{L}_0 = 2C_2\mathcal{L}$. We may divide the remaining points into at most $(\mathcal{L}_0/2)^{h+I_1}$ classes $\mathcal{S}(\mathbf{D},\mathbf{E})$ where $\mathbf{D} = (D_1,\ldots,D_h)$ and $\mathbf{E} = (E_1,\ldots,E_{I_1})$, for which

$$|X_{b_i}(s_n)| \in [D_i, 2D_i) \quad \text{and} \quad |Z^{(1)}_{d_i}(s_n)| \in [E_i, 2E_i). \tag{7.1}$$

We write

$$\mathcal{P}(\mathbf{D},\mathbf{E}) := \prod_i D_i \prod_i E_i.$$

As above, we estimate $W_2(s)$ trivially and conclude that

Lemma 15. *There exists a set $\mathcal{S}(\mathbf{D},\mathbf{E})$ of well-spaced points $\tau_n \in [T, 2T]$ satisfying* (7.1) *and such that*

$$J_1 \ll T + x_2\mathcal{P}(\mathbf{D},\mathbf{E})^2|\mathcal{S}(\mathbf{D},\mathbf{E})|\mathcal{L}_0^{h+I_1}.$$

We shall give upper bounds for $|\mathcal{S}|$ by means of Lemmas 13 and 14. Since these bounds are essentially the same as in [4] we simply quote the results.

LEMMA 16. *If the hypotheses of Lemma* 13 *hold for* $K(s) = X_i(s)$ *with* $K = 2X_i \geq T^{1/2}$ *then either*

$$\mathcal{K} \ll K^{1/2}T^{-1}(\log K)^3 \tag{7.2}$$

or

$$|\mathcal{S}| \ll \mathcal{K}^{-4}T(\log K)^9.$$

This is Lemma 18 of [4].

If (7.2) holds, the trivial bound $|\mathcal{S}| \ll T$ and Lemmas 15 and 16 imply

LEMMA 17. *If* $X_i \geq \frac{1}{2}T^{1/2}$ *for some* i *then either*

$$|\mathcal{S}| \ll \mathcal{K}^{-4}T(\log K)^9 \tag{7.3}$$

or

$$J_1 \ll T + x_1x_2T^{-1}\mathcal{L}_0^{3+h+I_1}. \tag{7.4}$$

The second estimate is proved taking $\mathcal{K} = D_i$ in (7.2) and observing that the definition implies that $\mathcal{P} \ll |W_1(s_n)|$. Since $\mathcal{L}_0^{3+h+I_1} \ll \mathcal{E}$ and $x_1x_2 \leq x$, (7.4) yields the conclusion of Lemma 6 and more.

Large factors of $W_1(s)$. The argument here is essentially the same as in Section 8 of [4], and Lemma 6 follows precisely in the same way, since the results in that section are bounds for $|\mathcal{S}|$. We take a factor of $W_1(s)$, $K(s) = X_{b_i}(s)$ or $Z_{d_i}^{(1)}(s)$, and let $K = 2X_i$ or $2Z_i$, $\mathcal{K} = D_i$ or E_i accordingly. We define σ by means of $\mathcal{K} = K^{\sigma-1/2}$. The argument in Section 8 of [4] is as follows: if φ is the maximum value of a σ occurring above then

$$\mathcal{P}(\mathbf{D},\mathbf{E})^2 \leq \prod_i D_i^{2\varphi-1}\prod_i E_i^{2\varphi-1} \leq x_1^{2\varphi-1}, \tag{7.5}$$

and by Lemma 15 we have

$$J_1 \ll T + xx_1^{2\varphi-2}\mathcal{L}_0^{h+I_1}|\mathcal{S}(\mathbf{D},\mathbf{E})|. \tag{7.6}$$

If $\varphi \geq 5/6$ then suitable choices of g in Lemma 13 yield

$$|\mathcal{S}(\mathbf{D},\mathbf{E})| \ll (T^{2-2\varphi} + z^{4-4\varphi})\mathcal{L}^{29}\mathcal{E}^{3/2},$$

and the upper bounds for T and z in the hypothesis of Lemma 6 together with (7.5) and (7.6) yield

$$J_1 \ll T + xx_1^{(\varphi-1)/6}\mathcal{L}_0^{29+h+I_1}\mathcal{E}^{3/2}.$$

The upper bound for $x_1^{\varphi-1}$ which we need is provided by Lemma 14 and the inequality $K \ll x$. In conclusion, since $\mathcal{L}_0^A \ll_A \mathcal{E}$, we see that Lemma 6 follows if $\varphi \geq 5/6$.

Conclusion of the proof of Lemma 6. In the remaining case, Heath-Brown's argument leads to the stronger inequality

$$J_1 \ll x^{1-\gamma} \tag{7.7}$$

for some $\gamma > 0$. This follows from several bounds for $|\mathcal{S}|$ which are essentially the same as in our case. We very briefly sketch the argument, without entering into the details. First the hypotheses of Lemma 6 ensure that

$$J_1 \ll T + x^{o(1)}\mathcal{P}^2|\mathcal{S}|.$$

By means of Lemma 13 we prove the following bounds: If $K(s) = X_{b_i}(s)$ then

$$|\mathcal{S}| \ll \begin{cases} T^{12(1-\sigma)/5}x^{o(1)} & \text{in any case,} \\ (T/X_i)^{4-4\sigma}x^{o(1)} & \text{if } T^{2/5} \le X_i \le T^{1/2}, \\ T^{2-2\sigma}x^{o(1)} & \text{if } X_i \ge T^{1/2}, \end{cases}$$

and if $K(s) = Z_{d_i}^{(1)}(s)$ then

$$|\mathcal{S}| \ll T^{12(1-\sigma)/5}x^{o(1)}.$$

Using these bounds we see that (7.7) holds provided that the following conditions hold.

First case. If $X_i \ge x^{1/3+\delta}$ for some $\delta \ge \beta$ and $\sigma \ge \varphi - \varepsilon$ we need to have

$$\gamma < \min\left(\tfrac{1}{6} - \beta, \tfrac{1}{18} - \tfrac{1}{3}\beta - 2\varepsilon, \tfrac{2}{3}\delta - \tfrac{2}{3}\beta - 2\varepsilon\right).$$

Second case. If $X_i \ge x^{1/3+\delta}$ for some $\delta \ge \beta$ and $\sigma \le \varphi - \varepsilon$ we need to have

$$\gamma < \min\left(\tfrac{1}{6} - \beta, \tfrac{2}{3}\varepsilon - \beta\right).$$

Third case. If $X_i \le x^{1/3+\delta}$ for all i we need

$$\gamma < \min\left(\tfrac{1}{6} - \beta, \tfrac{2}{3}\varepsilon - \beta - 4\delta\varepsilon, \tfrac{1}{6}\alpha - \tfrac{1}{3}\beta - 2\varepsilon\right).$$

Now, we easily see that the choices

$$\delta = \tfrac{1}{30}, \qquad \beta = \tfrac{1}{30}\alpha, \qquad \varepsilon = \tfrac{1}{15}\alpha$$

allow the choice $\gamma = \alpha/50$ and satisfy the hypotheses of Lemma 6.

8. Proof of Lemma 10. This lemma is proved in a similar fashion to Lemma 11 in [4] and we simply sketch the argument, with the necessary changes. As in Section 10 of [4], let $\mathbf{F} = (F_1, F_2, F_3)$ and $\mathcal{S}(\mathbf{F})$ be a set of well-spaced points $\tau_n \in [T, 2T]$ such that

$$F_i \le |P_{e_i}(1/2 + i\tau_n)| < 2F_i \qquad \text{for } i = 1, 2, 3.$$

The same argument of Section 7 gives

$$\int_T^{2T} |P(1/2 + i\tau)|^2\, d\tau \ll T_2 + \mathcal{L}^3|\mathcal{S}(\mathbf{F})| \prod_{i=1}^{3} F_i^2 \tag{8.1}$$

for some $\mathbf{F}$. Fix an index i and set $\mathcal{K} = F_i = V_i^{\sigma-1/2}$ and $K = 2V_i$. We remark that our choice of parameters implies that

$$T_2^{1/3} \ll K \ll T_2^{1/2}. \tag{8.2}$$

We use Lemma 13 with several different values of g. First, if $\varphi = \max\sigma \geq 5/6$, we choose $g = 2$ and (8.2) implies that

$$|\mathcal{S}(\mathbf{F})| \ll T_2^{2-2\varphi}\mathcal{L}^{29},$$

and Lemma 10 easily follows as in [4], on substituting into (8.1), since $\prod F_i^2 \leq \prod V_i^{2\varphi-1} \leq x^{2\varphi-1}$. An upper bound for $x^{\varphi-1}$ is provided by Lemma 14. In the other case, choose $g = 3$ to obtain

$$|\mathcal{S}(\mathbf{F})| \ll K^{6-6\sigma}\mathcal{L}^{59} \tag{8.3}$$

or g in such a way that $T_2K^{-1/2} \leq K^g \leq T_2K^{1/2}$. In the latter case we have

$$|\mathcal{S}(\mathbf{F})| \ll (TK^{1/2})^{2-2\sigma}\mathcal{L}^{59} \tag{8.4}$$

since $g \leq 3$ anyway. Since now $\sigma \leq 5/6$, (8.3) and (8.4) imply

$$|\mathcal{S}(\mathbf{F})| \ll K^{6-6\sigma}(T_2K^{-5/2})^{1/3}\mathcal{L}^{59}$$

when $K \leq T_2^{2/5}$ and when $K \geq T_2^{2/5}$ respectively. This means that

$$\begin{aligned}
&F_i^6|\mathcal{S}(\mathbf{F})| \ll (K^{\sigma-1/2})^6K^{6-6\sigma}\mathcal{L}^{59} = K^3\mathcal{L}^{59},\\
&F_i^6|\mathcal{S}(\mathbf{F})| \ll (K^{\sigma-1/2})^6K^{6-6\sigma}(T_2K^{-5/2})^{1/3}\mathcal{L}^{59} = K^3(T_2K^{-5/2})^{1/3}\mathcal{L}^{59}.
\end{aligned}$$

We use the former for $i = 1, 2$, and the latter for $i = 3$, take their geometric mean, and from (8.1) we obtain Lemma 10 in this case too, since $F_i^2 \leq V_i^{2\sigma-1} \leq V_i$.

9. Some comments. The knowledgeable reader sees at once that we had to make a different choice for the Dirichlet polynomials from Heath-Brown [4]. Indeed, the choice therein leads to too large error terms in Lemma 4 since we have a larger z than Heath-Brown and a much smaller h. This is due to the fact that we need z to be almost $x^{1/3}$, since we have the same problems he encounters in Section 9 when the product W has 6 factors, but already with only 3 factors. The slight additional difficulty is more than compensated by the fact that we only have to save a little over the estimate given by Montgomery's theorem, since our problem leads naturally to estimating the mean-square of a Dirichlet polynomial.

We did not use Watt's mean-value bound (Theorem 2 of [12]) in proving Lemma 6, because the hypothesis $T \geq K^4$ (in our notation) limits the former's usefulness in this problem to a subrange of the values of the parameters in Lemma 6. In particular, the case when some function $X_{b_i}(s)$ or $Z_{d_i}(s)$ has length K ($= X_i$ or Z_i resp.) bounded by $x^{1/6-\alpha}$ can be more easily handled by means of Montgomery's theorem alone. Compare the comment following the proof of Proposition 2.2 in [12] with the hypothesis of our Lemma 17. Even the more general Theorem 1 of Watt's paper [11] has, essentially, the same disadvantage.

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